Spin foam model for Lorentzian General Relativity

Alejandro Perez and Carlo Rovelli
Centre de Physique Théorique - CNRS, Case 907, Luminy, F-13288 Marseille, France, and
Physics Department, University of Pittsburgh, Pittsburgh, Pa 15260, USA

We present a spin foam formulation of Lorentzian quantum General Relativity. The theory is based on a simple generalization of an Euclidean model defined in terms of a field theory over a group. Its vertex amplitude turns out to be the one recently introduced by Barrett and Crane. As in the case of its Euclidean relatives, the model fully implements the desired sum over 2-complexes which encodes the local degrees of freedom of the theory.

I. INTRODUCTION

Spin foam models provide a well defined framework for background independent diffeomorphism invariant quantum field theory. A surprising great deal of approaches have led to this type of models [1–6]. In particular, due to their non perturbative features, spin foam models appear as a very attractive framework for quantum gravity.

Spin foam models provide a rigorous implementation of the Wheeler-Misner-Hawking [7,8] sum over geometries formulation of quantum gravity. The 4-geometries summed over are represented by foam-like structures known as spin foams. They are defined as colored 2-complexes. A 2-complex $J$ is a (combinatorial) set of elements called “vertices” $v$, “edges” $e$ and “faces” $f$, and a boundary relation among these, such that an edge is bounded by two vertices, and a face is bounded by a cyclic sequence of contiguous edges (edges sharing a vertex). A spin foam is a 2-complex plus a “coloring” $N$, that is an assignment of an irreducible representation $N_I$ of a given group $G$ to each face $f$ and of an intertwiner $i_e$ to each edge $e$. The model is defined by the partition function

$$Z = \sum_J N(J) \sum_N \prod_{f \in J} A_f(N_f) \prod_{e \in J} A_e(N_e) \prod_{v \in J} A_v(N_v),$$

where $A_f$, $A_e$ and $A_v$ correspond to the amplitude associated to faces, edges, and vertices respectively (they are given functions of the corresponding colors). $N(J)$ is a normalization factor for each 2-complex.

Spin foam models related to gravity have been obtained as modifications of topological quantum field theories (corresponding to BF theory) by implementation of the constraints that reduce BF theory to general relativity [6,9–11]. So far, these constructions were restricted to the Euclidean sector. A crucial step towards the definition of a physical Lorentzian model has been taken by Barrett and Crane in [12]. In this work, Barrett and Crane construct a well defined vertex amplitude for Lorentzian quantum gravity, based on the representation theory of $SL(2,C)$.

Based on the work of Barrett and Crane, in this letter we complete the definition of a Lorentzian spin foam model for gravity. That is, we give an explicitly formula for the partition function of the model. To this aim, we use the technology provided by field theory over group manifolds, developed in in [13,14]. In this language, spin foams (quantum 4-geometries) appear as the Feynman diagrams of a certain nonlocal scalar field theory over a group. Strikingly, the Barrett-Crane Lorentzian vertex appears completely naturally in this context.

Two important points should be emphasized. First, the theory defined in this way implements automatically the sum over 2-complexes $J$ in (1), and in particular, fixes the $N(J)$ value. This sum is necessary to restore full general covariance of a theory with local degrees of freedom such as GR [3,13]. Indeed, in the case of a topological field theory [15–18] the sum over 2-complexes in (1) can be dropped (for fixed topology) due to the triangulation invariance of the partition function. This is a consequence of the absence of local degrees of freedom in the topological theory. When the constraints are implemented, however, the theory acquires the local degrees of freedom of gravity and different 2-complexes carry physical information. In the language of standard QFT, they represent higher order radiative corrections. In our model, the sum over 2-complexes is automatically implemented by the formalism.

The second point is about divergences. The Euclidean model in [6] is defined in terms of a quantum deformation of the gauge group $(SO_q(4),$ with $q^n = 1)$. The quantum deformation is needed to regularize divergences in (1). In the limit in which the quantum deformation is removed $(q \to 1)$, these divergences appear whenever the 2-complex $J$ includes bubbles [11]. In reference [11], using the field theory over group technology, we have defined a variant of the model, in which the basic bubble amplitudes are finite for $q = 1$. The definition of the Lorentzian model presented here corresponds to this variant. Although further study is certainly needed, we suspect that the Lorentzian model presented here might be finite even with $q = 1$.

