Superconformal constraints for QCD conformal anomalies.

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Abstract

Anomalous superconformal Ward identities and commutator algebra in $\mathcal{N} = 1$ super-Yang-Mills theory give rise to constraints between the QCD special conformal anomalies of conformal composite operators. We evaluate the superconformal anomalies that appear in the product of renormalized conformal operators and the trace anomaly in the supersymmetric spinor current and check the constraints at one-loop order. In this way we prove the universality of QCD conformal anomalies, which define the non-diagonal part of the anomalous dimension matrix responsible for scaling violations of exclusive QCD amplitudes at the next-to-leading order.

Keywords: superconformal algebra, Ward identities, anomalous dimensions, conformal operators, superconformal anomalies

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1 Introduction.

Supersymmetric models [1], having higher space-time symmetry as compared to conventional ones, provide a strong consistency requirement on theoretical predictions. For the purposes of testing massless QCD calculations an especially illuminating example is $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, since both models have, up to a difference in colour representation of fermion fields, the same Lagrangian. Thus, we can map a QCD result to an $\mathcal{N} = 1$ super-Yang-Mills theory one by identifying the colour Casimir operators in corresponding representations, i.e. $C_A = C_F = 2N_fT_F ≡ N_c$. After this procedure a QCD result has to fulfill constraints arising from supersymmetry which connects gluonic and quark sectors of the theory. In this way the use of supersymmetry has allowed to find a set of identities [2, 3] between the entries of the forward anomalous dimension matrices of leading twist-two composite operators. They were valuable to clarify subtleties appearing in two-loop computations of anomalous dimensions. On tree level both theories are invariant under conformal transformations. Thus, the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory is also invariant under superconformal transformations [4, 5], which can give rise to a new set of constraints for certain conformal quantities which appear in the special conformal Ward identities for composite operators, the so-called special conformal matrix. Besides the breaking of conformal symmetry on quantum level by the trace anomaly in the energy-momentum tensor, we also have to deal with a superconformal anomaly due to the non-vanishing trace in a supersymmetric spinor current. Nevertheless, an explicit calculation of this anomaly will allow to check the special conformal anomalies calculated in QCD.

Composite operators appear in various QCD applications by means of the operator product expansion and consequently their hadronic matrix elements contain a non-perturbative input, which is needed as an initial condition for the solution of the evolution equations. In the case of exclusive processes the off-forwardness of hadronic matrix elements, given in terms of distribution amplitudes and skewed parton distributions, requires operators with total derivatives. To ensure that the twist-two operators do not mix under renormalization at leading order in coupling constant, i.e. their anomalous dimension matrix has the diagonal form $\gamma_{j\delta}j_{jk}$, it is necessary to arrange the operators in such a way that they have a covariant behaviour under conformal transformations. This can be easily done. However, beyond one-loop approximation the anomalous dimension matrix $\gamma_{jk}$ develops non-zero non-diagonal, $j > k$, elements $\gamma_{jk}^{\text{ND}} \propto O(\alpha_s^2)$.

The ordinary conformal algebra provides severe restrictions [6, 7] on the non-forward anomalous dimensions $\gamma$ of the conformal operators. In Refs. [6, 7] we have developed a formalism based on the use of the broken conformal Ward identities for evaluation of the non-diagonal part, $\gamma^{\text{ND}}$, of the complete anomalous dimensions matrix $\gamma = \gamma^D + \gamma^{\text{ND}}$. This non-diagonal part arises entirely due to the violation of the special conformal symmetry in perturbation theory. The corresponding
anomalies have been calculated to one-loop order accuracy in the minimal subtraction scheme using dimensional regularization, which imply the two-loop approximation for \( \gamma^{\text{ND}} \). To check our results, one can employ \( \mathcal{N} = 1 \) super Yang-Mills constraints, valid in a renormalization scheme that respects supersymmetry, for the entries of the non-forward anomalous dimension matrix, derived in [3]. Unfortunately, this is not the case for the dimensionally regularized theory. Thus, one has to find finite renormalization constants from the latter to the dimensional reduction scheme, which is expected to preserve the supersymmetry. But there arises a subtlety in the evaluation of this rotation matrix for the gluonic sector\(^1\) which prevents it to be unambiguously fixed [3]. Nevertheless, our result for two-loop non-forward anomalous dimensions is supported by the fact that the constraints can be fulfilled by a finite multiplicative renormalization, which proves the existence of a supersymmetric regularization scheme.

Alternatively, we derive in this paper constraints directly for the special conformal anomalies at one-loop level and show that they are indeed satisfied. Our consequent presentation is organized as follows. In section 2 we define conformal operators, their anomalous dimensions, and relations of the latter to conformal anomalies. Section 3 is devoted to the study of translational and conformal super-anomalies on Lagrangian level in the dimensional regularization scheme. Then in section 4 we present transformation properties of conformal operators under relevant superconformal variations required for a derivation of the Ward identities discussed in the same section. In section 5 we give a derivation of relations between the scale and special conformal anomalies of conformal operators. Furthermore, we show that the latter acquire anomalous contributions originating from the product of the trace anomaly in the spinor current and conformal operators. They are explicitly evaluated in section 6, where it is demonstrated that indeed the anomalous constraints are fulfilled with special conformal anomalies from [7]. Finally, we conclude. A few appendices are devoted to technical details that we found inappropriate to include in the body.

2 Preliminaries.

In this paper we discuss relations between the QCD scale and special conformal anomalies of conformal operators implied by the \( \mathcal{N} = 1 \) supersymmetry. In \( \mathcal{N} = 1 \) super Yang-Mills theory we introduce the conformal operators (for chiral even sector discussed throughout)

\[
Q^a_{\mu l} = \frac{1}{2} \bar{\psi}^a(i\partial_+)^l C^3_{j l} \left( \frac{\partial_+}{\partial_+} \right) \Gamma^i \psi^a, \quad G^a_{\mu l} = G^a_{+\mu}(i\partial_+)^{l-1} C^5_{j l-1} \left( \frac{\partial_+}{\partial_+} \right) T_{\mu
u} G^a_{\nu+},
\]

where \( C^\nu_j \) are the Gegenbauer polynomials and the tensor structures are \( \Gamma^{(V;A)} = (\gamma_+; \gamma_+ \gamma_5) \), \( T^{(V;A)}_{\mu\nu} = (g^{(V;A)}_{\mu\nu} - n_\mu n^*_\nu - n^*_\mu n_\nu; i\epsilon_{\mu\nu\rho\sigma} n^*_\rho n_\sigma) \). We use the convention \( \partial = \partial^+ + \partial^- \) and \( \bar{\partial} = \bar{\partial}^- \bar{\partial}^+ \)

\(^1\)This complication does not show up in the forward kinematics.
with adjoint covariant derivative defined by $D^a_{\mu} = \partial_{\mu} \delta^a + g f^{acb} B_{\mu}^c$. The ‘+’-sign as a subscript stands for contraction with the light-like vector $n_{\mu}$ which specifies a direction along the light cone. For the latter purposes we introduce another vector $n^*_{\mu}$ such as $n^2 = n^{*2} = 0$ and $n \cdot n^* = 1$. Obviously, the only difference from QCD arises in the gluino, which we loosely call quark, sector, which now belongs to the adjoint representation of the colour group. The factor $\frac{1}{2}$ in Eq. (1) is related to the Majorana nature of the quarks in the model.

The renormalization group equation for these operators looks like

$$\frac{d}{d \ln \mu} [\mathcal{O}_{jl}] = - \sum_{k=0}^{j} \gamma_{jk} [\mathcal{O}_{kl}],$$

(2)

where the square brackets will denote the renormalized operators defined by $[\mathcal{O}_{jl}] = \sum_{k=0}^{j} Z_{jk} \mathcal{O}_{kl}$, with renormalization constant matrix $Z_{jk}$, which generate finite Green functions with elementary field operators $\phi = \{\psi, B_{\mu}\}$. We use everywhere the matrix notation and introduce the vector $\mathcal{O} = \left(\frac{Q}{G}\right)$ of quark and gluon conformal operators, which mix with each other under renormalization. As mentioned in the introduction, the only modification of a given QCD result is to identify the colour Casimir operators. Since at leading order the conformal anomalies have a unique colour structure\(^2\) it presents no difficulty to disentangle the separate components.

We introduce as well the fermionic operator which is related to the bosonic ones (1) by supersymmetry

$$F \mathcal{O}_{jl}^i = G_{+\mu}(i \partial_\mu)^l P_{j}^{(2)} \left(\frac{\mathcal{D}_+}{\partial_+}\right) \mathcal{F}_{-\mu}^i \psi_+^a.$$  

(3)

Here $P_{j}^{(a,b)}$ are the Jacobi polynomials and the vertices read $\mathcal{F}^{(V;A)} = (\gamma_{\mu}^+; \gamma_{\mu}^+ \gamma_5)$. The operators that form a representation of the supersymmetry algebra are defined by linear combinations of (1)

$$\left\{ \begin{array}{l}
S^a_{jl} = Q^a \mathcal{O}_{jl}^F \quad \mathcal{P}^a_{jl} = G^a \mathcal{O}_{jl}^F,
\end{array} \right.$$

(4)

with $\Gamma = V(A)$ standing for the $S$ ($\mathcal{P}$) operator, and coefficients $Q^1 = 1$, $G^1 = \frac{\epsilon_j}{j}$, $Q^2 = -\frac{j+3}{j+1}$, $G^2 = \frac{6}{j+1}$ and $\theta_j = \frac{(j+2)(j+3)}{(j+1)}$. Obviously, $U = -\gamma_5 V$. Note that the bosonic and fermionic conformal operators form the $\mathcal{N} = 1$ chiral superfield

$$\Phi = \mathcal{A} + 2 \theta \chi - \theta^2 \mathcal{F},$$

(5)

with operators $S^1_{jl}$ and $\mathcal{P}^1_{jl}$ ($S^2_{jl}$ and $\mathcal{P}^2_{jl}$) being the real and imaginary parts of the $\mathcal{A}$ ($\mathcal{F}^\dagger$) complex fields, and $V_{j-1,l}$ identified with the Majorana fermion $(\chi_a^\dagger \chi_a)$ constructed from the Weyl spinor $\chi$.

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\(^2\)More precisely, the anomalous dimensions in the gluon-gluon channel have in addition to the $C_A$ term also trivial $N_f$ dependent contributions, which arise from the self-energy insertion.
Transformation between operators under supersymmetry arises from the conventional equation
\[ \{\bar{\zeta} Q, \Phi\}_- = [\zeta Q + \bar{Q} \bar{\zeta}, \Phi\}_- = \{\zeta r + \bar{r} \bar{\zeta}\} \Phi, \]
with \( r = i \frac{\partial}{\partial \theta} \) and \( \bar{r} = -i \frac{\partial}{\partial \bar{\theta}} + 2 \theta \phi \).

Now let us shortly point out, how the non-diagonal part of the anomalous dimension matrix is induced by the special conformal anomaly matrix. In four dimensional space-time the fifteen-parameter conformal group \( SO(4, 2) \) is defined by its algebra containing the Poincaré, dilatation \( \mathcal{D} \) and special conformal \( \mathcal{K}_\mu \) generators. The conformal anomalies are defined by the renormalized Ward identities. The generic form of the latter, however, written in an unrenormalized cast, reads
\[ \langle [\mathcal{O}_{ij}] \delta \mathcal{X} \rangle = -\langle \delta [\mathcal{O}_{ij}] \mathcal{X} \rangle - \langle i [\mathcal{O}_{ij}] \delta S \mathcal{X} \rangle, \]  
where \( \mathcal{X} = \prod \phi(x_\ell) \) is a product of elementary fields appearing in the classical Lagrangian. Here \( \delta \) is any of the variations from the symmetry algebra in question. When the transformation is a symmetry of the theory on quantum level then \( \delta S = 0 \) up to possible BRST exact operators. In the (dimensionally) regularized theory the action does not vanish anymore for conformal, i.e. both scaling, \( \delta^S \phi(x) = i[\phi(x), \mathcal{D}] = - (d_\phi + x_\nu \partial_\nu) \phi(x) \), and special conformal,
\[ \delta^C_\mu \phi(x) = i[\phi(x), \mathcal{K}_\mu] = -(2d_\phi x_\mu - x^2 \partial_\mu + 2x_\mu x_\nu \partial_\nu - 2ix_\nu \Sigma_{\mu\nu}) \phi(x), \]
variations, where \( d_\phi \) and \( \Sigma_{\mu\nu} \) are the canonical dimension and the spin operator of the field \( \phi \), respectively. Thus, the renormalization of the operator product \( i[\mathcal{O}_{ij}] \delta S \) is responsible for the conformal anomalies.

