Anomaly and Nonplanar Diagrams in Noncommutative Gauge Theories

Farhad Ardalan\textsuperscript{1} and Néda Sadooghi\textsuperscript{2}

Institute for Studies in Theoretical Physics and Mathematics (IPM)
School of Physics, P.O. Box 19395-5531, Tehran-Iran

and

Department of Physics, Sharif University of Technology
P.O. Box 11365-9161, Tehran-Iran

Abstract

Anomalies in relation to nonplanar triangle diagrams of noncommutative gauge theory are studied. Local chiral gauge anomaly for noncommutative U(1) and U(N) gauge theories with adjoint matter fields is determined perturbatively and shown to vanish for both planar and nonplanar diagrams. For U(1) gauge theory with fundamental matter field coupling, three different noncommutative currents are found, which give the same global symmetry and lead to the same classically conserved charge. Only one of these currents yields a nonplanar contribution and leads to a finite global anomaly for light-like noncommutativity parameter $\theta_{\mu\nu}$, of a novel form.

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\textsuperscript{1}Electronic address: ardalan@theory.ipm.ac.ir

\textsuperscript{2}Electronic address: sadooghi@theory.ipm.ac.ir
Noncommutative geometry [1] has made a dramatic appearance in string theory recently [2] and has made noncommutative gauge theory [1] an active field of study. The various aspects of noncommutative Yang-Mills (NCYM) theory have been extensively studied [3, 4] and a number of novel phenomena discovered [5, 6, 7, 8]. Of special importance is a certain duality between ultraviolet (UV) and infrared (IR) behavior of nonplanar loop diagrams of NC-field theories and in particular NCYM theories, which manifests itself in the singularity of amplitudes in two limits of small noncommutativity parameter $\theta$ and large momentum cutoff $\Lambda$ of the theory [7].

Recently Seiberg [9] has observed that NCYM theory appears as a manifestation of a matrix model and the underlying model forces the coupling of the matter to gauge fields to be in the adjoint representation of the noncommutative algebra. But then, it is known that only in the adjoint coupling there appear nonplanar diagrams and there is a chance of observing the UV/IR mixing [8].

Anomalies of NCYM have been studied in a number of papers [10, 11] and it has been found that for the fundamental and anti-fundamental fermion coupling replacing the usual products by $\star$-products will give the anomaly modulo certain subtleties [10]. This is because only planar diagrams appear in the triangle anomaly for this coupling. This result was also confirmed in the Fujikawa path integral method [10, 11].

In this paper we study the effect of nonplanar diagrams on anomalies. We will find that nonplanar diagrams do not contribute to gauge anomalies. However, global symmetries reflect the unconvemional behavior of the nonplanar diagrams in the form of a contribution to the global anomaly which involves a new $\star$-product.

In the first part of the paper, we will discuss the anomalies in gauge symmetry of the noncommutative U(1) and U(N) theories. Since we are interested in the effect of nonplanar integrals on the anomaly, the matter fields in both theories are taken to be in the adjoint representation. In Section II, we will first calculate the chiral anomaly for U(1) theory with matter fields in the adjoint representations [Sec. II.1] where planar as well as nonplanar triangle diagrams will appear in the lowest order. But, both contributions to the triangle anomaly vanish independently. In Section II.2, the chiral anomaly of U(N) theory with matter fields in the adjoint representation of both gauge group and the noncommutative algebra, is calculated. Here also, planar as well as nonplanar contributions to the triangle diagrams vanish due to the group structure of the U(N) theory. Hence both theories are free of gauge anomaly$^3$.

