We present a simple experimental scheme which can be used to demonstrate an all-or-nothing type contradiction between non-contextual hidden variables and quantum mechanics. The scheme, which is inspired by recent ideas by Cabello and García-Alcaine, shows that even for a single particle, path and spin information cannot be predetermined in a non-contextual way.

Most predictions of quantum mechanics are of a statistical nature, predictions for individual events are probabilistic. The question as to whether one can go beyond quantum mechanics in this respect, i.e. whether there could be hidden variables determining the results beyond quantum mechanics in this respect, i.e. whether there could be hidden variables determining the results of all individual measurements, has been answered to the affirmative by Bell’s theorem [1]. Locality means that in such theories the results of measurements in a certain space-time region are independent of what happens in a space-time region that is spacelike separated, in particular independent of the settings of a distant measuring apparatus.

Bell’s theorem refers to a situation where there are two particles and where the predictions of quantum mechanics are statistical. Furthermore, even definite (non-statistical) predictions of quantum mechanics are in conflict with a local realistic picture for systems of three particles or more [2].

The Kochen-Specker (KS) theorem [3] states that non-contextual theories (NCT) are incompatible with quantum mechanics. Non-contextuality means that the value for an observable predicted by such a theory does not depend on the experimental context, i.e. which other co-measurable observables are measured simultaneously. In quantum mechanics, observables have to commute in order to be co-measurable. Non-contextuality is a more stringent demand than locality because it requires mutual independence of the results for commuting observables even if there is no spacelike separation.

So far there has not been an experimental test of non-contextuality based on the original formulation of the KS theorem, which refers to a single spin-1 particle (cf. [4]). However, experimental tests of local hidden variable theories, such as tests of Bell’s inequality and of the GHZ paradox [2], can also be seen as tests of NCT. Note that such experiments in general involve several particles.

Recently, in a very interesting paper Cabello and García-Alcaine (CG) [5] have proposed an experimental test of the KS theorem based on two two-level systems (qubits).

In this paper we present a simple experimental scheme to test non-contextuality which is inspired by the CG argument. The experiment can be realized with single particles, using both their path and their spin degrees of freedom. It leads to a non-statistical test of non-contextuality versus quantum mechanics. In this respect it is similar to the GHZ argument against local realism.

In the following, we first show how a very direct experimental test of non-contextuality can be found, then we discuss our operational realization.

Consider four binary observables $Z_1, X_1, Z_2,$ and $X_2$. Let us denote the two possible results for each observable by ±1. In a NCT these observables have predetermined non-contextual values $+1$ or $−1$ for individual systems, denoted as $v(Z_1), v(Z_2), v(X_1),$ and $v(X_2).$ This means e.g. that for an individual system the result of a measurement of $Z_1$ will always be $v(Z_1)$ irrespective of which other co-measurable observables are measured with it. We will show that the existence of such non-contextual values is incompatible with quantum mechanics.

Imagine an ensemble $E$ of systems for which one always finds equal results for $Z_1$ and $Z_2$, and also for $X_1$ and $X_2$. (Clearly, in order for this statement to be meaningful, $Z_1$ and $Z_2$, and $X_1$ and $X_2$ have to be co-measurable.) In a NCT this means that

$$v(Z_1) = v(Z_2) \quad \text{and} \quad v(X_1) = v(X_2) \quad (1)$$

for each individual system of the ensemble. Then there are only two possibilities: either $v(Z_1) = v(X_2)$, which implies $v(X_1) = v(Z_2)$; or $v(Z_1) \neq v(X_2)$, which implies $v(X_1) \neq v(Z_2)$. We will see that this elementary logical deduction is already sufficient to establish a contradiction between NCT theories and quantum mechanics.

To this end, let us express the above argument in a slightly different way. Eq. (1) can be written as

$$v(Z_1)v(Z_2) = v(X_1)v(X_2) = 1. \quad (2)$$

Multiplying by $v(X_2)v(Z_2)$ it immediately follows that

$$v(Z_1)v(X_2) = v(X_1)v(Z_2). \quad (3)$$

A feasible “Kochen-Specker” experiment with single particles.

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Let us now introduce the notion of product observables such as $Z_1X_2$. By definition, one way of measuring $Z_1X_2$ is to measure $Z_1$ and $X_2$ separately and multiply the results; in general, there are other ways. In particular, if another compatible observable (e.g. $X_1Z_2$, cf. below) is measured simultaneously, it will in general not be possible to obtain separate values for $Z_1$ and $X_2$. However, in a non-contextual theory, the result of a measurement of an observable must not depend on which other observables are measured simultaneously. Therefore the predetermined value $v(Z_1X_2)$, for example, in a NCT has to follow the rule \[ v(Z_1X_2) = v(Z_1)v(X_2). \] (4)

In this new language, our above argumentation can be resumed in the following way:

\[ v(Z_1Z_2) = v(X_1X_2) = 1 \Rightarrow v(Z_1X_2) = v(X_1Z_2) \] (5)

i.e. if our systems have the property expressed in Eq. (1), then the two product observables $Z_1X_2$ and $X_1Z_2$ must always be equal in a NCT. Note that in general this prediction of NCT can only be tested if $Z_1X_2$ and $X_1Z_2$ are co-measurable.