Moreover, the commutator \( [\mathcal{D}, \mathcal{K}_-]_- = i\mathcal{K}_- \), where \( \mathcal{K}_- \) being the \( n^*_\mu \)-light-cone projection of \( \mathcal{K}_\mu \), provides a connection between the conformal anomalies. In Ref. [7] the non-diagonal elements of the next-to-leading anomalous dimensions
\[ \gamma^{ND(1)} = [\gamma^{D(0)}, d (\beta_0 - \gamma^{D(0)}) + g]_-, \]  
of QCD quark and gluon conformal operators were found in terms of one loop special conformal anomaly matrix
\[ a^{-1}_{jk}(B)\gamma^{c(0)}_{jk} \equiv -d_{jk} \left( \gamma^{D(0)}_k - \beta_0 P^G \right) + g_{jk}. \]  
It is constructed out of the leading order anomalous dimensions of conformal operators \( \gamma^{D(0)} \) and the first expansion coefficient of the QCD \( \beta \)-function \( \beta_0 = \frac{4}{3} T_F N_f - \frac{11}{3} C_A \) times \( a_{jk}(B)d_{jk}(F)|_{j>k} = -2(2k + 3) \) with \( a_{jk} \) matrix from the conformal transformation of \( \mathcal{O}_{jk} \) (see Eq. (29) below). The projector \( P^G = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \) in Eq. (8) singles out the gluonic component. Finally, the \( g \)-matrix has appeared from the renormalization of the product of the conformal operator \( [\mathcal{O}_{jk}] \) and the integrated trace anomaly \( \delta^-S \propto - \int d^4 x 2x_\mu \Theta_{\mu\nu}(x) \) in the energy-momentum tensor
\[ [\mathcal{O}_{ij}] \delta^-S = i \sum_{k=0}^j \gamma^{c}_j [\mathcal{O}_{kl-1}] + \cdots. \]  
In the dimensionally regularized theory, i.e. \( d = 4 - 2\varepsilon \), the conformal variations of the QCD action can be calculated in a straightforward manner. Choosing the scaling dimensions of the physical
fields equal to their canonical values in four dimensions\(^3\) \((d_\psi = \frac{d}{2}, d_B = 1)\) and setting the scaling dimensions of the ghost fields as \(d_\bar{\omega} = 0\) and \(d_\bar{\bar{\omega}} = d - 2\), the result

\[
\delta^B S = \int d^4x \ w^B(x) \left\{ - \frac{d - 4}{2} \left( O_A(x) + O_B(x) + \Omega_\bar{\omega}(x) - \bar{\omega}_\psi(x) - \Omega_D(x) \right) + (d - 2) \partial_\mu O_{B\mu}(x) \right\}, \quad \text{with} \quad B = \{D, C\},
\]

is decomposed in operators that can be easily classified according to their renormalization properties. Here the weight function reads \(w^D = 1\) and \(w^C = 2x\) for scale and special conformal transformations, respectively. We introduced the following set of type A and B operators

\[
O_A(x) = \frac{1}{2} \left( G^a_{\mu\nu} \right)^2, \quad O_B(x) = \delta_{\text{BRST}} \left( \bar{\omega}^a \partial_\mu B^a_\mu \right), \quad O_{B\mu}(x) = \delta_{\text{BRST}} \left( \bar{\omega}^a B^a_\mu \right),
\]

as well as class C equation-of-motion operators

\[
\Omega_G(x) = B^a_\mu \delta_S \frac{\delta S}{\delta B^a_\mu}, \quad \Omega_{\bar{\psi}\psi}(x) = \frac{\delta S}{\delta \bar{\psi}} \psi + \frac{\delta S}{\delta \psi} \bar{\psi}, \quad \Omega_\bar{\omega}(x) = \bar{\omega}^a \frac{\delta S}{\delta \bar{\omega}^a}, \quad \Omega_D(x) = D^a \frac{\delta S}{\delta D^a}.
\]

The renormalization of Eq. (10) is straightforward and the renormalization of the operator products and the resulting renormalized conformal Ward identities are given in Ref. [7].

As a side remark let us note that in spite of the fact that the conformal field transformation laws for the dimensionally reduced, from \(d = 4 - 2\varepsilon\) (and \(\varepsilon < 0\)) to 4 dimensions, theory differs from the ones in dimensional regularization by the presence of \(\varepsilon\)-scalar contributions, e.g.\(^4\) \(\delta^D B^a_\mu = x_\nu G^a_{\mu\nu} - B^a_\nu\) and \(\delta^C B^a_\mu = (2x_\nu x_\rho - x_\mu x_\nu)n^a_\rho G^a_{\mu\nu} - 2x_\nu B^a_\mu\) for the gauge covariant variations of four-dimensional fields. Nevertheless, the final result for the variation of the action takes the same form as in Eq. (10) but with boson fields being four-dimensional instead.

### 3 Superconformal anomalies.

In four-dimensional space-time the classical action of the \(\mathcal{N} = 1\) \(SU(N_c)\) super-Yang-Mills theory in the Wess-Zumino gauge, i.e.

\[
S_{\text{cl}} \equiv \int d^4x \mathcal{L}_{\text{cl}}(x) = \int d^4 x \left\{ -\frac{1}{4} \left( G^a_{\mu\nu} \right)^2 + \frac{i}{2} \bar{\psi}^{\alpha} \bar{\psi}^{\beta} + \frac{1}{2} \left( D^a \right)^2 \right\},
\]

contains the Yang-Mills field strength \(G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + g f^{abc} B^b_\mu B^c_\nu\), the Majorana field \(\psi^a\) satisfying the conventional condition \(\psi^T C(+) = \bar{\psi}\), and an auxiliary field \(D^a\). It is invariant

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\(^3\)This choice is legitimate since the infinitesimal conformal variation is linear in \(d_\phi\) and thus does not affect the Ward identities, since the anomalous part will show up as a renormalization counterterm of the product of conformal and equation-of-motion operators.

\(^4\)The indices \(\mu, \bar{\mu}\) and \(\bar{\mu}\) refer to the 4, \(d\) and \(2\varepsilon\) dimensional spaces, respectively.
under transformation of the superconformal group which consists besides the conformal group also of the translational, $Q$, and conformal, $S$, supertransformations. The latter two are defined infinitesimally by their action on field operators as

$$
\delta F \psi^a = \frac{i}{2} C^a_{\mu \nu} \sigma_{\mu \nu} \zeta - i D^a \gamma_5 \zeta, \quad \delta F B^a_{\mu} = -i \bar{\zeta} \gamma_\mu \psi^a, \quad \delta F D^a = \bar{\zeta} \mathcal{D}^{ab} \gamma_5 \psi^b,
$$

where $\zeta \equiv \zeta_0 - i \bar{\zeta} \zeta_1$ is the Grassmann parameter. For $\zeta_1 = 0$ ($F = Q$) we have restricted supertransformations, while for $\zeta_0 = 0$ ($F = S$) these equations define the superconformal variations.

The superconformal group is defined by its algebra from which we will be interested in one particular commutator

$$
[Q, K_{-}] = \gamma_{-} S,
$$

with $Q$ ($S$) being super (conformal) generators. Note, however, that for the short supermultiplet $(B^a_{\mu}, \psi^a, D^a)$ this commutation relation is modified for action on fermions. To restore it one has to use Jackiw’s gauge covariant conformal transformation $\bar{\delta}_C^C \equiv \delta_C^C + \delta_{\text{gauge}}^C$ [8], where the gauge transformation is defined with field-dependent parameter $c_\mu^a \equiv (2x_\mu x_\nu - x^2 g_{\mu \nu}) B^a_{\nu}$, instead of the conventional $\delta_C^C$ variation defined above. For the action on a space spanned by gauge invariant operators this modification is irrelevant. The commutator (15), when applied on a Green function with conformal operator insertion, will provide in the supersymmetric limit (identifying colour factors) non-trivial relations between the afore mentioned QCD special conformal anomalies.

To quantize the theory described by the action (13) we have to add a gauge fixing and a ghost term. We do it via the covariant gauge fixing

$$
S_{gf} = \int d^4 x \left\{ -\frac{1}{2\xi} (\partial_\mu B^a_{\mu})^2 + \partial_\mu \bar{\omega}^a \mathcal{D}^{ab}_{\mu} \omega^b \right\}.
$$

Although it explicitly breaks the supersymmetry on Lagrangian level, it will not affect gauge invariant quantities since the supersymmetry variations (14) commute with BRST transformations on the gauge fixing function$^5$, i.e. $[\delta^F, \delta_{\text{BRST}}]_{-}(\partial_\mu B^a_{\mu}) = 0$.

Translational and conformal supervariation of the action $S = S_{cl} + S_{gf}$ regularized by means of the dimensional regularization, $4 \rightarrow d = 4 - 2\varepsilon$, leads to

$$
\begin{align*}
\delta^Q S &= - (\bar{\zeta}_0 \mathcal{O}_{3\psi}) - \mathcal{O}_{Q}^{\text{BRST}}, \\
\delta^S S &= \frac{d-4}{2} (\bar{\zeta}_1 A) + \left(3\bar{\zeta}_1 \mathcal{O}_{3\psi}^{-} \right) - \mathcal{O}_{S}^{\text{BRST}},
\end{align*}
$$

$^5$We write for brevity $\delta_{\text{BRST}}$ instead of $\delta_{\text{BRST}}/\delta \lambda$, i.e. after transformation the infinitesimal Grassmann variable is canceled from the right. We recall that the BRST transformations are given by the set of equations $\delta_{\text{BRST}} B^a_{\mu} = \mathcal{D}^{ab}_{\mu} \omega^b$, $\delta_{\text{BRST}} \psi^a = g f^{abc} \omega^b \psi^c$, $\delta_{\text{BRST}} \omega^a = \frac{g}{2} f^{abc} \omega^b \omega^c$ and $\delta_{\text{BRST}} \bar{\omega}^a = \frac{1}{\xi} \partial_\mu B^a_{\mu}$. 

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where \( O \equiv \int d^d x \mathcal{O}(x) \) and the operator insertions read

\[
\mathcal{A}(x) = \sigma_{\mu
u} G^{a}_{\mu
u} \bar{\psi}^a,
\]

\[
\mathcal{O}^{\text{BRST}}_Q(x) = i \delta^{\text{BRST}} \delta Q \left( \bar{\psi}^a \partial_{\mu} B^a_{\mu} \right), \quad \mathcal{O}^{\text{BRST}}_S(x) = i \delta^{\text{BRST}} \delta S \left( \bar{\psi}^a \partial_{\mu} B^a_{\mu} \right),
\]

\[
\mathcal{O}_{3\psi}(x) = i \frac{g}{2} f^{abc} \left( \bar{\psi}^a \gamma_{\mu} \psi^b \right) \left( \gamma_{\mu} \psi^c \right), \quad \mathcal{O}_{3\psi}^-(x) = \frac{g}{2} f^{abc} \left( \bar{\psi}^a \gamma_{\mu} \psi^b \right) \left( \not\gamma_{\mu} \psi^c \right),
\]

with the operator \( \mathcal{A} \) being the superconformal anomaly [10, 11] in the trace of the supersymmetric current, i.e. \( Q_{\mu} = \frac{1}{2} G^{a}_{\mu
u} \sigma_{\mu\nu} \gamma_{\mu} \bar{\psi}^a \). We used in Eq. (20) the identity

\[
\delta^F \delta^{\text{BRST}} \left( \bar{\psi}^a \partial_{\mu} B^a_{\mu} \right) = 2 \delta^{\text{BRST}} \delta^F \left( \bar{\psi}^a \partial_{\mu} B^a_{\mu} \right) + \delta^F \Omega.
\]

Note that the three-fermion operators \( \mathcal{O}_{3\psi} \) and \( \mathcal{O}_{3\psi}^- \) vanish in four dimensions by means of Fierz rearrangement. Moreover, \( \mathcal{O}_{3\psi}^- \) can be generated by a special conformal variation of the operator \( \mathcal{O}_{3\psi} \), namely, \( \delta^{\text{C}}_\mu \mathcal{O}_{3\psi} = i \gamma_{\mu} \mathcal{O}_{\bar{\psi} \psi} + \mathcal{O}(\epsilon^2) \), where at one-loop level we can safely neglect the remainder.