Many issues remain open. In particular: (i) Can we get stronger evidence that the model gives general relativity in the classical limit? (ii) Can finiteness be proven? (iii) What is the physical meaning and the physical regime of
for integration variables in the projector

In this equation, $e$ comes from the projector $P$. The vertex and propagator of the theory are simply given by a set of delta functions on the group, $\phi(g_1, g_2, g_3, g_4)$. The action (4) is proportional to the trivial diverging factor $\int d\gamma$. This divergence could be fixed easily, for instance by gauge fixing and just dropping one of the group integrations. For the clarity of the presentation, however, we have preferred to keep gauge invariance manifest, use the action formally to generate the Feynman expansion, and drop the redundant group integration whenever needed.

\[ A(J) = \int d\gamma \prod_e \prod_f \delta(\gamma_e^{(1)} u_{1f} \gamma_e^{(2)} u'_{1f} \gamma_e^{(3)} u_{2f}) \cdots \gamma_N^{(1)} u_N^{(2)} \gamma_N^{(3)} u'_{Nf} \gamma_N^{(4)} u_{Nf}). \]
Next, we rewrite this equation in terms of the matrix elements $D_{j_1j_2j_3j_4}^{n\rho}(\gamma)$ of the representation $(n, \rho)$ in the canonical basis, defined in the appendix. The trace becomes

$$
\text{Tr} \left[ T_{n_1n_2} \left( \gamma^{(1)}_{\rho_1} u_{n_1} \gamma^{(2)}_{\rho_2} u_{n_2} \gamma^{(3)}_{\rho_3} u_{n_3} \gamma^{(4)}_{\rho_4} u_{n_4} \right) \right] = D^{n_1}_{j_1j_2j_3j_4, \rho_1}(\gamma^{(1)}) D^{n_2}_{j_2j_3j_4, \rho_2}(u_{j_1}) D^{n_3}_{j_3j_4, \rho_3}(\gamma^{(2)}) \ldots D^{n_4}_{j_4, \rho_4}(\gamma^{(3)}).
$$

(Repeated indices are summed, and the range of the $j_i$ and $q_i$ indices is specified in the appendix.) According to equation (A27), each $u_{j_1}$ integration produces a projection into the subspace spanned by the simple representations $(0, \rho)$. That is, after the integration over $u_{j_1}$, the matrix $D_{j_1j_2j_3j_4}^{n_1\rho_1}(u_{j_1})$ collapses to $\delta_{j_1,0} \delta_{j_2,0}$. One of these two Kronecker deltas appears always contracted with the indices of the $D(\gamma)$ associated to a vertex; while the other is contracted with a propagator. We observe that the representation matrices associated to propagators $(\gamma^{(2)}_e)$ appear in four faces in (7). The ones associated to vertices appear also four times but combined in the ten corresponding faces converging at a vertex. Consequently, they can be paired according to the rule $D_{j_1j_2j_3j_4}^{n_{q_1\rho_1}}(\gamma^{(1)}_e) D_{j_2j_3j_4, \rho_2}(\gamma^{(2)}_e) \ldots D_{j_4, \rho_4}(\gamma^{(3)}_e)$. In Fig. (1) we represent the structure described above. A continuous line represents a representation matrix, while a dark dot a contraction with a projector ($\delta_{j_0,0}$). Taking all this into account, we have

$$
A(J) = \sum_{n_f} \int_{\rho_f} d\rho_f \prod_f (\rho_f^2 + n_f^2) \prod_e A_e(\rho_{e_1}, \ldots, \rho_{e_4}; n_{e_1}, \ldots, n_{e_4}) \prod_v A_v(\rho_{v_1}, \ldots, \rho_{v_{10}}; n_{v_1}, \ldots, n_{v_{10}}),
$$

where $A_e$ is given by

$$
A_e(\rho_{e_1}, \ldots, \rho_{e_4}; n_{e_1}, \ldots, n_{e_4}) = \delta_{n_{e_1}0} \ldots \delta_{n_{e_4}0} \int d\gamma \, D_{0000}^{\rho_1}(\gamma) \ldots D_{0000}^{\rho_4}(\gamma),
$$