In the second part of the paper, we will discuss the global symmetry of the U(1) theory with matter

$^3$When this study was almost done an article by C. P. Martin [12] appeared, with the same result for U(N) theory calculated in a different manner.
same global symmetry of the noncommutative action, lead to the same classically conserved charge. The triangle diagrams for two of them include only planar contributions. The axial anomaly for them is therefore the usual anomaly from commutative field theory, where all products are replaced by $\star$-products [10]. But, there is also a third current, which includes not only planar but also nonplanar phases. In Sec. III.2, we will determine the axial anomaly related to this global current and note the appearance of an unusual form of the axial anomaly in U(1) gauge theory for light-like noncommutativity parameter. As it was noted in [10], the noncommutative anomaly is gauge invariant only after integrating over all those space-time coordinates where the noncommutativity does not vanish. Here, we will also show, that taking the commutative limit and integrating over space-time coordinates do not commute. Sec. IV is devoted to our conclusions. We speculate that the nonplanar contribution to the global anomaly is a consequence of the noncommutative electric dipole in QED [13].

II. Chiral Gauge Anomaly in Noncommutative Gauge Theory

II.1 Noncommutative U(1) with Adjoint Matter Fields

We follow the notation of [10] and recall that the noncommutative (NC) gauge theory is characterized by the replacement of the familiar product of functions with the $\star$-product defined by:

$$f (x) \star g (x) \equiv e \left( \frac{i}{2} \theta_{\mu \nu} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \zeta} f (x + \xi) g (x + \zeta) \right) \bigg|_{\xi = \zeta = 0},$$

where $\theta_{\mu \nu}$ is a real constant antisymmetric background, and reflects the noncommutativity of the coordinates

$$[x_\mu, x_\nu] = i \theta_{\mu \nu}.$$

In NC-U(1) gauge theory the matter fields can couple to gauge fields in fundamental, anti-fundamental, or adjoint representation. The fundamental and anti-fundamental covariant derivatives are given by

$$D_\mu \psi_L (x) \equiv \partial_\mu \psi_L (x) + ig A_\mu (x) \star \psi_L (x), \quad \text{and} \quad D_\mu \bar{\psi} (x) \equiv \partial_\mu \bar{\psi}_L (x) - ig \bar{\psi}_L (x) \star A_\mu (x),$$

respectively. In the adjoint representation, it is defined by

$$D_\mu \psi_L (x) \equiv \partial_\mu \psi_L (x) + ig [A_\mu (x), \psi_L (x)] \star.$$
field in the adjoint representation is given by:

$$S_F[\bar{\psi}, \psi] = \int d^Dx \left\{ -\frac{i}{2} \bar{\psi}_\alpha (x) \gamma^\mu \partial_\mu \psi_\beta (x) - g \bar{\psi}_\alpha (x) \star [A_\mu (x), \psi_\beta (x)] \right\} (\gamma_\mu P_+)^{\alpha \beta} .$$

(2.1.5)

This action is invariant under the following local \(\star\)-gauge transformations

$$\bar{\psi} (x) \rightarrow \bar{\psi}^\prime (x) = U (x) \star \bar{\psi} (x) \star U^{-1} (x) , \quad \psi (x) \rightarrow \psi' (x) = U (x) \star \psi (x) \star U^{-1} (x) ,$$

(2.1.6)

and

$$A'_\mu (x) = U (x) \star A_\mu (x) \star U^{-1} (x) - \frac{i}{g} U (x) \partial_\mu U^{-1} (x) .$$

(2.1.7)

Here, \(U (x) \equiv \left( e^{ig\alpha (x)} \right)_\star\) and \(\alpha (x)\) is an arbitrary function. The local current corresponding to the above action is given by:

$$J_\mu (x) \equiv -i (\gamma_\mu P_+)^{\alpha \beta} \left\{ \bar{\psi}_\alpha (x), \psi_\beta (x) \right\} \star ,$$

(2.1.8)

where \(\left\{ \bar{\psi}_\alpha (x), \psi_\beta (x) \right\}_\star \equiv \bar{\psi}_\alpha (x) \star \psi_\beta (x) + \psi_\beta (x) \star \bar{\psi}_\alpha (x)\).