Later on we will concentrate on the use of the \( \xi = -3 \) gauge in the derivation of the constraints, which ensures renormalized supersymmetry at one loop order [12]. This means that the quark and gluon anomalous dimensions are equal \( \gamma_{\phi} \equiv \gamma_{G} = \gamma_{\psi} \). Furthermore, at one-loop order it was found that anomaly \( \mathcal{A} \) does not acquire gauge variant counterterms [11], provided one uses this particular value of the gauge fixing parameter. Therefore, we can write to this accuracy \( \epsilon \mathcal{A} \) in terms of the renormalized operator

\[
\frac{d - 4}{2} \mathcal{A} = \frac{\beta_{\epsilon}}{g} [\mathcal{A}] + \mathcal{O}(\alpha_s^2),
\]

with \( d \)-dimensional \( \beta \)-function \( \beta_{\epsilon} = -\epsilon g + \beta \). Finally, we write the superconformal variation of the action to one-loop accuracy as

\[
i \delta^S S = \frac{\beta_{\epsilon}}{g} \bar{\zeta}_{\xi} [\mathcal{A}] + \bar{\zeta}_{\xi} \mathcal{O}_{\bar{\psi} \psi}^- - \mathcal{O}^{\text{BRST}}_S + \mathcal{O}(\alpha_s^2).
\]

Let us point out a further consequence of the \( \xi = -3 \) gauge, which also leads to the equality of anomalous dimensions \( \gamma_{\phi} \) and \( \beta \)-function, namely \( \beta/g = \gamma_{\phi} = -\alpha_s/(4\pi)3N_c \). Consequently, the renormalization of the conformal variation of the action will be simplified and its integrand in Eq. (10) reads in one-loop approximation (see e.g. Ref. [7])

\[
-\frac{d - 4}{2} \left( \mathcal{O}_A(x) + \mathcal{O}_B(x) + \Omega_{\omega}(x) - \Omega_{\bar{\psi} \psi}(x) - \Omega_D(x) \right)
= -\frac{\beta_{\epsilon}}{g} \left( [\mathcal{O}_A(x)] - \Omega_{\bar{\psi} \psi}(x) \right) - \frac{d - 4}{2} [\mathcal{O}_B(x) + \Omega_{\omega}(x) - \Omega_D(x)]
+ 2 \gamma_{\omega} \Omega_{\omega}(x) + (d - 2) \partial_{\mu} [\mathcal{O}_{B\mu}(x)],
\]

with ghost anomalous dimension \( \gamma_{\omega} \).
4 Superconformal Ward identities.

In order to derive Ward identities we have to know the change of conformal operators under the superconformal symmetry. Using the rules in (14) one finds that the translational supersymmetry transformation laws are given by (here and everywhere $\sigma_j = \frac{1}{2}[1 - (-1)^j]$)

$$\delta^Q S^1_{jl} = \sigma_j \tilde{\zeta}_0 V_{j-l}, \quad \delta^Q S^2_{jl} = \sigma_j \tilde{\zeta}_0 V_{jl}, \quad \delta^Q P^1_{jl} = \sigma_j+1 \tilde{\zeta}_0 U_{jl}, \quad \delta^Q P^2_{jl} = \sigma_j+1 \tilde{\zeta}_0 U_{jl},$$

$$\delta^Q V_{j-l-1} = -\gamma - \tilde{\zeta}_0 \left( S^1_{jl} + S^2_{j-l-1} \right) - \gamma - \tilde{\zeta}_0 \left( P^1_{jl} + P^2_{j-l-1} \right).$$

These equations make our comment about the formation of the conformal operators into the chiral superfield apparent: Eqs. (25,26) are in one-to-one correspondence with the supersymmetric rules for the Wess-Zumino multiplet [4]. Under superconformal variations conformal operators behave as follows

$$\delta^S S^1_{jl} = -\sigma_j (j + l + 3) \tilde{\zeta}_1 \gamma_+ V_{j-l-1}, \quad \delta^S S^2_{jl} = -\sigma_j (l - j) \tilde{\zeta}_1 \gamma_+ V_{jl},$$

$$\delta^S P^1_{jl} = -\sigma_j+1 (j + l + 3) \tilde{\zeta}_1 \gamma_+ U_{j-l-1}, \quad \delta^S P^2_{jl} = -\sigma_j+1 (l - j) \tilde{\zeta}_1 \gamma_+ U_{jl},$$

$$\delta^S V_{j-l-1} = -\gamma - \tilde{\zeta}_1 \left( (l - j) S^1_{jl} + (j + l + 2) S^2_{j-l-1} \right) - \gamma - \tilde{\zeta}_1 \left( (l - j) P^1_{jl} + (j + l + 2) P^2_{j-l-1} \right).$$

Note, that Eqs. (25,27) do not require Fierz rearrangement and, therefore, they do not change their form when the theory is regularized via dimensional regularization. Finally, let us recall that the transformation laws of conformal operators under scaling and special conformal variations are given by

$$\delta^{D\Omega} O_{jl} = - (l + d(\Omega)) O_{jl}, \quad \delta^{C\Omega} O_{jl} = i a_{jl}(\Omega) O_{jl-1}.$$

Here $d(B) \equiv d(G) = d(Q) = 3$ and $d(F) = \frac{7}{2}$ as well as $a_{jl}(B) \equiv a_{jl}(Q) = a_{jl}(G) = a(j, l, 1, 1)$ and $a_{jl}(F) = a(j, l, 2, 1)$ with $a(j, l, \nu_1, \nu_2) = 2(j - l)(j + l + \nu_1 + \nu_2 + 1)$, where $\nu_\phi = d_\phi + s_\phi - 1$ and $s_\phi$ being the spin of the field $\phi$. Again the scale dimension $d_\phi$ is chosen to coincide to its canonical value in four dimensions.

Due to difficulties to preserve the supersymmetry of the theory with quantization and regularization procedures our modest goal will be, therefore, a derivation of the constraints for the special conformal anomalies of the QCD conformal operators stemming from the commutator equation (15) at one loop level only. We will choose the covariant gauge with $\xi = -3$, which gives us the advantages mentioned above.

The dilatation and special conformal Ward identities for a conformal operator $\Omega O$, which is either bosonic ($\Omega = B$) or fermionic ($\Omega = F$) one, look now very simple (cf. Ref. [7])

$$\langle [\Omega O_{jl}] \delta^D \mathcal{X} \rangle = \sum_{k=0}^{j} \{(l + d(\Omega)) 1 + \gamma(\Omega)\}_{jk} \langle [\Omega O_{kl}] \mathcal{X} \rangle + \frac{\beta}{g} \langle i [\Omega O_{jl} \left( \frac{1}{2} - \frac{1}{2} \Omega_{\bar{\psi}} \right)] \mathcal{X} \rangle,$$
\[
\langle [\delta^Q \mathcal{O}_{ji}] \delta^C \mathcal{X} \rangle = -i \sum_{k=0}^j \{a_{ji}(\Omega) 1 + \gamma^c(\Omega)\} \langle [\delta^Q \mathcal{O}_{kl}] \mathcal{X} \rangle + \frac{\beta}{g} \langle i[\delta^Q \mathcal{O}_{ji} (\mathcal{O}_A - \Omega_{\bar{\psi}\psi})] \mathcal{X} \rangle \\
-2 \langle i[\delta^Q \mathcal{O}_{ji} \Delta_{\text{ext}}] \mathcal{X} \rangle,
\]

with a two-by-two unit matrix \(1 \equiv 1_{[2 \times 2]}\). Here \(\gamma\) and \(\gamma^c\) are the scale and special conformal anomalies. It is well known, the scale Ward identity coincides with the Callan-Symanzik equation. Thus, \(\gamma\) is the conventional anomalous dimension matrix and the combination \((l + d) 1 + \gamma\) is the scale dimension matrix of conformal operators. Obviously \(O_A(x) - \Omega_{\bar{\psi}\psi}(x) = -2 \mathcal{L}_{\text{cl}}(x)\) is the classical Lagrangian (13) without auxiliary fields. In Eq. (30,31) we have also introduced a new conventions for the operator insertions weighed with different functions, i.e. \(O = \int d^4 x \mathcal{O}(x)\), \(O^- = \int d^4 x \ 2x_- \mathcal{O}(x)\) (analogously for equation-of-motion operators) and \(\Delta_{\text{ext}} = \int d^4 x 2x_- \partial_\mu \mathcal{O}_{B\mu}(x)\). The precise definition of the renormalized operator products is given in the next section.

Let us turn to the renormalized supersymmetric Ward identities. For definiteness let us consider parity even \(S\) operators. From unrenormalized Ward identity (6), the superconformal variations of the action (17,23) and the operators (25,27) we can immediately derive the renormalized \(Q\) and \(S\) supersymmetric Ward identities in one loop approximation:

\[
\langle [S^a_{ji}] \delta^Q \mathcal{X} \rangle = -\langle [\delta^Q S^a_{ji}] \mathcal{X} \rangle + \langle \tilde{\gamma}_0 O_{3\psi} \rangle [S^a_{ji}] \mathcal{X} \rangle + \langle [S^a_{ji}] \mathcal{O}_{\text{BRST}} \mathcal{X} \rangle,
\]

\[
\langle [S^a_{ji}] \delta^S \mathcal{X} \rangle = -\langle [\delta^S S^a_{ji}] \mathcal{X} \rangle + \sigma_j \tilde{\gamma}_1 \gamma + \sum_{k=0}^j r_{jk}^{a_{V[1]}} \langle [V_{kl-1}] \mathcal{X} \rangle - \frac{\beta}{g} \langle \tilde{\gamma}_1 \mathcal{A} \rangle [S^a_{ji}] \mathcal{X} \rangle \\
-\langle \langle \tilde{\gamma}_1 O_{3\psi} \rangle [S^a_{ji}] \mathcal{X} \rangle + \langle [S^a_{ji}] O_{\text{BRST}} \mathcal{X} \rangle.
\]

Here the superconformal anomaly \(r_{jk}^{a_{V[1]}}\) is the residue of the renormalization constant

\[
r_{jk} = r_{jk}^{[0]} + \frac{1}{\varepsilon} r_{jk}^{[1]} + \ldots,
\]

arisen from the renormalization of the operator product

\[
\langle \tilde{\gamma}_1 \mathcal{A} \rangle [S^a_{ji}] = \langle \tilde{\gamma}_1 \mathcal{A} S^a_{ji} \rangle + \sigma_j \tilde{\gamma}_1 \gamma + \sum_{k=0}^j r_{jk}^{a_{V[1]}} [V_{k,l-1}]\]

and induced by the \(\varepsilon\) term of \(\beta\). Note that since the three-fermion operators vanish in four dimensions, their product with the bosonic operators will give a finite contribution at one loop order. Similar equations hold for \(P_{ji}\) with the replacement of index \(V\) by \(A\) and \(\sigma_j\) by \(\sigma_{j+1}\).

We have neglected infinite terms in the above Ward identities, since they have to cancel each other. It is instructive to discuss this issue in more detail for the supersymmetric Ward identity of the two-vector \(S = \left(\begin{array}{c} S^1 \\ S^2 \end{array} \right)\)

\[
\langle [S_{ji}] \delta^Q \mathcal{X} \rangle = -\langle \delta^Q [S_{ji}] \mathcal{X} \rangle - \langle i[S_{ji}] (\delta^Q S) \mathcal{X} \rangle.
\]
The variation of “good” component of the fermion field, $\psi_+ \equiv \frac{1}{2} \gamma_+ \gamma^a \psi$, entering in $\mathcal{X}$ may cause a divergency on the l.h.s. since it contains a composite field strength $G_{\mu \nu}$. Fortunately, in the light-cone gauge the latter can be expressed in terms of elementary vector potential $B_\mu$ and, therefore, the l.h.s. is finite by definition. This gauge, together with the use of dimensional reduction which implies $\delta^Q S = 0$, leads to

$$\langle \delta^Q [\mathcal{S}_{jl}], \mathcal{X} \rangle = \text{finite}. \quad (37)$$

Since the renormalization of the composite operators is both gauge and scheme independent at leading order, Eq. (37) holds also true for our choice of scheme. Furthermore, in Eq. (32) the renormalization of the product $[\mathcal{S}_{jl}] \mathcal{O}_3 \psi$ is finite at leading order because $\mathcal{O}_3 \psi \sim \mathcal{O}(\varepsilon)$ and cancels a pole in one-loop diagrams. Consequently, the product $[\mathcal{S}_{jl}] \mathcal{O}_Q^{\text{BRST}} = \text{finite}$, or if divergent it cancels singularities in $[\mathcal{S}_{jl}] \delta^Q \psi$ mentioned above.