and $A_v$ by

3This projection implements the constraint that reduces BF theory to GR. Indeed, the generators of $SL(2, C)$ are identified with the classical two-form field $B$ of BF theory. The generators of the simple representations satisfy precisely the BF to GR constraint. Namely $B$ has the appropriate $e \wedge e$ form [6, 3]. Notice however that the representations $(0, \rho)$ are not the only simple representations; there are also simple representations of the form $(n, 0)$ with $n = 1, 2, \ldots$. The two sets have a simple geometrical interpretation in terms of space and time like directions (see [12]). We suspect that to recover full GR both set of simple representations should be included.
The spin foam model is finally given by integrations (see footnote 2 above). The vertex amplitude (16) is precisely the one defined by Barrett and Crane in [11]. We can now remove the trivial divergence (the integration over the gauge group) by dropping one of the group \( \gamma \)'s from our previous notation, since now they all take the value zero. This expression corresponds to the hyperbolic distance from the point \( \gamma \) to the apex of the hyperboloid (boost parameter). \( \gamma \) is finite, and its explicit value is computed in [12]. Finally, the vertex amplitude (11) results

\[
A_v(\rho_1, \ldots, \rho_5; n_1, \ldots, n_{10}) = \delta_{n_10} \ldots \delta_{n_{10}0}
\]

\[
\int \prod_{i=1}^5 d\gamma^*_i \ D^{\rho_1}_{\gamma^*_1}(\gamma^*_1\gamma^{-1})D^{\rho_2}_{\gamma^*_2}(\gamma^*_2\gamma^{-1})D^{\rho_3}_{\gamma^*_3}(\gamma^*_3\gamma^{-1})D^{\rho_4}_{\gamma^*_4}(\gamma^*_4\gamma^{-1})D^{\rho_5}_{\gamma^*_5}(\gamma^*_5\gamma^{-1}) \ D^{\rho_0}_{\gamma^*_0}(\gamma\gamma^{-1})
\]

\[
D^{\rho_0}_{\gamma^*_0}(\gamma\gamma^{-1}) = D^\rho_{\gamma_0}(\gamma(x_1, x_2)) = K_\rho(\eta(1 \gamma^{-1}) = K_\rho(x_1, x_2).
\]

Finally, the invariant measure on \( SL(2, C) \) is simply the product of the invariant measures of the hyperboloid and \( SU(2) \), that is \( d\gamma = du dx \). Using all this, the vertex and edge amplitudes can be expressed in simple form. The edge amplitude (10) becomes

\[
A_e(\rho_1, \ldots, \rho_4) = \int dx \ K_{\rho_1}(x)K_{\rho_2}(x)K_{\rho_3}(x)K_{\rho_4}(x),
\]

where we have dropped the \( n \)'s from our previous notation, since now they all take the value zero. This expression is finite, and its explicit value is computed in [12]. Finally, the vertex amplitude (11) results

\[
A_v(\rho_1, \ldots, \rho_5; n_1, \ldots, n_{10}) = \int dx_1 \ldots dx_5 \ K_{\rho_1}(x_1, x_5)K_{\rho_2}(x_1, x_4)K_{\rho_3}(x_1, x_3)K_{\rho_4}(x_1, x_2)K_{\rho_5}(x_2, x_5)K_{\rho_6}(x_2, x_4)K_{\rho_7}(x_2, x_3)K_{\rho_8}(x_3, x_5)K_{\rho_9}(x_3, x_4)K_{\rho_{10}}(x_4, x_5).
\]

We can now remove the trivial divergence (the integration over the gauge group) by dropping one of the group integrations (see footnote 2 above). The vertex amplitude (16) is precisely the one defined by Barrett and Crane in [12]. The spin foam model is finally given by

\[
A(J) = \int d\rho_j \prod_f \rho_j^F \ \prod_v A_v(\rho_{v_1}, \ldots, \rho_{v_4}) \ \prod_e A_e(\rho_{\epsilon_1}, \ldots, \rho_{\epsilon_10}),
\]

It corresponds to the Lorentzian generalization to the one defined in [11].