To study the Ward-identities and to determine the chiral anomaly in NC-U(1) with adjoint matter fields we consider the three-point function

$$\Gamma_{\mu \lambda \nu} (x, y, z) \equiv \left\langle T \left( J_\mu (x) J_\lambda (y) J_\nu (z) \right) \right\rangle ,$$

(2.1.9)

where the currents are given in the Eq. (2.1.8). Contracting the fermionic fields gives two types of triangle diagrams in the lowest order of perturbative expansion of the vacuum expectation value (2.1.9) [See Fig. 1.1]. The corresponding Feynman integrals are given by:

$$\Gamma_{\mu \lambda \nu} (x, y, z) = - \int_{-\infty}^{+\infty} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} e^{-i(k_2+k_3)x} e^{i k_2 y} e^{i k_3 z} \times \left\{ \left[ \text{Tr} \left( D^{-1} (\ell + k_3) \gamma_\mu P_+ D^{-1} (\ell + k_2) \gamma_\lambda D^{-1} (\ell) \gamma_\nu \right) \right] F_a (\ell) + \left[ (k_2, \lambda) \leftrightarrow (k_3, \nu) \right] F_b (\ell) \right\} .$$

(2.1.10)

For massless fermions the inverse fermion propagator is \(D (\ell) \equiv \ell\). The first expression on the second line is the contribution of diagram A, and the second expression that of diagram B. The phase factors are

$$F_a (\ell) = e^{i k_2 \times k_3} \left[ 1 - e^{2i\ell \times k_3} - e^{2i\ell \times k_2} + e^{2i\ell \times (k_3+k_2)} \right]$$

$$+ e^{-i k_2 \times k_3} \left[ -1 + e^{-2i\ell \times k_3} + e^{-2i\ell \times k_2} - e^{-2i\ell \times (k_2+k_3)} \right] ,$$

(2.1.11a)
Gauge Invariance: To check the gauge invariance let us calculate $\partial^\lambda \Gamma_{\mu\lambda\nu}$. Using the decomposition $k_2 = D (\ell + k_2) - D (\ell)$ in the contribution of diagram $A$ and $k_2 = D (\ell) - D (\ell - k_2)$ in the contribution of diagram $B$, we arrive at:

$$
\partial^\lambda \Gamma_{\mu\lambda\nu} (x, y, z) = -i \int_{-\infty}^{+\infty} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^Dell}{(2\pi)^D} e^{-i(k_2+k_3)x} e^{ik_2y} e^{ik_3z}
\times \left\{ \left[ \text{Tr} \left( D^{-1} (\ell - k_3) \gamma_\mu P_+ D^{-1} (\ell) \gamma_\nu \right) - \text{Tr} \left( D^{-1} (\ell - k_3) \gamma_\mu P_+ D^{-1} (\ell + k_2) \gamma_\nu \right) \right] F_a (\ell)
+ \left[ \text{Tr} \left( D^{-1} (\ell - k_2) \gamma_\mu P_+ D^{-1} (\ell + k_3) \gamma_\nu \right) - \text{Tr} \left( \gamma_\mu P_+ D^{-1} (\ell + k_3) \gamma_\nu D^{-1} (\ell) \right) \right] F_b (\ell) \right\},
$$

(2.1.12)

where the expression on the second (third) line is the contribution of the diagram $A$ ($B$). After a shift $\ell \rightarrow \ell + k_3$ ($\ell \rightarrow \ell - k_2 + k_3$) in the first (second) integral on the second line one can show, that it cancels only partly the corresponding contribution of the second (first) integral on the third line.

What remains are the contributions corresponding to the phases

$$
f_a^{(1)} (\ell) = e^{ik_2 \times k_3} \left( 1 - e^{2i\ell \times k_3} \right) - e^{-ik_2 \times k_3} \left( 1 - e^{-2i\ell \times k_3} \right), \quad \text{and}
$$

$$
f_a^{(2)} (\ell) = e^{ik_2 \times k_3} \left( 1 + e^{2i\ell \times (k_2 + k_3)} \right) - e^{-ik_2 \times k_3} \left( 1 + e^{-2i\ell \times (k_2 + k_3)} \right),
$$

(2.1.13)

in the first and second integrals of the second line, respectively, and the contributions of $f_a^{(1)} (\ell) = -f_a^{(2)} (\ell \rightarrow -\ell)$ and $f_b^{(2)} (\ell) = -f_a^{(1)} (\ell \rightarrow -\ell)$ in the first and second integrals of the third line.