Now we discuss the consequences of Eq. (37). The components of the renormalized operators $[\mathcal{S}_{jk}]$ are defined in terms of unrenormalized ones, constructed though from the renormalized fields $\phi$, by

$$[\mathcal{S}^a_{jl}] = \sum_{b=1}^{2} \sum_{k=0}^{j} \{a \mathcal{S}^b_{kl} \} jk, \quad \mathcal{S}^a_{kl} = Z_a^{-1} \mathcal{S}^a_{kl} \langle 0 \rangle,$$

where $\mathcal{S}^a_{jl}$ is expressed in terms of the bare fields $\phi^{(0)} = \sqrt{Z_{\phi}} \phi$ and coupling $g^{(0)} = \mu^g / \sqrt{Z_{\phi}}$. The anomalous dimension matrix of the vector $[\mathcal{S}]$ is defined as usual

$$\frac{d}{d \ln \mu} [\mathcal{S}]_{jl} = \sum_{k=0}^{j} \gamma^S_{jk} [\mathcal{S}]_{kl}, \quad \text{with} \quad \gamma^S = \gamma^Z + 2 \gamma_\phi = \left( \begin{array}{cc} 11\gamma & 12\gamma \\ 21\gamma & 22\gamma \end{array} \right). \quad (39)$$

In our scheme the anomalous dimensions

$$\gamma_\phi = \frac{1}{2} \frac{d}{d \ln \mu} \ln Z_{\phi} = \frac{1}{2} \frac{\partial}{\partial \ln g} Z^{[1]}_\phi, \quad \text{and} \quad \gamma^S = \left( \frac{d}{d \ln \mu} Z^S \right) Z^{-1}_S = \frac{\partial}{\partial \ln g} Z^{[1]}_S, \quad (40)$$

are expressed by the residues of the Laurent expansion of the $Z$ factors $Z = 1 + Z^{[1]}_\phi / \varepsilon + \mathcal{O}(\varepsilon^{-2})$. The renormalized fermionic operator is defined by the same equation (38), however, with the $\epsilon$-poles have to cancel. This is ensured by the relations

$$\sum_{k=0}^{j} \{11Z^S_{jk} \} jk \sigma_k \{ \mathcal{V}_{k-1,l} \} + \sum_{k=0}^{j} \{12Z^S_{jk} \} jk \sigma_k \{ \mathcal{V}_{kl} \} = \sigma_j \sum_{k=0}^{j} \{Z^S_{j-1,k} \} jk \{ \mathcal{V}_{kl} \}, \quad (41)$$

where we implied that $Z_{jk} = 0$ for $k > j$. Substituting the Laurent series into this result, the $\frac{1}{\varepsilon}$-poles have to cancel. This is ensured by the relations

$$\sum_{k=0}^{j} \{11Z^S_{jk} \} jk \sigma_k \{ \mathcal{V}_{k-1,l} \} + \sum_{k=0}^{j} \{12Z^S_{jk} \} jk \sigma_k \{ \mathcal{V}_{kl} \} = \sigma_j \sum_{k=0}^{j} \{Z^S_{j-1,k} \} jk \{ \mathcal{V}_{kl} \}, \quad (42)$$

$$\sum_{k=0}^{j} \{21Z^S_{jk} \} jk \sigma_k \{ \mathcal{V}_{k-1,l} \} + \sum_{k=0}^{j} \{22Z^S_{jk} \} jk \sigma_k \{ \mathcal{V}_{kl} \} = \sigma_j \sum_{k=0}^{j} \{Z^S_{1,j} \} jk \{ \mathcal{V}_{kl} \},$$
where \([V_{kl}]\) are independent operators. These when combined together with the analogous results for parity odd operators (replace \(S \rightarrow P\), \(V \rightarrow U\), and \(\sigma_j \rightarrow \sigma_{j-1}\)) and \(Z_V = Z_U\) provide constraints for anomalous dimensions (when differentiated w.r.t. \(\ln g\)) \([2, 3]\):

\[
\begin{align*}
\{11Z_S^{[1]}\}_{2n+1,2m+1} &= \{22Z_P^{[1]}\}_{2n,2m} = \{Z_V^{[1]}\}_{2n,2m}, \\
\{11Z_P^{[1]}\}_{2n,2m} &= \{22Z_S^{[1]}\}_{2n-1,2m-1} = \{Z_V^{[1]}\}_{2n-1,2m-1}, \\
\{12Z_S^{[1]}\}_{2n+1,2m+1} &= \{21Z_P^{[1]}\}_{2n,2m+2} = \{Z_V^{[1]}\}_{2n,2m+1}, \\
\{21Z_P^{[1]}\}_{2n+1,2m+1} &= \{12Z_S^{[1]}\}_{2n+2,2m} = \{Z_V^{[1]}\}_{2n+1,2m+1}.
\end{align*}
\tag{43}
\]

Now let us turn to the renormalization of the superconformal Ward identities. An important consequence of these constraints is that the operators are multiplicatively renormalizable in the one-loop approximation, e.g. \(\{12Z_S^{[1]}\}_{jj} = \{21Z_P^{[1]}\}_{jj} = 0\) and \(\{11Z_S^{[1]}\}_{jj} = \{22Z_P^{[1]}\}_{jj} = \{Z_V^{[1]}\}_{jj}\).

Thus, in this approximation the classical transformation laws \((27)\) for superconformal variation remain true also for the renormalized operators. Consequently, the superconformal variation of the renormalized operator provide finite Green functions

\[
\left\langle \delta^S[\mathcal{S}_{jl}] \mathcal{X} \right\rangle = \langle [\delta^S \mathcal{S}_{jl}] \mathcal{X} \rangle = \text{finite}. \tag{44}
\]

Since \(O_{\bar{3}\psi}^{-}\) vanishes in four dimensional space-time, \(\left\langle (\bar{\zeta}_l O_{\bar{3}\psi}^{-}) [\mathcal{S}_{jl}] \mathcal{X} \right\rangle\) is finite. Thus, also the superconformal Ward identity \((33)\) is renormalized up to possible divergencies on the l.h.s. that are canceled by the renormalization with BRST-exact operators on the r.h.s. of this Ward identity. Note that in our leading order approximation the anomalous term proportional to the \(\beta\) function is given by a tree approximation.

### 5 Constraint equalities for conformal anomalies.

Having derived Ward identities we are now able to discuss consequences of the superconformal algebra. To demonstrate the method, we derive at first the set of relations \((43)\) between the anomalous dimensions arising from the commutator of super and scaling variations

\[
[\delta^Q, \delta^D]_- = \frac{1}{2} \delta^Q, \tag{45}
\]

which is deduced from the commutator algebra \([Q, \mathcal{D}]_- = \frac{i}{2} Q\). Next we deal in the same conceptual manner with the commutator \((15)\) of super and special conformal transformations, written in a symbolical form as

\[
[\delta^Q, \delta^C_-]_- = -i\gamma_- \delta^S. \tag{46}
\]
This provides us the desired constraints for the special conformal anomalies of the conformal operators. We mostly concentrate on the even parity sector and just state the results for the odd one.

Before we start, let us argue that the three-fermion operators and BRST-exact operators will not contribute to the constraints. As already mentioned, the product of the three-fermion operators with composite operators provide in one-loop approximation a finite part, which could possibly contribute to the constraints. However, its evaluation gives a result that depends on the $\gamma_5$ and $\text{tr}\, 1$ prescriptions. Let us explain this point in more detail. Calculating the contributions of this operator product one deals with a quark loop that due to different Wick pairings has three terms: two of them contain a string of Dirac matrices while the last one is a trace. Obviously, in four dimensions the sum gives zero result, recall that $O^{3\psi}$ and $O_{3\psi}^{\psi}$ vanish by means of Fierz rearrangement. However, because of $\varepsilon^{-1}$ pole in the loop momentum integration, we have to keep $\varepsilon$-contributions from the spinor algebra that cancel this pole. Obviously, the $\varepsilon$ part is ambiguous for axial channel: the result depends on the handling of $\gamma_5$ in the string of Dirac matrices as well as in the trace. Next, since one of the three contributions is given by a trace of Dirac matrices, the result depends on the prescription for the trace of the unit matrix $\text{tr}\, 1$. One of the choices made in most calculations is to adopt a fiction for $d$-dimensional gamma matrices that still $\text{tr}\, 1 = 4$. However, in those computations the trace appears as a single overall factor and the above choice is permissible. It results into scheme dependence for finite part of e.g. one-loop diagrams. In our case, since we have an additive trace contribution, we have to continue the Clifford algebra in $d$ dimensional space as well, this results into the rule $\text{tr}\, 1 = 2^{[d/2]}$. This convention produces a term involving $\ln 2$ reflecting scheme dependence. On the other hand we certainly know that in leading order the conformal anomalies do not depend on these ambiguities. So we conclude that the contribution of the operator products in questions can not affect the constraints.

Now we come to the operator products containing unphysical BRST operators. Of course, one expects that these operators do not contribute to the physical sector, however, they may be responsible for the cancellation of unphysical pieces appearing in the renormalization of products containing only gauge invariant operators. On the very end we are interested on relations for physical quantities and as we already know from the constraints on anomalous dimensions these operator products have to be die out in Eq. (45). For the commutator (46), we have the superconformal anomaly and the only difficulty could be a gauge dependent term that is cancelled by the operator products in questions. From our previous experience in Ref. [6, 7], we expect that such a contribution is absent and this will be shown by explicit calculation in section 6. So it is justified to neglect the whole unphysical sector from the very beginning.
5.1 Commutator constraints for anomalous dimensions.

First let us demonstrate the derivation of the relations (43) for anomalous dimensions at leading order from the commutator of scale and supersymmetric variations, given in Eq. (45), applied to the Green functions of composite operators

\[ \langle [S_{jl}] (\delta^Q, \delta^D) \rangle = \frac{1}{2} \langle [S_{jl}] \delta^Q \rangle. \]  

(47)

The r.h.s. of this equality is obviously given by supersymmetric Ward identity (32) times \( \frac{1}{2} \). To calculate the l.h.s. of the commutator we employ the Ward identities (30) and (32), with BRST-exact operators being omitted, and find the following contributions

\[
\langle [S_{jl}] \delta^Q \delta^D \rangle = -\frac{\beta}{g} (i \left[ \left( \delta^Q S_{jl} \right) \left( O_A - \Omega_{\bar{\psi} \psi} \right) \right] \langle \mathcal{X} \rangle + \left( l + \frac{7}{2} \right) \langle (\bar{\zeta}_0 O_{3\psi}) [S_{jl}] \mathcal{X} \rangle
- \sigma_j \bar{\zeta}_0 \sum_{k=0}^j \left( l + \frac{7}{2} \right) \left( \delta_{j-1,k}^0 0 \delta_{jk} \right) + \left( \gamma_j^{\nu} \right)_{0}^{0,\nu} \langle \left( \mathcal{V}_{kl} \right) \rangle \langle \mathcal{X} \rangle, \]  

(48)

\[
\langle [S_{jl}] \delta^D \delta^Q \rangle = -\frac{\beta}{g} (i \delta^Q [S_{jl} \left( O_A - \Omega_{\bar{\psi} \psi} \right)] \right] \langle \mathcal{X} \rangle + (l + 3) \langle (\bar{\zeta}_0 O_{3\psi}) [S_{jl}] \mathcal{X} \rangle
- \bar{\zeta}_0 \sum_{k=0}^j \left( l + 3 \right) \left( \delta_{jk} 0 \delta_{jk} \right) + \left( \gamma_j^S \right)_{1,1}^{1,1} \langle \left( \mathcal{V}_{kl} \right) \rangle \langle \mathcal{X} \rangle. \]  

(49)

To derive the r.h.s. of Eq. (48) we have used for \( \langle (\bar{\zeta}_0 O_{3\psi}) [S_{jl}] \delta^D \mathcal{X} \rangle \) the scaling Ward identities with \( \delta^D O_{3\psi} = -\frac{1}{2} O_{3\psi} \) and the variation (29) of conformal operators. The variation of the action proportional to \( \varepsilon \) is neglected, since this Green function is already finite at one-loop order. As it was already expected from our previous result (43) and a discussion about scheme dependence, the three-fermion operator insertion will not affect the constraints at one loop level.