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4We denote by \( \gamma[x] \) the usual action of \( SL(2, C) \) matrices on \( x \) defined as an hermitian spinor, namely, \( \gamma[x] = \gamma x \gamma^\dagger \).
We have carried over the generalization of the model defined in [11] to the Lorentzian signature. The model is given by an $SL(2,C)$ BF quantum theory plus a quantum implementation of the additional constraints that reduce BF theory to Lorentzian general relativity.

The analog model in the Euclidean $SO(4)$ case was shown to be finite up to first bubble corrections. It would be very interesting to study this issue in the Lorentzian case. Evidence in favor of the conjecture of finiteness comes from the fact that, as in the Euclidean case, the edge contribution in the model tends to regularize the amplitudes. Divergences appear when compatibility conditions at edges fail to prevent colors associated to faces to get arbitrarily large. This happens when there are close surfaces in the spin foam, namely, bubbles. In [11] this divergences were cured by the dumping effect of edge amplitudes. As in its Euclidean relative, in the Lorentzian model presented here the edge amplitude goes to zero for large values of the colors. More precisely, the amplitude (15) behaves like $\rho_i \sim 1$ for $\rho_i \to \infty$.

The state sum contains only representations of the form $(0, \rho)$. These correspond to the simple irreducible representations representing space-like directions [12]. To obtain full general relativity, it might be necessary to generalize the present construction to include the others simple representations; that is, those of the form $(n, 0)$, with $n$ an arbitrary integer, which correspond to time-like directions. A simple modification of the action (4) should allow these other balanced representation to be included.

These important issues will be investigated in the future.

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APPENDIX A: REPRESENTATION THEORY OF $SL(2,C)$

We review a series of relevant facts about $SL(2,C)$ representation theory. Most of the material of this section can be found in [19,20]. For a very nice presentation of the subject see also [21].

We denote an element of $SL(2,C)$ by

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

(A1)

with $\alpha, \beta, \gamma, \delta$ complex numbers such that $\alpha\delta - \beta\gamma = 1$. All the finite dimensional irreducible representations of $SL(2,C)$ can be cast as a representation over the set of polynomials of two complex variables $z_1$ and $z_2$, of order $n_1 - 1$ in $z_1$ and $z_2$ and of order $n_2 - 1$ in $\bar{z}_1$ and $\bar{z}_2$. The representation is given by the following action

$$T(g)P(z_1, z_2) = P(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2).$$

(A2)

The usual spinor representations can be directly related to these ones.

The infinite dimensional representations are realized over the space of homogeneous functions of two complex variables $z_1$ and $z_2$ in the following way. A function $f(z_1, z_2)$ is called homogeneous of degree $(a, b)$, where $a$ and $b$ are complex numbers differing by an integer, if for every $\lambda \in C$ we have

$$f(\lambda z_1, \lambda z_2) = \lambda^a \bar{\lambda}^b f(z_1, z_2),$$

(A3)

where $a$ and $b$ are required to differ by an integer in order to $\lambda^a \bar{\lambda}^b$ be a single valued function of $\lambda$. The infinite dimensional representations of $SL(2,C)$ are given by the infinitely differentiable functions $f(z_1, z_2)$ (in $z_1$ and $z_2$ and their complex conjugates) homogeneous of degree $(\frac{a+n}{2}, \frac{a-n}{2})$, with $n$ an integer and $\mu$ a complex number. The representations are given by the following action

$$T_{\mu}(g)f(z_1, z_2) = f(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2).$$

(A4)

One simple realization of these functions is given by the functions of one complex variables defined as

$$\phi(z) = f(z, 1).$$

(A5)
On this set of functions the representation operators act in the following way

\[ T_{n\mu}(g)\phi(z) = (\beta z + \delta)^{\mu+n-1}(\bar{\beta} \bar{z} + \bar{\delta})^{\mu+n-1}\phi \left( \frac{\alpha z + \gamma}{\beta z + \delta} \right). \]  

(A6)

Two representations \( T_{n_1\mu_1}(g) \) and \( T_{n_2\mu_2}(g) \) are equivalent if \( n_1 = -n_2 \) and \( \mu_1 = -\mu_2 \).

