Formation of Root Singularities on the Free Surface of a Conducting Fluid in an Electric Field

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Abstract

The formation of singularities on a free surface of a conducting ideal fluid in a strong electric field is considered. It is found that the nonlinear equations of two-dimensional fluid motion can be solved in the small-angle approximation. This enables us to show that for almost arbitrary initial conditions the surface curvature becomes infinite in a finite time.

Electrohydrodynamic instability of a free surface of a conducting fluid in an external electric field \([1,2]\) plays an essential role in a general problem of the electric strength. The interaction of strong electric field with induced charges at the surface of the fluid (liquid metal for applications) leads to the avalanche-like growth of surface perturbations and, as a consequence, to the formation of regions with high energy concentration which destruction can be accompanied by intensive emissive processes.

In this Letter we will show that the nonlinear equations of motion of a conducting fluid can be effectively solved in the approximation of small perturbations of the boundary. This allows us to study the nonlinear dynamics of the electrohydrodynamic instability and, in particular, the most physically meaningful singular solutions.

Let us consider an irrotational motion of a conducting ideal fluid with a free surface, \(z = \eta(x, y, t)\), that occupies the region \(-\infty < z \leq \eta(x, y, t)\), in an external uniform electric field \(E\). We will assume the influence of gravitational and capillary forces to be negligibly small, which corresponds to the condition

\[ E^2 \gg 8\pi \sqrt{g\alpha\rho}, \]

where \(g\) is the acceleration of gravity, \(\alpha\) is the surface tension coefficient, and \(\rho\) is the mass density.

The potential of the electric field \(\varphi\) satisfies the Laplace equation,

\[ \Delta \varphi = 0, \]

with the following boundary conditions,

\[ \varphi \to -Ez, \quad z \to \infty, \]

\[ \varphi = 0, \quad z = \eta. \]
The velocity potential $\Phi$ satisfies the incompressibility equation

$$\Delta \Phi = 0,$$

which one should solve together with the dynamic and kinematic relations on the free surface,

$$\frac{\partial \Phi}{\partial t} + \frac{(\nabla \Phi)^2}{2} = \frac{(\nabla \varphi)^2}{8\pi \rho} + F(t), \quad z = \eta,$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z} - \nabla \eta \cdot \nabla \Phi, \quad z = \eta,$$

where $F$ is some function of variable $t$, and the boundary condition

$$\Phi \to 0, \quad z \to -\infty.$$

The quantities $\eta(x, y, t)$, $\psi(x, y, t) = \Phi|_{z=\eta}$ are canonically conjugated, so that the equations of motion take the Hamiltonian form [3],

$$\frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \quad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi},$$

where the Hamiltonian

$$H = \int_{z \leq \eta} \frac{(\nabla \psi)^2}{2} d^3r - \int_{z \geq \eta} \frac{(\nabla \varphi)^2}{8\pi \rho} d^3r$$

coincides with the total energy of a system. With the help of the Green formula it can be rewritten as the surface integral,

$$H = \int_s \left[ \frac{\psi}{2} \frac{\partial \Phi}{\partial n} + \frac{E \eta}{8\pi \rho} \frac{\partial \tilde{\varphi}}{\partial n} \right] ds,$$

where $\tilde{\varphi} = \varphi + Ez$ is the perturbation of the electric field potential; $ds$ is the surface differential.

Let us assume $|\nabla \eta| \ll 1$, which corresponds to the approximation of small surface angles. In such a case we can expand the integrand in a power series of canonical variables $\eta$ and $\psi$. Restricting ourselves to quadratic and cubic terms we find after scale transformations

$$t \to t E^{-1}(4\pi \rho)^{1/2}, \quad \psi \to \psi E/(4\pi \rho)^{1/2}, \quad H \to HE^2/(4\pi \rho)$$

the following expression for the Hamiltonian,

$$H = \frac{1}{2} \int \left[ \psi \frac{\partial \varphi}{\partial n} + \eta \left( (\nabla \psi)^2 - (\nabla \varphi)^2 \right) \right] d^2r$$

$$- \frac{1}{2} \int \left[ \eta \frac{\partial \varphi}{\partial n} - \eta \left( (\nabla \eta)^2 - (\nabla \varphi)^2 \right) \right] d^2r.$$
Here $\hat{k}$ is the integral operator with the difference kernel, whose Fourier transform is the modulus of the wave vector,

$$\hat{k} f = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{f(x',y')}{{(x' - x)^2 + (y' - y)^2}^{3/2}} \, dx' \, dy'. $$