Now we come to the terms in the above equations proportional to the \( \beta \) function. To derive the r.h.s. of (48,49) we have used an equation (modulo infinite constants which again do not affect the constraints, since the latter are basically relations between finite contributions)

\[ \langle i [S_{jl} \left( O_A - \Omega_{\bar{\psi} \psi} \right) \mathcal{X} \rangle = \langle i [S_{jl} \left( [O_A] - \Omega_{\bar{\psi} \psi} \right) \mathcal{X} \rangle - 2 \langle i [S_{jl}] \mathcal{X} \rangle, \]  

(50)

to get rid of the variation sign on the field monomial \( \delta^Q \mathcal{X} \). It results from the study of the differential vertex operator insertions in the Green function \( \langle [O_{jl}] \mathcal{X} \rangle \) with bosonic conformal operator \( O = \left( \frac{\delta^Q}{\delta^D} \right) \), and one finds [7] that the renormalization constant of the operator product \( [O_{jl}][O_A] \),

\[
i [O_A(x)] [O_{jl}] = i [O_A(x) O_{jl}] - \delta^{(d)}(x) \sum_{k=0}^{j} \left\{ Z_A \right\}_{jk} [O_{kl}] - \frac{i}{2} \partial_\mu \delta^{(d)}(x) \sum_{k=0}^{j} \left\{ Z_A \right\}_{jk} [O_{kl}] \right] - \ldots
- \left( g \frac{\partial \ln X}{\partial g} - 2 \xi \frac{\partial \ln X}{\partial \xi} \right) B^a_\mu(x) \frac{\delta}{\delta B^a_\mu(x)} [O_{jl}], \]  

(51)

13
contains a finite contribution (second term on the r.h.s.)

\[
Z_A = \left( g \frac{\partial Z}{\partial g} - 2\xi \frac{\partial Z}{\partial \xi} \right) Z^{-1} - 2Z \mathcal{P}_G G^{-1} Z^{-1} - 2 \left( g \frac{\partial \ln X}{\partial g} - 2\xi \frac{\partial \ln X}{\partial \xi} \right) Z \mathcal{P}_G G Z^{-1}.
\] (52)

The constant \(X\) is related to the charge and gluon wave function renormalization constants by the relation \(X = Z_g \sqrt{Z_G}\). For the \([\mathcal{S}_{jl}] \Omega_{\bar{\psi}}\), we have to use the identity

\[
i[\mathcal{O}_{jl}] \Omega_{\phi}(x) = i[\mathcal{O}_{jl}]\Omega_{\phi}(x) - \phi(x) \frac{\delta}{\delta \phi(x)} [\mathcal{O}_{jl}],
\] (53)

where it is obvious that

\[
\langle [\mathcal{O}_{jl}] \Omega_{\phi}(x) \rangle = i \langle [\mathcal{O}_{jl}]\phi(x) \frac{\delta}{\delta \phi(x)} \rangle.
\] (54)

So that in Eq. (50) the finite piece appears as a consequence of two contributions: \(\mathcal{O}\) part from the \([\mathcal{S}_{jl}] [\mathcal{O}_A]\) product due to finite part in Eqs. (51,52) and \(\mathcal{O}\) part from the \([\mathcal{S}_{jl}] \Omega_{\bar{\psi}}\) by means of Eq. (53). Similarly, we have for the fermion operators a finite contribution

\[
i[\mathcal{V}_{jl}] [\mathcal{O}_A] = i[\mathcal{V}_{jl}\mathcal{O}_A] + [\mathcal{V}_{jl}] + \mathcal{O}(\varepsilon^{-r}),
\] (55)

for the trace anomaly, and

\[
i[\mathcal{V}_{jl}] \Omega_{\bar{\psi}} = i[\mathcal{V}_{jl}\Omega_{\bar{\psi}}] - [\mathcal{V}_{jl}] + \mathcal{O}(\varepsilon^{-r}),
\] (56)

for the equation-of-motion insertion. In both cases one should note the factors of one in front of the second term on the r.h.s. not 2 as for the quark and gluon operators. Although \(\Omega_{\bar{\psi}} = \Omega_{\bar{\psi}}\) for Majorana fermions (recall \(\bar{\psi}^{a} \gamma_{\mu} \psi^{a} = 0\) due to the Majorana flip properties), \(\psi\) and \(\bar{\psi}\) are treated as independent variables in the functional integral. Finally, we have

\[
\langle i\delta^{\mathcal{Q}} [\mathcal{S}_{jl} (\mathcal{O}_A - \Omega_{\bar{\psi}})] \rangle = \langle i[\delta^{\mathcal{Q}} \mathcal{S}_{jl}] (\mathcal{O}_A - \Omega_{\bar{\psi}}) \rangle,
\] (57)

which is almost a trivial result. To derive it one uses (50) and observe that the variation of the last term \(\langle [\mathcal{S}_{jl}] \chi \rangle\) in it cancels with the second terms in Eqs. (55,56) so that we are left with the r.h.s. of (57). All other terms in the commutator of Ward identities are relatively straightforward to handle. Obviously, with the equality (57) the contributions proportional to the \(\beta\) function in Eqs. (48) and (49) cancel each other in the commutator relation.

Subtracting Eq. (48) from (49) and comparing the difference with the supersymmetric Ward identity (32) we get, after extraction of independent combinations and identifying both parity sectors for fermionic operators, the known supersymmetric relations [3] (see also [2]):

\[
\begin{align*}
11_{S}^{S} & = 22_{S}^{S} = \gamma_{2n+1,2m+1}\gamma_{2n,2m} = \gamma_{2n,2m}, & m \leq n, \\
12_{S}^{S} & = 21_{S}^{S} = \gamma_{2n+1,2m+2}\gamma_{2n,2m+2} = \gamma_{2n,2m+1}, & m \leq n - 1, \\
21_{S}^{S} & = 12_{S}^{S} = \gamma_{2n,2m+1}\gamma_{2n+1,2m+1} = \gamma_{2n+1,2m}, & m \leq n, \\
22_{S}^{S} & = 11_{S}^{S} = \gamma_{2n+1,2m+2}\gamma_{2n,2m+2} = \gamma_{2n+1,2m+1}, & m \leq n, \\
12_{S}^{S} & = 0, & 12_{S}^{S} = 0.
\end{align*}
\] (58)
Obviously, these are the same relations as given in Eq. (43). If one relies on a supersymmetry preserving scheme, these equations can be derived in any order of perturbation theory and they have been checked at two-loop order\textsuperscript{6} [3].

### 5.2 Commutator constraints for special conformal anomalies.

The constraints for the special conformal anomalies of the conformal operators result from the commutator (46) of supersymmetric and special conformal variations applied to the Green function:

\[
\langle [S_{jl}] (\delta^Q, \delta^D) \mathcal{X} \rangle = -i\gamma_- \langle [S_{jl}] \delta^S \mathcal{X} \rangle.
\]

The derivation runs along the same line as above up to the appearance of the superconformal anomaly on the r.h.s., given by the Ward identity (33), and the absence of finite contributions to the renormalization of the product \([O_{jl}][O_A]\). Again omitting the BRST-exact operator insertions, the commutator of the l.h.s. is given by the two equations

\[
\langle [S_{jl}] \delta^Q \delta^C \mathcal{X} \rangle = -\frac{\beta}{g} \langle i \left( \delta^Q S_{jl} \left( O_A - O^-_{\bar{\psi}\psi} \right) \right) \mathcal{X} \rangle + \langle \left( \left[ \bar{\zeta}_0 O_{3\bar{\psi}} \right] S_{jl} \right) \delta^C \mathcal{X} \rangle - i\sigma_j \bar{\zeta}_0 \sum_{k=0}^j \left\{ a_{jl}(F) \begin{pmatrix} \delta_{j-1,k} & 0 \\ 0 & \delta_{jk} \end{pmatrix} + \begin{pmatrix} \gamma_{j-1,k}^c, \gamma_{jk}^c \\ 0 & \gamma_{jk}^c \end{pmatrix} \right\} \langle \left[ \mathcal{V}_{k,l-1} \right] \mathcal{X} \rangle,
\]

\[
\langle [S_{jl}] \delta^C \delta^Q \mathcal{X} \rangle = -\frac{\beta}{g} \langle i \delta^Q \left( S_{jl} \left( O_A - O^-_{\bar{\psi}\psi} \right) \right) \mathcal{X} \rangle - i a_{jl}(B) \langle \left[ \left[ \bar{\zeta}_0 O_{3\bar{\psi}} \right] S_{jl-1} \right] \mathcal{X} \rangle - i\sigma_j \bar{\zeta}_0 \sum_{k=0}^j \sigma_k \left\{ a_{jl}(B) \begin{pmatrix} \delta_{jk} & 0 \\ 0 & \delta_{jk} \end{pmatrix} + \begin{pmatrix} 11 \gamma_{jk}^c, \gamma_{jk}^c \\ 21 \gamma_{jk}^c, \gamma_{jk}^c \end{pmatrix} \right\} \langle \left[ \mathcal{V}_{k,l-1} \right] \mathcal{X} \rangle.
\]

The conformal variation of the Green function with three-fermion operator appearing in Eq. (60) can be calculated at tree level:

\[
\langle \left[ \bar{\zeta}_0 O_{3\bar{\psi}} \right] S_{jl} \rangle \delta^C \mathcal{X} \rangle = -ia_{jl}(B) \langle \left[ \left[ \bar{\zeta}_0 O_{3\bar{\psi}} \right] S_{jl-1} \right] \mathcal{X} \rangle - \langle \left[ \left[ \bar{\zeta}_0 O_{3\bar{\psi}} \right] S_{jl} \right] \mathcal{X} \rangle.
\]

Here we again neglected the BRST-exact operator that arises from the conformal variation of the action. Taking into account the Ward identity (33) and the relation \(\delta^C O_{3\bar{\psi}} = i\gamma_- O^-_{\bar{\psi}\psi} \), we observe again that the three-fermion operator contributions cancel each other in the commutator constraint.

\textsuperscript{6}Here we evaluated the rotation matrices from the conventional \(\overline{\text{MS}}\), in which all next-to-leading anomalous dimensions are available, to dimensional reduction scheme. As mentioned in the introduction this procedure does not completely fix the non-diagonal part of the rotation matrices.
Let us now consider the operator product proportional to the $\beta$ function, which is defined by [7]

$$
\langle i \left[ \mathcal{O}_{jl} \left( \mathcal{O}_A - \Omega_{\bar{\psi}\psi}^- \right) \right] \mathcal{X} \rangle = \langle i \left[ \mathcal{O}_{jl} \right] \left( \mathcal{O}_A - \Omega_{\bar{\psi}\psi}^- \right) \mathcal{X} \rangle - \langle \int d^d x 2 x_+ \left( \psi \frac{\delta}{\delta \psi} + \bar{\psi} \frac{\delta}{\delta \bar{\psi}} \right) \left[ \mathcal{O}_{jl} \right] \mathcal{X} \rangle.
$$

(62)

The supersymmetric variation of this product gives

$$
\langle i \delta^Q \left[ \mathcal{S}_{jl} \left( \mathcal{O}_A - \Omega_{\bar{\psi}\psi}^- \right) \right] \mathcal{X} \rangle = \langle i \left[ \delta^Q \mathcal{S}_{jl} \right] \left( \mathcal{O}_A - \Omega_{\bar{\psi}\psi}^- \right) \mathcal{X} \rangle + i \langle \left[ \left( \bar{\zeta}_0 \gamma_A \right) \mathcal{S}_{jl} \right] \mathcal{X} \rangle
$$

$$+ \langle \left( \int d^d x 2 x_+ \psi \frac{\delta}{\delta \psi} \delta^Q \left[ \mathcal{S}_{jl} \right] \mathcal{X} \rangle - \langle \int d^d x 2 x_+ \left( \psi \frac{\delta}{\delta \psi} + \bar{\psi} \frac{\delta}{\delta \bar{\psi}} \right) \left[ \mathcal{S}_{jl} \right] \mathcal{X} \rangle \rangle.
$$

(63)

Here we have used $\delta^Q \left( \mathcal{O}_A - \Omega_{\bar{\psi}\psi}^- \right) = \bar{\zeta}_0 \gamma_A + \ldots$, with ellipsis standing for the $\mathcal{O}_{3\psi}$ operator which again is irrelevant in one-loop approximation, since the whole contribution (63) is multiplied by the $\beta$-function and thus starts from $\alpha_s$. The first line of this equation ensures the cancellation with the same terms appearing in Eqs. (33,60), however, the remaining equation of motion operators in the second line will contribute to the constraints for the special conformal anomaly. We can absorb these additional pieces by a redefinition of the special conformal anomaly matrix. The action of the equation-of-motion operators is

$$
\int d^d x 2 x_+ \left( \psi \frac{\delta}{\delta \psi} + \bar{\psi} \frac{\delta}{\delta \bar{\psi}} \right) \mathcal{S}_{jl} = 2 i \sum_{k=0}^j b_{jk}(B) \mathcal{Q}_{k,l-1},
$$

$$
\int d^d x 2 x_+ \psi \frac{\delta}{\delta \psi} \mathcal{V}_{jl} = 2 i \sum_{k=0}^j b_{jk}(F) \mathcal{V}_{k,l-1},
$$

(64)

where $b_{jk}(B)$ matrix can be found in [6, 7] and the fermionic one is evaluated in Appendix A. Now we shift the $QQ$-entry by $\mathcal{Q}_{Q} \gamma^c_{jk} \rightarrow [\mathcal{Q}]_{Q} \gamma^c_{jk} = \mathcal{Q}_{Q} \gamma^c_{jk} + 2\beta \frac{\delta}{\delta \psi} b_{jk}(B)$ and set for the remaining channels $\mathcal{A} \mathcal{B} \gamma^c_{jk} \equiv \mathcal{A} \mathcal{B} \gamma^c_{jk}$. This redefinition treats the $\beta$ term equivalently for quarks and gluons and instead of Eq. (9) we have at leading order

$$
a^{-1}_{jk}(B) \mathcal{P}^{c(0)}_{jk}(B) \equiv -d_{jk} \left( \gamma_{k}^{(0)} - \beta \delta \right) + g_{jk}.
$$