It follows from the results of [5] that the prediction (5) leads to an observable contradiction with quantum mechanics. To see this, consider a system of two qubits and the observables [5]

\[ Z_1 := \sigma_z^{(1)}, X_1 := \sigma_x^{(1)}, Z_2 := \sigma_z^{(2)}, X_2 := \sigma_x^{(2)}, \] (6)

where $\sigma_z^{(1)}$ means the z-component of the “spin” of the first qubit etc. It is easy to check that this set of observables satisfies all the properties required above. In particular, while $Z_1$ and $X_1$, and $Z_2$ and $X_2$, do not commute, the two product observables $Z_1X_2$ and $X_1Z_2$ do. Furthermore, the quantum-mechanical two-qubit state

\[ |\psi_1\rangle = \frac{1}{\sqrt{2}}((|+z\rangle|+z\rangle + |-z\rangle|-z\rangle)) \]

\[ = \frac{1}{\sqrt{2}}((|+x\rangle|+x\rangle + |-x\rangle|-x\rangle)) \] (7)

is a joint eigenstate of the commuting product observables $Z_1Z_2$ and $X_1X_2$ with both eigenvalues equal to +1. Therefore, on the one hand the ensemble described by this state possesses the property of the ensemble $E$ discussed above (cf. (1)): the measured values of $Z_1Z_2$ and $X_1X_2$ are equal to +1 for every individual system. On the other hand, quantum mechanics predicts for the state $|\psi_1\rangle$, that the measured value of $Z_1X_2$ will always be opposite to the value of $X_1Z_2$. This can be seen by decomposing $|\psi_1\rangle$ in the basis of the joint eigenstates of the two commuting product observables $Z_1X_2$ and $X_1Z_2$:

\[ |\psi_1\rangle = \frac{1}{\sqrt{2}}(|\chi_{1,-1}\rangle + |\chi_{-1,1}\rangle), \] with

\[ |\chi_{1,-1}\rangle = \frac{1}{\sqrt{2}}((|+z\rangle|+z\rangle + |-z\rangle|-z\rangle) \]

\[ +(|+z\rangle|-z\rangle - |-z\rangle|+z\rangle) \]

\[ = \frac{1}{\sqrt{2}}(|+z\rangle|+z\rangle - |-z\rangle|-z\rangle) \]

\[ = \frac{1}{\sqrt{2}}(|-x\rangle|+z\rangle + |+x\rangle|-z\rangle)) \] (9)

\[ |\chi_{-1,1}\rangle = \frac{1}{\sqrt{2}}(|+z\rangle|+z\rangle + |-z\rangle|-z\rangle) \]

\[ -(|+z\rangle|-z\rangle + |-z\rangle|+z\rangle) \]

\[ = \frac{1}{\sqrt{2}}(|+z\rangle|-z\rangle - |-z\rangle|+z\rangle) \]

\[ = \frac{1}{\sqrt{2}}(|+x\rangle|+z\rangle - |-x\rangle|-z\rangle)). \] (10)

and

\[ Z_1X_2|\chi_{1,-1}\rangle = +|\chi_{1,-1}\rangle \]

\[ X_1Z_2|\chi_{1,-1}\rangle = -|\chi_{1,-1}\rangle \]

\[ Z_1X_2|\chi_{-1,1}\rangle = -|\chi_{-1,1}\rangle \]

\[ X_1Z_2|\chi_{-1,1}\rangle = +|\chi_{-1,1}\rangle \] (11)

From (8) and (11) one sees that $|\psi_1\rangle$ is a linear combination of exactly those joint eigenstates of $Z_1X_2$ and $X_1Z_2$ for which the respective eigenvalues are opposite, which means, of course, that in a joint measurement the two observables will always be found to be different. With Eq. (5) in mind, this implies that the ensemble described by $|\psi_1\rangle$ cannot be described by any non-contextual theory.

Note that one would already have a contradiction if quantum mechanics only predicted that the observed values of $Z_1X_2$ and $X_1Z_2$ are sometimes different, but in fact the result is even stronger, with QM and NCT predicting exactly opposite results. Thus, we have conflicting predictions for observable effects on a non-statistical level [6] (cf. [2]).

According to the argument presented in the previous paragraph, an experimental test of non-contextuality can be performed in the following way: (i) Show that $Z_1Z_2 = 1$ and $X_1X_2 = 1$ for systems prepared in a certain way. (ii) Determine whether $Z_1X_2$ and $X_1Z_2$ are equal for such systems. Note that in steps (i) and (ii) the observables $Z_1, X_1, Z_2,$ and $X_2$ appear in two different contexts.

Quantum mechanics predicts that step (i) can be accomplished by constructing a source of systems described by the state $|\psi_1\rangle$ and measuring $Z_1Z_2$ and $X_1X_2$ on these systems. According to QM, both $Z_1Z_2$ and $X_1X_2$ will always be found to be equal to +1. This can e.g. be verified by measuring the pairs $Z_1$ and $Z_2$ and $X_1$ and $X_2$ separately on many systems, and obtaining the values of $Z_1Z_2$ and $X_1X_2$ by multiplication. Alternatively, one could also perform joint measurements of $Z_1Z_2$ and