In the following we will use the current algebra of NC-U(1) with adjoint matter fields in order to trace the origin of these contributions to $\partial^\lambda \Gamma_{\mu\lambda\nu}$. By calculating the divergence of the three-point function explicitly, a separation between the "formal" and the "potentially anomalous" part of $\partial^\lambda \Gamma_{\mu\lambda\nu}$ will occur [10], which reads

$$
\partial^\lambda \Gamma_{\mu\lambda\nu} (x, y, z) = \left[ \partial^\lambda \Gamma_{\mu\lambda\nu} (x, y, z) \right]_{\text{formal}} + \left[ \partial^\lambda \Gamma_{\mu\lambda\nu} (x, y, z) \right]_{\text{anom.}},
$$

(2.1.14a)

where

$$
\left[ \partial^\lambda \Gamma_{\mu\lambda\nu} (x, y, z) \right]_{\text{formal}} = \delta \left( y^0 - z^0 \right) \left\langle T \left( J_\mu (x) \left[ J_0 (y), J_\nu (z) \right] \right) \right\rangle
+ \delta \left( y^0 - x^0 \right) \left\langle T \left( \left[ J_0 (y), J_\mu (x) \right] J_\nu (z) \right) \right\rangle,
$$

(2.1.14b)

and

$$
\left[ \partial^\lambda \Gamma_{\mu\lambda\nu} (x, y, z) \right]_{\text{anom.}} = \left\langle T \left( J_\mu (x) \left( \partial^\lambda J_\lambda (y) \right) J_\nu (z) \right) \right\rangle.
$$

(2.1.14c)

\footnote{A finite shift of integration variables is only allowed after dimensional regularization.}
\[ [J_0 (y), J_\nu (z)] = (\gamma_\mu P_+) \gamma \int_{-\infty}^{+\infty} \frac{dp_1}{(2\pi)^4} \frac{dp_1'}{(2\pi)^4} \gamma (p_1) \overline{\psi} (p_1) \left[ e^{iy(p_1-k)} e^{iz(k-p_1')} - e^{iy(k-p_1')} e^{iz(p_1-k)} \right] \]

\[ \times \left[ e^{+ik\times(p_1-p_1')} - e^{+ik\times(p_1+p_1')} - e^{-ik\times(p_1+p_1')} + e^{-ik\times(p_1-p_1')} \right], \]  

(2.1.15)

it is possible to show that the remaining integrals contributing to \( \partial^5 \Gamma_{\mu\lambda\nu} \) arise from the formal part, so that even if they do not vanish, they do not indicate any break down of the gauge invariance of the theory. Hence, the potentially anomalous part of \( \partial^5 \Gamma_{\mu\lambda\nu} \) vanishes and the local gauge invariance of the theory is intact.

**Chiral Gauge Anomaly:** To determine the chiral anomaly we consider \( \partial^5 \Gamma_{\mu\lambda\nu} \) and dimensionally regularize the corresponding Feynman integrals. Recall that in the dimensional regularization, \( \gamma_5 \) is defined so that it anticommutes with \( \gamma_\mu \)'s for \( \mu = 0, 1, 2, 3 \) but commutes with them for other values of \( \mu \), and that the loop momenta have components in all dimensions, whereas the external momenta \( k_2 \) and \( k_3 \) are only four dimensional. Using \( \ell = \ell_{||} + \ell_\perp \), where \( \ell_{||} \) has nonzero components in dimensions 0, 1, 2, 3 and \( \ell_\perp \) has nonzero components in the other \( D - 4 \) dimensions, and the identity

\[ (k_2 + k_3) P_+ = P_- D (\ell + k_2) - D (\ell - k_3) P_+ + \gamma_5 \ell_\perp, \]

(2.1.16a)

in the corresponding integrals of diagram A and

\[ (k_2 + k_3) P_+ = P_- D (\ell + k_3) - D (\ell - k_2) P_+ + \gamma_5 \ell_\perp. \]