(65)

The new special conformal anomaly matrix

$$
\mathcal{P}^c = \begin{pmatrix}
\frac{1}{k} & \frac{1}{k+1} \\
\frac{1}{2k} & \frac{1}{k+1}
\end{pmatrix}
\begin{pmatrix}
1 & \frac{k+3}{6} & \frac{6}{j} & \frac{k+3}{j+1} \\
-1 & \frac{k}{6} & \frac{6}{j} & \frac{k}{j+1}
\end{pmatrix}
\begin{pmatrix}
QQ \gamma^c_{jk} \\
GG \gamma^c_{jk}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{2k+3} \\
\frac{1}{k+1}
\end{pmatrix}
\begin{pmatrix}
1 & \frac{k+3}{6} & \frac{6}{j} & \frac{k+3}{j+1} \\
-1 & \frac{k}{6} & \frac{6}{j} & \frac{k}{j+1}
\end{pmatrix}
\begin{pmatrix}
QQ \gamma^c_{jk} \\
GG \gamma^c_{jk}
\end{pmatrix}.
$$

(66)
An analogous convention is introduced for the special conformal anomaly matrix of the fermionic operators
\[ \Gamma^c_{jk}(F) \equiv \gamma^c_{jk}(F) + \frac{\beta}{g}b_{jk}(F). \]  
(68)

If one compares now both sides of Eq. (59) in terms of these new conventions we find that besides the special conformal anomaly \( \Gamma^c \) only the superconformal anomaly,
\[ a\Delta^i_{jk} \equiv -2\alpha r^{[1]}_{jk}, \]  
(69)
arising from the renormalization of \([A]S_{jl}\) in Eq. (35), contributes to the desired constraints:

\[ \sum_{k=0}^{j} 11\Gamma^c_{jk}(B)\sigma_k[\mathcal{V}_{k-1,l-1}] + \sum_{k=0}^{j} 12\Gamma^c_{jk}(B)\sigma_k[\mathcal{V}_{kl-1}] \]  
\[ -\sigma_j \sum_{k=0}^{j} \left\{ \Gamma^c_{j-1,k}(F) - \frac{1}{2}\Delta^V_{jk} \right\} [\mathcal{V}_{kl-1}] = 0, \]  
(70)

\[ \sum_{k=0}^{j} 22\Gamma^c_{jk}(B)\sigma_k[\mathcal{V}_{kl-1}] + \sum_{k=0}^{j} 21\Gamma^c_{jk}(B)\sigma_k[\mathcal{V}_{k-1,l-1}] \]  
\[ -\sigma_j \sum_{k=0}^{j} \left\{ \Gamma^c_{jk}(F) - \frac{3}{2}\Delta^V_{jk} \right\} [\mathcal{V}_{kl-1}] = 0, \]  
(71)

and for \( \mathcal{P}_{jl} \) they read

\[ \sum_{k=0}^{j} 11\Gamma^c_{jk}(B)\sigma_{k+1}[\mathcal{U}_{k-1,l-1}] + \sum_{k=0}^{j} 12\Gamma^c_{jk}(B)\sigma_{k+1}[\mathcal{U}_{kl-1}] \]  
\[ -\sigma_{j+1} \sum_{k=0}^{j} \left\{ \Gamma^c_{j-1,k}(F) - \frac{1}{2}\Delta^A_{jk} \right\} [\mathcal{U}_{kl-1}] = 0, \]  
(72)

\[ \sum_{k=0}^{j} 22\Gamma^c_{jk}(B)\sigma_{k+1}[\mathcal{U}_{kl-1}] + \sum_{k=0}^{j} 21\Gamma^c_{jk}(B)\sigma_{k+1}[\mathcal{U}_{k-1,l-1}] \]  
\[ -\sigma_{j+1} \sum_{k=0}^{j} \left\{ \Gamma^c_{jk}(F) - \frac{3}{2}\Delta^A_{jk} \right\} [\mathcal{U}_{kl-1}] = 0, \]  
(73)

where we have implied \( \Gamma^c_{jk} = 0 \) for \( k > j \). Extracting the independent components from Eqs. (70-73) we finally obtain four equalities for the non-diagonal elements

\[ 11\Gamma^c_{2n+1,2m+1}(B) + \frac{1}{2}\Delta^V_{2n+1,2m} = 22\Gamma^c_{2n,2m}(B) + \frac{1}{2}\Delta^A_{2n,2m} = \Gamma^c_{2n,2m}(F), \]  
(74)

\[ 22\Gamma^c_{2n+1,2m+1}(B) + \frac{1}{2}\Delta^V_{2n+1,2m+1} = 11\Gamma^c_{2n+2,2m+2}(B) + \frac{1}{2}\Delta^A_{2n+2,2m+1} = \Gamma^c_{2n+1,2m+1}(F), \]  
(75)

\[ 12\Gamma^c_{2n+1,2m+1}(B) + \Delta^V_{2n+1,2m+1} = 21\Gamma^c_{2n,2m+2}(B) + \Delta^A_{2n,2m+1} = \Gamma^c_{2n,2m+1}(F), \]  
(76)

\[ 21\Gamma^c_{2n+1,2m+1}(B) + \Delta^V_{2n+1,2m+1} = 12\Gamma^c_{2n+2,2m}(B) + \Delta^A_{2n+2,2m} = \Gamma^c_{2n+1,2m}(F), \]  
(77)

\footnote{Here the fermionic conformal anomaly \( \gamma^c(F) \) has the same structure as the quark one, namely, \( \gamma^c(F) = -b(F)\gamma(F) + w(F) \), with anomalous dimension \( \gamma(F) \) of the quark-gluon operator \( \mathcal{V} \) (\( \mathcal{U} \)) and a part \( w(F) \) deduced from the renormalization of \( \mathcal{V} \) (\( \mathcal{U} \)) with the trace of the energy-momentum tensor.}
with $n > m$, and six equations for the diagonal elements, which are of no relevance in prediction (7) for the anomalous dimension matrix. Eqs. (74-77) are the main results of this section. The rest of the paper is devoted to a consistency check of our results for the special conformal anomalies by evaluating the $\Delta$-anomalies at one-loop order.

6 Evaluation of superconformal anomaly.

To simplify the calculation of the renormalization constants $\Delta$, it is advantageous to use the light-cone position formalism and to this end we introduce the non-local light-ray operators

\[ QO^i(\kappa_1, \kappa_2) = \frac{1}{2} \bar{\psi}^a_+ (\kappa_2 n) \Gamma^i \Phi^{ab}[\kappa_2, \kappa_1] \psi^b_+ (\kappa_1 n), \]

\[ GO^i(\kappa_1, \kappa_2) = G^a_{+\mu}(\kappa_2 n) T^i_{\mu\nu} \Phi^{ab}[\kappa_2, \kappa_1] G^b_{\nu+}(\kappa_1 n), \]

\[ FO^i(\kappa_1, \kappa_2) = G^a_{+\mu}(\kappa_2 n) F^i_{\mu} \Phi^{ab}[\kappa_2, \kappa_1] \psi^b_+ (\kappa_1 n). \] (78)

The calculations of the mixing kernels for the product $[A][O]$ with fermionic operators are straightforward and go along the line of Ref. [13] (see also appendix B of Ref. [7] for a recent review of this formalism in the case when the non-forwardness is essential). The diagrams are represented in Figs. 1 and 2 for quark and gluon operators, respectively. Since these renormalization constants must be gauge-independent, we have performed the computation with a general $\xi$ and indeed observed its cancelation in the final result. This shows that the BRST-exact operators can not
Figure 2: Same as in Fig. 1 but for the operator product $i[A][^G O_{ji}]$.

contribute to the constraints. The generic form of the result is

$$
[^P O](\kappa_1, \kappa_2) \left( \zeta_1[A] \right) = \frac{1}{\varepsilon} \frac{\alpha_s}{2\pi} N_c \int_0^1 dx \int_0^1 dy \theta(1-x-y) \mathcal{O}^i(x,y) \times \left( \bar{\zeta}_1 \gamma_5 F^i \right) \left( \bar{x} \kappa_1 + x \kappa_2, y \kappa_1 + \bar{y} \kappa_2 \right) \mp (\kappa_1 \leftrightarrow \kappa_2),
$$

(79)

where the $-(+)$-sign in the second term corresponds to the parity even quarks and parity odd gluons (parity even gluons and parity odd quarks) and we use throughout the convention $\bar{x} \equiv 1-x$.

Skipping details (see Appendix B) we give the result

$$
-k_2^+ Q^V(x,y) = - \left[ \frac{1}{y} \right]_+ \delta(x), \quad Q^A(x,y) = Q^V(x,y),
$$

(80)

$$
G^V(x,y) = - \left[ \frac{1}{x} \right]_+ \delta(y) - 2 + \delta(y) + (1-y)\delta(x), \quad G^A(x,y) = G^V(x,y) + 4y,
$$

(81)

with the $+$-prescription defined conventionally by

$$
\left[ \frac{1}{x} \right]_+ = \frac{1}{x} - \delta(x) \int_0^1 \frac{dy}{y}.
$$

We have kept in the quark sector the gluon momentum $k_{2+}$ unintegrated, which stems from the exponential in the Feynman rules for the operator insertion Vertex $\times \exp \left( -i\kappa_1 k_{1+} - i\kappa_2 k_{2+} \right)$. We merely substitute it by the corresponding parton momentum fraction when passing to exclusive type kernels (see Appendix C). Note that the result for the polarized quark kernel is the same as for the parity even case. This could be anticipated, since for divergent parts we can use anticommutativity of $\gamma_5$. However, there is a sign change in the contribution of the diagram in Fig. 1 (c) but it gives the contribution $\pm x\delta(1-x-y)$ with $+(−)$ sign for even (odd) parity.
and together with $\bar{x}\delta(1-x-y)$ from the Fig. 1 (d) gives a vanishing result when the symmetry property of the corresponding quark string operators is used. For gluons the difference between the parity even and odd sectors comes entirely from the contributions of the graphs in Figs. 2 (b), (c) and (e). The contributions from (d) and (g) annihilate each other, while remaining diagrams give identical\(^8\) results for $V$ and $A$ channels.

Evaluation of the conformal moments according to the method spelled in Appendix C gives for gluons

$$
\left(\frac{\alpha_s}{2\pi N_c}\right)^{-1} G R_{jk}^{[1]} = \frac{1}{3} \theta_{j-1,k} \left\{ \frac{(k+1)(k+3)\mp 2}{k+1} + (k+2)(k+3)\Psi_{jk} + (-1)^{j+k}(k+2)(k+3)\Xi_{jk} \right\}, \tag{82}
$$

with $- (+)$ sign for vector (axial) channel; and for quarks

$$
\left(\frac{\alpha_s}{2\pi N_c}\right)^{-1} Q R_{jk}^{[1]} = 2 \frac{\theta_{j-1,k}}{(k+1)} \left\{ (-1)^{j+k}(k+2) - 1 - (-1)^{j+k}(j+1)(j+2)\Psi_{jk} - (j+1)(j+2)\Sigma_{jk} \right\}. \tag{83}
$$

To simplify the presentation we have introduced the matrices, depending only on logarithmic derivative of the Euler integral $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$, via

$$
\Psi_{jk} = \psi \left( \frac{j + k + 4}{2} \right) - \psi \left( j + k + 4 \right) - \psi \left( \frac{j - k}{2} \right) + \psi \left( j - k \right),
$$

$$
\Xi_{jk} = \psi \left( j + k + 4 \right) + \psi \left( j - k \right) - \psi \left( k + 4 \right) - \psi \left( k + 1 \right),
$$

$$
\Sigma_{jk} = \psi \left( j + k + 4 \right) + \psi \left( j - k \right) - 2\psi \left( j + 2 \right).
$$

The conformal moments $R_{jk}$ of the kernels $R(x, y)$ are related to those in Eq. (69) by a normalization factor and, therefore, the superconformal anomaly is

$$
a \Delta_{jk} = -\frac{2}{g_k} \left( g^a_j Q R_{jk}^a + g^a_j G R_{jk}^a \right) \quad \text{with} \quad a = 1, 2. \tag{85}
$$

We should note that for even and odd $j - k$ the $\psi$-functions with argument depending on both $j$ and $k$ enter only in the particular combination

$$
A_{jk} = \psi \left( \frac{j + k + 4}{2} \right) - \psi \left( \frac{j - k}{2} \right) + 2\psi(j - k) - \psi(j + 2) - \psi(1),
$$

which arises in all special conformal anomalies $\gamma^c$, too [7].