Unitary representations of \( SL(2, C) \) are infinite dimensional. They are a subset of the previous ones corresponding to the two possible cases: \( \mu \) purely imaginary (\( T_{n, i\rho}(g) \mu = i\rho, \rho = \bar{\rho} \) known as the principal series), and \( n = 0 \), \( \mu = \bar{\mu} = \rho, \rho \neq 0 \) and \(-1 < \rho < 1 \) \( (T_{0\rho}(g) \) the supplementary series). From now on we concentrate on the principal series unitary representations \( T_{n\rho}(g) \) which we denote simply as \( T_{n\rho}(g) \) (dropping the \( i \) in front of \( \rho \)).

The invariant scalar product for the principal series is given by

\[ \langle \phi, \psi \rangle = \int \bar{\phi}(z)\psi(z)dz, \]

(A7)

where \( dz \) denotes \( dRe(z)dIm(z) \).

There is a well defined measure on \( SL(2, C) \) which is right-left invariant and invariant under inversion (namely, \( dg = d(gg_0) = d(g_0g) = d(g^{-1}) \)). Explicitly, in terms of the components in (A1)

\[ dg = \left( i \right)^3 \frac{d\beta d\gamma d\delta}{|\delta|^2} = \left( i \right)^3 \frac{d\alpha d\gamma d\bar{\delta}}{|\bar{\gamma}|^2} = \left( i \right)^3 \frac{d\beta d\alpha d\bar{\delta}}{|\beta|^2} = \left( i \right)^3 \frac{d\beta d\gamma d\delta}{|\alpha|^2}, \]

(A8)

where \( d\alpha, d\beta, d\gamma \), and \( d\delta \) denote integration over the real and imaginary part respectively.

Every square-integrable function, i.e, \( f(g) \) such that

\[ \int |f(g)|^2 dg \leq \infty, \]

(A9)

has a well defined Fourier transform defined as

\[ F(n, \rho) = \int f(g)T_{n, \rho}(g)dg. \]

(A10)

This equation can be inverted to express \( f(g) \) in terms of \( T_{n, \rho}(g) \). This is known as the Plancherel theorem which generalizes the Peter-Weyl theorem for finite dimensional unitary irreducible representations of compact groups as \( SU(2) \). Namely, every square-integrable function \( f(g) \) can be written as

\[ f(g) = \frac{1}{8\pi^n} \sum_n \int \text{Tr}[F(n, \rho)T_{n, \rho}(g^{-1})](n^2 + \rho^2)d\rho, \]

(A11)

where only components corresponding to the principal series are summed over (not all unitary representations are needed)\(^5\), and

\[ \text{Tr}[F(n, \rho)T_{n, \rho}(g^{-1})] = \int F_{n\rho}(z_1, z_2)T_{n\rho}(z_2, z_1; g)dz_1dz_2. \]

(A12)

\( F_{n\rho}(z_1, z_2) \) and \( T_{n\rho}(z_2, z_1; g) \) correspond to the kernels of the Fourier transform and representation respectively defined by their action on the space of functions \( \phi(z) \) (they are analogous to the momenta components and representation matrix elements in the case of finite dimensional representations), namely

\[ F(n, \rho)\phi(z) := \int f(g)T_{n\rho}(g)\phi(z)dg := \int F_{n\rho}(z, \bar{z})\phi(z)d\bar{z}, \]

(A13)

and

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\(^5\)If the function \( f(g) \) is infinitely differentiable of compact support then it can be shown that \( F(n, \rho) \) is an analytic function of \( \rho \) and an expansion similar to (A11) can be written in terms of non-unitary representations.
\[ T_{n,\rho}(g)\phi(z) := \int T_{n,\rho}(z, \bar{z}; g)\phi(\bar{z})d\bar{z}. \] (A14)

From (A6) we obtain that
\[ T_{n,\rho}(z, \bar{z}; g) = (\beta z + \delta) e^{i\bar{\alpha}z + \gamma} \delta \left( \bar{z} - \frac{\alpha z + \gamma}{\beta z + \delta} \right). \] (A15)

The “resolution of the identity” takes the form
\[ \delta(g) = \frac{1}{8\pi^4} \sum_n \int \text{Tr}[T_{n,\rho}(g)](n^2 + \rho^2) \, dp. \] (A16)

This is a key formula that we use in the paper.