The equations of motion, corresponding to this Hamiltonian, take the following form,

$$\psi_t - \hat{k} \eta = \frac{1}{2} \left[ (\hat{k} \psi)^2 - (\nabla \psi)^2 + (\hat{k} \eta)^2 - (\nabla \eta)^2 \right] + \hat{k}(\eta \hat{k} \psi) + \nabla(\eta \nabla \eta), $$

$$\eta_t - \hat{k} \psi = -\hat{k}(\eta \hat{k} \psi) - \nabla(\eta \nabla \psi). $$

(1)  

Subtraction of Eqs. (2) and (1) gives in the linear approximation the relaxation equation

$$ (\psi - \eta)_t = -\hat{k}(\psi - \eta), $$

whence it follows that we can set $\psi = \eta$ in the nonlinear terms of Eqs. (1) and (2), which allows us to simplify the equations of motion. Actually, adding Eqs. (1) and (2) we obtain an equation for a new function $f = (\psi + \eta) / 2$,

$$ f_t - \hat{k} f = \frac{1}{2} (\hat{k} f)^2 - \frac{1}{2} (\nabla f)^2, $$

which corresponds to the consideration of the growing branch of the solutions. As $f = \eta$ in the linear approximation, Eq. (3) governs the behavior of the elevation $\eta$.

First we consider the one-dimensional case when function $f$ depends only on $x$ (and $t$) and the integral operator $\hat{k}$ can be expressed in terms of the Hilbert transform $\hat{H}$,

$$ \hat{k} = -\frac{\partial}{\partial x} \hat{H}, \quad \hat{H} f = \frac{1}{\pi} \text{P} \int_{-\infty}^{+\infty} \frac{f(x')}{x' - x} \, dx', $$

where $\text{P}$ denotes the principal value of the integral. As a result, Eq. (3) can be rewritten as

$$ f_t + \hat{H} f_x = \frac{1}{2} (\hat{H} f_x)^2 - \frac{1}{2} (f_x)^2. $$

(4)

It should be noted that if one introduces a new function $\tilde{f} = \hat{H} f$, then Eq. (4) transforms into the equation proposed in Ref. [4] for the description of the nonlinear stages of the Kelvin-Helmholtz instability.

For further consideration it is convenient to introduce a function, analytically extendable into the upper half-plane of the complex variable $x$,

$$ v = \frac{1}{2} (1 - i\hat{H}) f_x. $$

Then Eq. (4) takes the form

$$ \text{Re} (v_t + iv_x + 2vv_x) = 0, $$

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that is, the investigation of integro-differential equation (4) amounts to the analysis of the partial differential equation

\[ v_t + iv_x + 2vv_x = 0, \tag{5} \]

which describes the wave breaking in the complex plane. Let us study this process in analogy with [5,6], where a similar problem was considered. Eq. (5) can be solved by the standard method of characteristics,

\[ v = Q(x'), \tag{6} \]
\[ x = x' + it + 2Q(x')t. \tag{7} \]

where the function \( Q \) is defined from initial conditions. It is clear that in order to obtain an explicit form of the solution we must resolve Eq. (7) with respect to \( x' \). A mapping \( x \rightarrow x' \), defined by Eq. (7), will be ambiguous if \( \partial x/\partial x' = 0 \) in some point, i.e.

\[ 1 + 2Q_x't = 0. \tag{8} \]

Solution of (8) gives a trajectory \( x' = x'(t) \) on the complex plane \( x' \). Then the motion of the branch points of the function \( v \) is defined by an expression

\[ x(t) = x'(t) + it + 2Q(x'(t))t. \]

At some moment \( t_0 \) when the branch point touches the real axis, the analyticity of \( v(x,t) \) at the upper half-plane of variable \( x \) breaks, and a singularity appears in the solution of Eq. (4).