The final step is to insert our findings for the superconformal anomalies given in Eqs. (82-85) and our results for the eight special conformal anomaly matrices from Ref. [7], rotated to $^{ab}\Gamma^c$-basis by means of (67), into the four conformal constraints (74-77). Indeed, we find that all of them are identically fulfilled. Thus, the universality of the special conformal anomaly matrix for even and odd parity sectors arises from the form of superconformal anomaly in $\mathcal{N} = 1$ Yang-Mills theory.

\(^8\)Discarding the fact that contributions with $\kappa_1 \leftrightarrow \kappa_2$ enter with opposite signs.
7 Conclusions.

The use of $\mathcal{N} = 1$ supersymmetry allows to find non-trivial relations for both scale and special conformal anomalies of QCD conformal operators. In the former case, the constraints, at leading order in dimensional regularization scheme as well as to all orders in a supersymmetry preserving scheme, involve only anomalous dimensions. There are six constraints for the diagonal and four for the non-diagonal elements of the four entries in the anomalous dimension matrices in the parity even and odd case. Although, in the supersymmetric limit the colour factors have to be identified, the constraints have predictive power also beyond the leading order approximation. As it has been observed in Ref. [3] there is a subtlety in finding the supersymmetry preserving regularization scheme in the non-forward case. Standard dimensional reduction [9] does not serve its purpose. However, an existence of a supersymmetry preserving scheme was proven by the ability to fulfill the constraints by a multiplicative renormalization of anomalous dimensions.

There exist four constraints for the special conformal anomalies. They contain new ingredients, $\Delta$, due to the superconformal symmetry breaking by the trace anomaly, $A$, in the spinor current. The four symmetry violating entries arise as a counterterm in the product of $A$ and conformal operators. Thus, there is no predictive power in these constraints, however, they serve for a consistency check of the special conformal anomalies.

Let us mention for completeness, the situation with the two further leading twist-two conformal operators appearing in the definition of the so-called transversity distributions. It is well known that these quark and gluon operators do not mix with each other under renormalization due to different spin representations w.r.t. Lorentz group. Their forward anomalous dimensions are related in the supersymmetric limit by one constraint. However, unfortunately, there exist no constraints for the non-diagonal elements of conformal anomalies that do not involve anomalous contributions of fermionic operators [14].

We have evaluated the superconformal anomalies in one-loop approximation and found that our relations are indeed fulfilled with the known results for the special conformal anomalies from Refs. [6, 7]. In other words, we can now reconstruct the special conformal anomalies of the gluon-gluon and gluon-quark sector from those in the remaining two channels, provided the former two would not be already known. Furthermore, we know that the latter give us the rotation matrix to the conformal scheme in which the conformal operator product expansion of two electromagnetic currents is valid. Since the Wilson coefficients of this expansion in both of these remaining channels are fixed by the ones of the forward kinematics and coincide with the explicit calculation in the minimal subtraction scheme [15], rotated to the conformal scheme [16], we have a complete consistency check of all special conformal anomaly matrices evaluated at one-loop level. Since these conformal anomalies induce, together with terms proportional to the $\beta$ function, the non-diagonal
elements of the anomalous dimension matrix at two-loop level, we have also a complete but indirect check on their correctness from the superconstraints (58). Of course, in the flavour non-singlet sector the anomalous dimensions arising from our prediction coincide with the conformal moments of evolution kernel calculated at two-loop level [17].

Altogether, we have a complete list of consistency checks for the field theoretical treatment of conformal anomalies, their evaluation at one-loop order and next-to-leading predictions arising from their use. This also supports our results for the transversity sector [18], which has been treated in the same manner.

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A Evaluation of the fermionic $b$-matrix.

The fermionic $b$-matrix is defined by Eq. (64). In order to evaluate the matrix it proves convenient to make the Fourier transform on the fields, i.e. $\phi(x) = \int d^d k e^{-k \cdot x} \tilde{\phi}(k)$ and introduce a set of new variable of the integrand: $Y \equiv k_2^+ + k_1^+$ and $X = \frac{k_2^+ - k_1^+}{k_2^+ + k_1^+}$, so we get

$$
\int d^d x \frac{\delta}{\delta \bar{\psi}} \psi \delta \mathcal{V}_{jl} = 2i \delta_{jl} \int d^d k_1 d^d k_2 \left\{ \hat{\mathcal{L}} P_{(2,1)}^j(X) \right\} \tilde{G}_+ \gamma_+ \bar{\psi}(k_2)
\int d^d x \frac{\delta}{\delta \bar{\psi}} \psi \delta \mathcal{V}_{kl-1},
$$

(A.1)

with differential operator $\hat{\mathcal{L}} \equiv l + (1 - X) \frac{d}{dX}$ and $B_{jk}$ being defined by the integral

$$
B_{jk}(\alpha, \beta) = \int_{-1}^1 dX \frac{w(X|\alpha, \beta)}{n_k(\alpha, \beta)} P_{k}^{(\alpha, \beta)}(X) \hat{\mathcal{L}} P_{j}^{(\alpha, \beta)}(X).
$$

(A.2)

The weight and normalization factors are given by standard equations

$$
w(X|\alpha, \beta) = (1 - X)^{\alpha}(1 + X)^{\beta},
$$

$$
n_k(\alpha, \beta) = 2^{\alpha+\beta+1} \frac{\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)}{(2k + \alpha + \beta + 1)\Gamma(k + 1)\Gamma(k + \alpha + \beta + 1)}.
$$

Using Rodriga’s formula for Jacobi polynomials and integrating $k$ (and $k + 1$) times by parts we come to integrals which can be easily evaluated by means of the result

$$
\int_{-1}^1 dX w(X|\alpha, \gamma) P_j^{(\alpha, \beta)}(X) = (-1)^j 2^{\alpha+\gamma+1} \frac{\Gamma(\alpha + j + 1)\Gamma(\beta - \gamma + j)\Gamma(\gamma + 1)}{\Gamma(j + 1)\Gamma(\beta - \gamma)\Gamma(\alpha + \gamma + j + 2)}.
$$

(A.3)
So that we finally obtain

$$B_{jk}(\alpha, \beta) = (l-k)\delta_{jk} - \theta_{j,k+1}(-1)^{j-k}(2k + \alpha + \beta + 1) \frac{\Gamma(j + \alpha + 1)\Gamma(k + \alpha + \beta + 1)}{\Gamma(k + \alpha + 1)\Gamma(j + \alpha + \beta + 1)}. \quad (A.4)$$

Identifying $\alpha = 2$ and $\beta = 1$ we get the result

$$b_{jk}(F) \equiv \frac{\theta_j}{\theta_k} \left\{ (l-k)\delta_{jk} - 2(-1)^{j-k}(k+2)(k+3)/(j+3)\theta_{j-1,k} \right\}, \quad (A.5)$$

where $\theta_{jk} = \{1, \text{ if } j \geq k; \ 0, \text{ if } j < k\}$.

**B Renormalization of the operator product $[A][O]$.**

For the calculation of $Z$-matrices one uses in the light-ray formalism the usual momentum space Feynman rules and $\text{Vertex} \times \exp(-i\kappa_1 k_1 - i\kappa_2 k_2)$ for the non-local operator where ‘$\text{Vertex}$’ stands for Dirac or Lorentz tensor. Introducing the Feynman parameters $x, y$ for the propagators we reduce Feynman integrals to the form

$$\mathcal{J} \times \mathcal{R}(x, y) \times \text{Vertex}, \text{ with } \mathcal{J} = \int_0^1 dx \int_0^1 dy \theta(1-x-y)e^{-ik_1((1-x)\kappa_1 + x\kappa_2) - ik_2(\gamma_1 + (1-y)\kappa_2)}, \quad (B.1)$$

where the exponential corresponds to the Fourier transform of the coordinate dependence of the string operator ‘after evolution’, $O((1-x)\kappa_1 + x\kappa_2, y\kappa_1 + (1-y)\kappa_2))$. To calculate the divergent part of the operator product $[A][O]$, we have in addition to take the Feynman rule for the anomaly $[A]$ defined in Eq. (19).

Our calculation has been performed with arbitrary gauge fixing parameter $\xi$, which has canceled in the sum of diagrams. Therefore, we present the results corresponding to the definition (79) for the separate contributions in $\xi = 1$ gauge.

In the quark parity even and odd sectors we get on the diagram-by-diagram basis from Fig. 1

$$-k_2^+ Q \mathcal{R}_{(a)} = \frac{1}{2},$$

$$-k_2^+ Q \mathcal{R}_{(b)} = -\frac{1}{2}(1-y)\delta(x),$$

$$-k_2^+ Q \mathcal{R}_{(c)} = \pm \frac{x}{2} \delta(1-x-y),$$

$$-k_2^+ Q \mathcal{R}_{(d)} = \frac{y}{2} \delta(1-x-y) - \frac{1}{2} \left(1 + \frac{y}{2}\right) \delta(x) - \frac{1}{2},$$

$$-k_2^+ Q \mathcal{R}_{(e)} = \left[ \frac{1}{y} \right]^{+} \delta(x) - \delta(x),$$

$$-k_2^+ Q \mathcal{R}_{(f)} = -\left[ \frac{1}{y} \right]^{+} \delta(x) + \frac{1}{2}(3-y)\delta(x),$$

$$-k_2^+ Q \mathcal{R}_{(g)} = -\left[ \frac{1}{y} \right]^{+} \delta(x) + \frac{1}{2} \left(1 + \frac{y}{2}\right) \delta(x), \quad (B.2)$$
where $+(-)$ sign in the contribution of the diagram (c) stands for even (odd) case. In these results we dropped the contributions of the type $\text{const} \cdot \delta(x)\delta(y)$, since they do not enter into the physical part of the constraints (74-77), namely for $k < j$. In the sum of Eqs. (B.2) the term $\frac{1}{2} \cdot \delta(1 - x - y)$ for vector case and $\frac{3}{2}(1 - 2x) \cdot \delta(1 - x - y)$ for the axial one cancels with the $\kappa_1 \leftrightarrow \kappa_2$ contribution (‘+’ and ‘−’ sign, respectively) in Eq. (79) and we get the result in Eq. (80).

The gluon case is calculated from the graphs in Fig. 2 and reads for vector channel

$$-k_2^+ G\mathcal{R}_{(\Phi)}^V = k_2^+ \delta(y) \left\{ \left[ \frac{2}{x} \right]_+ - 2 + 2\delta(x) \right\},$$

$$-k_2^+ G\mathcal{R}_{(a,N.A)}^V = \delta(y) \left\{ \frac{3}{2}(1 - x^2)k_{1+} + \frac{i}{2} \kappa x(1 - x)^2k_{1+}^2 - \frac{3}{2} \delta(x)k_{2+} \right\},$$

$$-k_2^+ G\mathcal{R}_{(b)}^V = \left( 2 - \frac{3}{2} - \frac{3}{2}(1 - x)k_{1+} + \frac{i}{2} \kappa ((1 - y)k_{2+} + xk_{1+}) (yk_{2+} + (1 - x)k_{1+}) \right)$$

$$+ \frac{1}{2}(1 - x)\delta(y) \left\{ (1 - 3x)k_{1+} - 2k_{2+} - i\kappa(1 - x)k_{1+} (xk_{1+} + k_{2+}) \right\},$$

$$-k_2^+ G\mathcal{R}_{(c)}^V = yk_{2+} - (1 + x)k_{1+},$$

$$-k_2^+ G\mathcal{R}_{(d)}^V = \left( 2 - \frac{3}{2} - \frac{3}{2}(1 - x)k_{1+} + \frac{i}{2} \kappa ((1 - y)k_{2+} + xk_{1+}) (yk_{2+} + (1 - x)k_{1+}) \right),$$

$$-k_2^+ G\mathcal{R}_{(e)}^V = \frac{1}{2}(3 - x)k_{1+} - \left( 1 - \frac{y}{2} \right)k_{2+} - \frac{i}{2} \kappa ((1 - y)k_{2+} + xk_{1+}) (yk_{2+} + (1 - x)k_{1+}),$$

$$-k_2^+ G\mathcal{R}_{(f,\Phi)}^V = k_2^+ \delta(y) \left\{ - \left[ \frac{1}{x} \right]_+ + 1 - \delta(x) \right\},$$

$$-k_2^+ G\mathcal{R}_{(f,N.A)}^V = \delta(y) \left\{ -(1 - x)k_{1+} + \delta(x)k_{2+} \right\},$$

$$-k_2^+ G\mathcal{R}_{(g)}^V = k_2^+ G\mathcal{R}_{(d)},$$

(B.3)

with $\kappa \equiv \kappa_2 - \kappa_1$. The subscript $\Phi$ ($N.A$) stands for the contributions originating from the expansion of the path ordered exponential (non-abelian part of the field strength tensor). The axial case differs from the previous one only in the contributions of diagrams (b), (c) and (e) which are

$$-k_2^+ G\mathcal{R}_{(b)}^A = \left( 2 - \frac{7}{2} - \frac{3}{2} - \frac{7}{2} \right)k_{2+} - \left( \frac{3}{2} - \frac{7}{2} \right)k_{1+} + \frac{i}{2} \kappa ((1 - y)k_{2+} + xk_{1+}) (yk_{2+} + (1 - x)k_{1+})$$

$$+ \frac{1 - x}{2}\delta(y) \left\{ (1 - 3x)k_{1+} - 2k_{2+} - i\kappa x(1 - x)k_{1+}^2 - i\kappa(1 - x)k_{1+}k_{2+} \right\},$$

$$-k_2^+ G\mathcal{R}_{(c)}^A = -yk_{2+} - (1 - x)k_{1+},$$

$$-k_2^+ G\mathcal{R}_{(e)}^A = \left( \frac{3}{2} - \frac{5}{2} - \frac{5}{2} \right)k_{1+} - \left( 1 - \frac{5}{2} \right)k_{2+} - \frac{i}{2} \kappa ((1 - y)k_{2+} + xk_{1+}) (yk_{2+} + (1 - x)k_{1+}).$$

(B.4)

Summing the separate terms we have to use the formula (see Ref. [7] for a general result)

$$\mathcal{J} \ast \{ k_{1+} [1 - (1 - x)\delta(y)] + k_{2+} [1 - (1 - y)\delta(x)] \} = 0,$$

(B.5)

to reduce the result to its final form (81). Then we use the equation $k_{2+} B_{\mu}^a(k_2) = iG_{+\mu}^a(k_2)$, valid to leading order in the coupling, to reconstruct the field strength from the potential.
C Conformal moments of regularized kernels.