There exists an alternative realization of the representations in terms of the space of homogeneous functions \( f(z_1, z_2) \) defined above [20]. Because of homogeneity (A3) any \( f(z_1, z_2) \) is completely determined by its values on the sphere \( S^3 \)
\[ |z_1|^2 + |z_2|^2 = 1. \] (A17)

As it is well now there is an isomorphism between \( S^3 \) and \( SU(2) \) given by
\[ u = \begin{bmatrix} \bar{z}_2 - \bar{z}_1 \\ z_1 \\ z_2 \end{bmatrix} \] (A18)
for \( u \in SU(2) \) and \( z \) satisfying (A17). Alternatively we can define the the function \( \phi(u) \) of \( u \in SU(2) \) as
\[ \phi(u) := f(u_{21}, u_{22}), \] (A19)
with \( f \) as in (A3). Due to (A3) \( \phi(u) \) has the following “gauge” behavior
\[ \phi(\gamma u) = e^{i\omega(a-b)} \phi(u) = e^{i\omega n} \phi(u), \] (A20)
for \( \gamma = \begin{bmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{bmatrix} \). The action of \( T_{n,\rho}(g) \) on \( \phi(u) \) is induced by its action on \( f(z_1, z_2) \) (A4). We can now use Peter-Weyl theorem to express \( \phi(u) \) in terms of irreducible representations \( D_{q_1q_2}^j(u) \) of \( SU(2) \); however in doing that one notices that due to (A20) only the functions \( \phi_q^j(u) = (2j + 1)^{1/2} D_{q_1q_2}^j(u) \) are needed (where \( j = |n| + k, k = 0, 1, \ldots \)). Therefore \( \phi(u) \) can be written as
\[ \phi(u) = \sum_{j=n}^{\infty} \sum_{q=-j}^{j} \phi_q^j(u). \] (A21)

This set of functions is known as the canonical basis. This basis is better suited for generalizing the Euclidean spin foam models, since the notation maintains a certain degree of similarity with the one in [13,11]. We can use this basis to write the matrix elements of the operators \( T_{n,\rho}(g) \), namely
\[ D_{j_1j_2q_1q_2}^{n,\rho}(g) = \int_{SU(2)} D_{j_1q_1}^{j_2}(u) \, [T_{n,\rho}(g)\phi_{q_2}^{j_2}(u)] \, du. \] (A22)

Since \( T_{n_1n_2}(u_0)\phi(u) = \phi(u_0u) \), invariance of the \( SU(2) \) Haar measure implies that
\[ D_{j_1j_2q_1q_2}^{n,\rho}(u_0) = \delta_{j_1, j_2} D_{q_1q_2}^{j_1}(u_0). \] (A23)

In terms of these matrix elements equation (A11) acquires the more familiar form
\[ f(g) = \sum_{n=0}^{\infty} \sum_{\rho=0}^{\infty} \left[ \sum_{j_1, j_2=n}^{\infty} \sum_{q_1=-j_1}^{j_1} \sum_{q_2=-j_2}^{j_2} D_{j_1j_2q_1q_2}^{n,\rho}(g) f_{n_1n_2}^{j_1j_2}(u_0) \right] (n^2 + \rho^2) \, dp, \] (A24)
where

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\[
\frac{f_{j_1, q_1j_2q_2}}{\rho} = \int f(g) D_{j_1, q_1j_2q_2}^{n, \rho}(g) dg,
\]

(A25)

and the quantity in brackets represents the trace in (A11). In the same way we can translate equation (A16) obtaining

\[
\delta(g) = \sum_{n=0}^{\infty} \int_{0}^{\infty} \left[ \sum_{j=0}^{n} \sum_{\rho=0}^{j} D_{j, q,j q}^{n, \rho}(g) \right] (n^2 + \rho^2) d\rho.
\]

(A26)

Using equations (A22) and (A23), we can compute

\[
\int_{SU(2)} D_{j_1, q_1j_2q_2}^{n, \rho}(u) du = \delta_{j_1,0} \int_{SU(2)} D_{q_2}^{j_2}(u) du = \delta_{j_1,0} \delta_{j_2,0},
\]

(A27)

a second key equation for the paper.