(2.1.16b)

in the Feynman integral corresponding to the diagram B, we arrive at:

\[ \partial^5 \Gamma_{\mu\lambda\nu} = -i \int_{-\infty}^{+\infty} \frac{dk_2}{(2\pi)^4} \frac{dk_3}{(2\pi)^4} e^{-i(k_2+k_3)x} e^{ik_2y} e^{ik_3z} \left[ A_{\lambda\nu} (k_2, k_3; \theta) + R_{\lambda\nu} (k_2, k_3; \theta) \right], \]

(2.1.17a)

where

\[ A_{\lambda\nu} (k_2, k_3; \theta) \equiv \int_{-\infty}^{+\infty} \frac{d^D \ell}{(2\pi)^D} \left[ \operatorname{Tr} \left( D^{-1} (\ell - k_3) \gamma_5 \ell_\perp D^{-1} (\ell + k_2) \gamma_\lambda D^{-1} (\ell) \gamma_\nu \right) F_a (\ell) \right] \]

\[ + \left[ (k_2, \lambda) \leftrightarrow (k_3, \nu) \right] F_b (\ell), \]

(2.1.17b)

and

\[ R_{\lambda\nu} (k_2, k_3; \theta) \equiv \]

\[ \int_{-\infty}^{+\infty} \frac{d^D \ell}{(2\pi)^D} \left[ \left[ \operatorname{Tr} \left( D^{-1} (\ell - k_3) P_- \gamma_\lambda D^{-1} (\ell) \gamma_\nu \right) - \operatorname{Tr} \left( P_+ D^{-1} (\ell + k_2) \gamma_\lambda D^{-1} (\ell) \gamma_\nu \right) \right] F_a (\ell) \]

\[ + \left[ \operatorname{Tr} \left( D^{-1} (\ell - k_2) P_- \gamma_\nu D^{-1} (\ell) \gamma_\lambda \right) - \operatorname{Tr} \left( P_+ D^{-1} (\ell + k_3) \gamma_\nu D^{-1} (\ell) \gamma_\lambda \right) \right] F_b (\ell) \right], \]

(2.1.17c)
Anomalous Part: As it turns out the contributions of the planar phases to $A_{\lambda\nu}$ from Eq. (2.1.17b) vanish, because of the following arguments:

As we know from ordinary commutative U(1) the contributions of both relevant triangle diagrams are equal due to Bose symmetry. In the noncommutative U(1), the planar part of $A_{\lambda\nu}$ are just the same as their commutative counterparts, except here these integrals are to be modified with corresponding planar phase factors, which can be taken out of the integration. Hence in order to add the contributions of diagrams $A$ and $B$ to $A_{\lambda\nu}^{pl.}$ only the planar phases have to be added together. The planar phase for the diagram $A$ turns out to be $2i \sin (k_2 \times k_3)$ whereas for the diagram $B$ is $-2i \sin (k_2 \times k_3)$ [see Eqs. (2.1.11a) and (2.1.11b)]. Hence the planar contribution of $B$ cancels the planar contribution of $A$.

For the nonplanar part the above argument does not go through and the calculation must be performed explicitly. After appropriate shift of integration variables and after using the property (2.1.11b) for nonplanar phases, it turns out that the contribution of diagram $A$ cancels the contribution of diagram $B$, so that the nonplanar part of $A_{\lambda\nu}$ vanishes too. Hence, $\partial^\mu \Gamma_{\mu\lambda\nu}$ does not receive any anomalous contribution from $A_{\lambda\nu}$ [Eq. (2.1.17b)].