The conformal moments of the transition kernels derived in the body of the text are defined according to

\[ R_{jk}^i \equiv \int_{-1}^{1} dt \frac{w(2,1|t)}{n_k(2,1)} P_k^{(2,1)}(t) \int_{0}^{1} dx \int_{0}^{\infty} R(x,y) C_j^i \left( [1-x-y]t - x + y \right), \quad (C.1) \]

with the weight \( w(2,1|t) \equiv (1 - t)^2(1 + t) \) and normalization \( n_k(2,1) \equiv \frac{8}{(k+1)(k+3)} \). We have for quarks \( C_j^Q = C_j^{3/2} \) and \( C_j^G = C_j^{5/2} \) for gluons. From this equation it is straightforward to evaluate the moments of all parts of the kernels (along the line of Ref. [7]) except of the ones with \( + \)-prescription since in this case we obtain the result in terms of derivatives of hypergeometric function w.r.t. its indices which is not easy to handle. In these cases we have to modify our modus operandi and develop a more efficient machinery which leads to more tractable expressions. It can be achieved according to the re-expansion of the integrand making use of the orthogonality for Gegenbauer polynomials. To be more specific let us consider the gluon kernel \([1/x]_+ \delta(y)\) which in the momentum fraction formalism translates into \([\theta(t-t')/(t-t')]_+\). We use the following regularization of singular distributions

\[ \int_{0}^{1} \frac{dx}{[1-x]_+} \phi(x) \equiv \int_{0}^{1} \frac{dx}{(1-x)^{1-\varepsilon}} \left[ \phi(x) - \phi(1) \right]. \quad (C.2) \]

Then using the representation of the Gegenbauer polynomial in terms of hypergeometric function \( _2F_1 \) and using Rodriga’s formula for Jacobi polynomials we integrate \( k \) times by parts to get

\[ M_{jk}^G \equiv \left\{ \left[ \frac{\theta(t-t')}{(t-t')} \right]_+ \right\}_{jk} = (-1)^{j+k} \frac{(k+2)(k+3)}{6\varepsilon} \]

\[ - (-1)^{j} \frac{(k+2)(k+3)\Gamma(j+4)}{12\Gamma(j)\Gamma(k+2)} \int_{0}^{1} dx x^k (1-x)^{\varepsilon-1} \int_{0}^{1} dy y^{k+1}(1-y)^{k+2} \]

\[ \times \frac{d^k}{d(xy)^k} _2F_1 \left( -j+1, j+4 \bigg| \frac{3}{xy} \right). \quad (C.3) \]

The first term on the r.h.s. originates from the \( \phi(1) \) contribution in Eq. (C.2). The simplicity of the consequent analysis depends on the handling of derivatives acting on \( _2F_1 \). If we merely differentiate it \( k \) times as it appears and perform \( y \) integration this gives \( _3F_2 \). Finally, after \( x \) integration Eq. (C.3) will be proportional to the derivative of \( _4F_3 \) w.r.t. a low index [see later Eq. (C.12)]. Fortunately, it is possible to avoid this if one notices that \( k - 1 \varepsilon \)-differentions and \( y \) integration leads to a \( _2F_1 \) function with shifted indices. After the last integration is done one ends up with an expression with \( \varepsilon \) derivative acting on \( _3F_2 \). In order to achieve this let us re-expand the action of \( k \)-th derivative \( d/d(xy) \) on \( _2F_1 \)  w.r.t. the complete set of polynomials \( C_i^{k+3/2}(2x-1) \)

\[ \frac{d}{dx} _2F_1 \left( -j+k, j+k+3 \bigg| x \right) = \sum_{l=0}^{\infty} c_{j-k,l} _2F_1 \left( -l, l+2k+3 \bigg| x \right) \]

\[ \quad (C.4) \]

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with expansion coefficients
\[ c_{j-k,l} = -[1 - (-1)^{j-k-l}] \theta_{j-k,l+1}(2l + 2k + 3) \Gamma(l + 2k + 3) \Gamma(j - k + 1) \Gamma(l + 1) \Gamma(j + k + 3) \] (C.5)
easily obtained from the orthogonality of polynomials. Then we get
\[ M^G_{jk} = \frac{(-1)^{j+k} (k + 2)^2 (k + 3) \Gamma(j + k + 3)}{12 \Gamma(2k + 5)\Gamma(j - k + 1)} \left\{ \psi(1) - \psi(k + 1) + \frac{\partial}{\partial \varepsilon} \right\}_{\varepsilon = 0} \]
\[ \times \sum_{l=0}^{\infty} c_{j-k,l} \binom{3}{2} F_2 \left( \begin{array}{c} -l, l + 2k + 3, k + 1 \\ 2k + 5, k + 1 + \varepsilon \end{array} \right) \right) \] (C.6)
Now the way to handle the derivative of \( 3F_2 \) is rather straightforward. First we use the fundamental identity for \( 3F_2 \) \( l \in \mathbb{N} \)
\[ 3F_2 \left( \begin{array}{c} -l, l + \alpha, \beta \\ \gamma, \beta + \varepsilon \end{array} \right) \left( 1 \right) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha)}{\Gamma(\gamma + l)\Gamma(\gamma - \alpha - l)} 3F_2 \left( \begin{array}{c} -l, l + \alpha, \varepsilon \\ 1 + \alpha - \gamma, \beta + \varepsilon \end{array} \right) \right) \] (C.7)
where we substitute \( \gamma = \alpha + 2 - \rho \) with \( \rho \rightarrow 0 \). Then the expansion w.r.t. \( \varepsilon \) is easy to construct
\[ 3F_2 \left( \begin{array}{c} -l, l + \alpha, \varepsilon \\ \rho - 1, \beta + \varepsilon \end{array} \right) \left( 1 \right) = 1 + \varepsilon \left\{ \frac{l(l + \alpha)}{\beta} \right\} + \varepsilon \Gamma(\rho - 1) \left( 1 + (-1)^l [(l - 1)(l + \alpha + 1) + \beta] \Gamma(1 + l + \alpha - \beta)\Gamma(\beta) \right) + \mathcal{O}(\varepsilon^2) \] (C.8)
and together with the identity
\[ \frac{\Gamma(\rho - 1)}{\Gamma(2 - \rho - l)} = (-1)^l \frac{\Gamma(l - 1 + \rho)}{\Gamma(2 - \rho)} \] (C.9)
we find
\[ 3F_2 \left( \begin{array}{c} -l, l + \alpha, \beta \\ \alpha + 2, \beta + \varepsilon \end{array} \right) \left( 1 \right) = \frac{\Gamma(\alpha + 2)}{\Gamma(l + \alpha + 2)} \left\{ \frac{l + \alpha}{\beta} \right\} (\delta_{l,0} + \delta_{l,1}) \] (C.10)
\[ + \varepsilon \Gamma(l - 1) \left( (-1)^l [(l - 1)(l + \alpha + 1) + \beta] \Gamma(1 + l + \alpha - \beta)\Gamma(\beta) \right) \theta_{l,2} + \mathcal{O}(\varepsilon^2) \right\} \right). \]

Using the results we have just derived, we perform in a last step the summation in Eq. (C.6) according to the formula
\[ \frac{\partial}{\partial \varepsilon} \left\{ \sum_{l=0}^{\infty} c_{j-k,l} 3F_2 \left( \begin{array}{c} -l, l + 2k + 3, k + 1 \\ 2k + 5, k + 1 + \varepsilon \end{array} \right) \right) \right) \right) = \frac{1}{(k + 1)(k + 2)} - \frac{(j - k)(j + k + 3)}{(k + 1)(k + 2)(k + 3)} \]
\[ - \frac{1}{k + 2} \left\{ (-1)^{j+k}\psi \left( \frac{j + k + 4}{2} \right) + [1 - (-1)^{j+k}]\psi(j + k + 4) - (-1)^{j+k}\psi \left( \frac{j - k}{2} \right) \right\} + [1 + (-1)^{j+k}]\psi(j - k) - \psi(k + 4) - \psi(1) \right\}. \] (C.11)
As a by-product we verify the following formula for the derivative of $4F_3$, which is difficult to derive by other means

\[
\frac{\Gamma(j + k + 4)}{\Gamma(j - k)\Gamma(2k + 5)} \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon = 0} 4F_3\left( -j + k + 1, j + k + 4, k + 2, k + 1 \left| 1 \right. \right. \\
= \frac{(j - k)(j + k + 3)}{(k + 1)(k + 3)} - \frac{1}{k + 1} + (-1)^{j+k}\psi \left( \frac{j + k + 4}{2} \right) + [1 - (-1)^{j+k}] \psi(j + k + 4) \\
- (-1)^{j+k}\psi \left( \frac{j - k}{2} \right) + [1 + (-1)^{j+k}] \psi(j - k) - \psi(k + 4) - \psi(1). \quad (C.12)
\]

Slightly different procedure holds for quarks. In this case due to the presence of the momentum fraction $k_2^+ = \frac{1-t}{t}(k_1 + k_2)_+$ we re-expand the integrand, modified by adding a constant,

\[
M_{jk}^Q \equiv \left\{ \frac{2}{1-t} \left( \frac{t'}{t'} - t \right) \right\}_{jk} = (-1)^k \frac{(k + 2)(k + 3)(j + 1)(j + 2)}{k + 1} \\
\times \int_0^1 y^2(1 - y) P_k^{(1,2)}(2y - 1) \int_0^1 dx \left[ \frac{1}{x} + \frac{1}{y} \right] \left\{ 2F_1 \left( -\frac{j + 3}{2} \right| xy \right) - 2F_1 \left( -\frac{j + 3}{2} \right| 0 \right\},
\]

in the following series

\[
\frac{1}{x} \left\{ 2F_1 \left( -\frac{j + 3}{2} \right| x \right) - 2F_1 \left( -\frac{j + 3}{2} \right| 0 \right\} = \sum_{k=0}^{\infty} d_{jk} \left\{ 2F_1 \left( -\frac{k + 3}{2} \right| x \right\} \quad (C.14)
\]

with the expansion coefficients

\[
d_{jk} = -\theta_{j-1,k} \frac{3 + 2k}{(j + 1)(j + 2)} (j - k)(j + k + 3). \quad (C.15)
\]

Consequent integration and expansion in $\varepsilon$ requires Eq. (C.10) as well as the following result

\[
3F_2 \left( \begin{array}{c} -l, l + \alpha, \beta \\ 1 + \alpha, \beta + \varepsilon \end{array} \right| 1 \right) = \frac{\Gamma(1 + \alpha)}{\Gamma(1 + \alpha + l)} \left\{ \delta_{l,0} + \varepsilon \Gamma(l) \left( \frac{\Gamma(1 + \alpha - \beta + l)\Gamma(\beta)}{\Gamma(1 + \alpha - \beta)\Gamma(\beta + l)} - (-1)^l \theta_{l,1} \right) \right\} + O(\varepsilon^2), \quad (C.16)
\]

which can be deduced using the same recipe as presented above. Final summation gives us the result in Eq. (83).

References


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