Let us consider the solution behavior close to the singularity. Expansion of (6) and (7) at a small vicinity of \( x = x(t_0) \) up to the leading orders gives

\[ v = Q_0 - \delta x'/(2t_0), \]
\[ \delta x = i\delta t + 2Q_0\delta t + Q''t_0(\delta x')^2, \]

where \( Q_0 = Q(x'(t_0)) \), \( Q'' = Q''(x'(t_0)) \), \( \delta x = x - x(t_0), \delta x' = x' - x'(t_0) \), and \( \delta t = t - t_0 \). Eliminating \( \delta x' \) from these equations, we find that close to singularity \( v_x \) can be represented in the self-similar form (\( \delta x \sim \delta t \)),

\[ v_x = - \left[ 16Q''t_0^3(\delta x - i\delta t - 2Q_0\delta t) \right]^{-1/2}. \]

As \( \text{Re}(v) = \eta/2 \) in the linear approximation, we have at \( t = t_0 \)

\[ \eta_{xx} \sim |\delta x|^{-1/2}, \]

that is the surface curvature becomes infinite in a finite time. It should be mentioned that such a behavior of the charged surface is similar to the behavior of a free surface of an ideal fluid in the absence of external forces [5,6], though the singularities are of a different nature (in the latter case the singularity formation is connected with inertial forces).
Let us show that the solutions corresponding to the root singularity regime are consistent with the applicability condition of the truncated equation (3). Let \( Q(x') \) be a rational function with one pole in the lower half-plane,

\[
Q(x') = -\frac{is}{2(x' + iA)^2},
\]

which corresponds to the spatially localized one-dimensional perturbation of the surface \((s > 0 \text{ and } A > 0)\). The characteristic surface angles are thought to be small, \( \gamma \approx s/A^2 \ll 1 \).

It is clear from the symmetries of (9) that the most rapid branch point touches the real axis at \( x = 0 \). Then the critical moment \( t_0 \) can be found directly from Eqs. (7) and (8). Expansion of \( t_0 \) with respect to the small parameter \( \gamma \) gives

\[
t_0 \approx A \left[1 - 3(\gamma/4)^{1/3}\right].
\]

Taking into account that the evolution of the surface perturbation can be described by an approximate formula

\[
\eta(x, t) = \frac{s(A - t)}{(A - t)^2 + x^2},
\]

we have for the dynamics of the characteristic angles

\[
\gamma(t) \approx \frac{s}{(A - t)^2}.
\]

Then, substituting the expression for \( t_0 \) (10) into this formula, we find that at the moment of the singularity formation with the required accuracy

\[
\gamma(t_0) \sim \gamma^{1/3},
\]

that is, the angles remain small and the root singularities are consistent with our assumption about small surface angles.

In conclusion, we would like to consider the more general case where the weak dependence of all quantities from the spatial variable \( y \) is taken into account. One can find that if the condition \( |k_x| \ll |k_y| \) holds for the characteristic wave numbers, then the evolution of the fluid surface is described by an equation

\[
[v_t + iv_x + 2vv_x]_x = -iv_{yy}/2,
\]

which extends Eq. (5) to the two-dimensional case.

An interesting group of particular solutions of this equation can be found with the help of substitution \( v(x, y, t) = w(z, t) \), where

\[
z = x - \frac{i(y - y_0)^2}{2t}.
\]

The equation for \( w \) looks like

\[
w_t + iw_z + 2ww_z = -w/(2t).
\]
It is integrable by the method of characteristics, so that we can study the analyticity violation similarly to the one-dimensional case. Considering a motion of branch points in the complex plane of the variable $z$ we find that a singularity arises at some moment $t_0 < 0$ at the point $y_0$ along the $y$-axis. Close to the singular point at the critical moment $t = t_0$ we get

$$\eta_{xx}\big|_{\delta y = 0} \sim |\delta x|^{-1/2}, \quad \eta_{xx}\big|_{\delta x = 0} \sim |\delta y|^{-1}.$$

This means that in the examined quasi-two-dimensional case the second derivative of the surface profile becomes infinite at a single isolated point.

Thus, the consideration of the behavior of a conducting fluid surface in a strong electric field shows that the nonlinearity determines the tendency for the formation of singularities of the root character, corresponding to the surface points with infinite curvature. We can assume that such weak singularities serve as the origin of the more powerful singularities observed in the experiments [7,8].

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References