Rest Part: Now, let us consider the rest terms from Eq. (2.1.17c). It turns out that after a finite shift $\ell \to \ell + k_3$ ($\ell \to \ell - k_2$), the first (second) integral of the second line cancels the corresponding contribution of the second (first) integral on the third line only partly. The remaining contributions are from the phases

\[
\begin{align*}
 f_a^{(1)} (\ell) &= e^{+ik_2 \times k_3} \left(1 - e^{+2i\ell \times k_3} \right) - e^{-ik_2 \times k_3} \left(1 - e^{-2i\ell \times k_3} \right), \\
 f_a^{(2)} (\ell) &= e^{+ik_2 \times k_3} \left(1 - e^{+2i\ell \times k_2} \right) - e^{-ik_2 \times k_3} \left(1 - e^{-2i\ell \times k_2} \right),
\end{align*}
\]

(2.1.18)

in the first and the second integrals of the second line, respectively. Besides, the contributions of $f_a^{(1)} (\ell) = -f_a^{(2)} (\ell \to -\ell)$ and $f_b^{(2)} (\ell) = -f_a^{(1)} (\ell \to -\ell)$ in the first and second integrals of the third line do not vanish after the above shift of integration variables. But, as in the previous section, using the separation

\[
\partial^\mu \Gamma_{\mu\lambda\nu} (x, y, z) = [\partial^\mu \Gamma_{\mu\lambda\nu} (x, y, z)]^{formal} + [\partial^\mu \Gamma_{\mu\lambda\nu} (x, y, z)]^{anomal},
\]

(2.1.19a)

where

\[
[\partial^\mu \Gamma_{\mu\lambda\nu} (x, y, z)]^{formal} \equiv \delta \left(x^0 - y^0 \right) \left\langle T (\{J_0 (x), J_\lambda (y)\} | J_\nu (z)) \right\rangle \\
+ \delta \left(x^0 - z^0 \right) \left\langle T (J_\lambda \lambda (y) | J_0 (x), J_\nu (z)) \right\rangle,
\]

(2.1.19b)
and the commutation relation (2.1.15), and after some algebraic manipulations, involving mainly finite shift of integration variables, one can show that these remaining terms are fully reproduced by the formal part of $\partial^\mu \Gamma_{\mu\lambda\nu}$. Hence, they do not contribute to any anomaly, even if they do not vanish. As we have shown before, the anomalous part of $\partial^\mu \Gamma_{\mu\lambda\nu}$ vanishes too. Noncommutative U(1) with adjoint matter fields is therefore free of chiral gauge anomaly.

II.2 Noncommutative U(N) with Adjoint Matter Fields

Let us introduce the matter fields in NC-U(N) with the covariant derivative:

$$D_\mu \Psi_L (x) \equiv \partial_\mu \Psi_L (x) + ig[A_\mu (x), \Psi_L (x)]_*, \quad (2.2.1)$$

where $A_\mu \equiv A_\mu^a t^a$, $\Psi_L \equiv \psi_L^a t^a$ and $t^a$, $a = 0, \cdots N^2 - 1$, are the generators of U(N) and $\psi_L^a \equiv P_+ \psi^a$ and $P_+ = \frac{1}{2} (1 + \gamma_5)$, which is

$$D_\mu \psi_L^a \equiv \partial_\mu \psi_L^a + \frac{ig}{2} D^{abc} [A_\mu, \psi_L^b]_* - \frac{ig}{2} C^{abc} [A_\mu, \psi_L^c]_*, \quad (2.2.2)$$

explicitly. Here, we have used the identity $2 t^a t^b = D^{abc} t^c + i C^{abc} t^c$, where the $D^{abc}$ and $C^{abc}$ are given by $\{t^a, t^b\} \equiv D^{abc} t^c$ and $[t^a, t^b] \equiv i C^{abc} t^c$. Using the definitions (2.2.1) and (2.2.2) of the covariant derivative, the fermionic action of this model is given by:

$$S[\Psi_L, \psi_L, A_\mu] = i \int_{-\infty}^{+\infty} d^4 x \text{Tr} \left( \Psi_L (x) \star D^* \Psi_L (x) \right), \quad (2.2.3)$$

where the trace is over U(N) indices. Equivalently, we have:

$$S[\psi^a, \psi^a, A^a_\mu] = \int_{-\infty}^{+\infty} d^4 x \left( \gamma^\mu \partial^\mu \right)_{\alpha\beta} N \left[ i \left( \overline{\psi}_\alpha (x) \star \partial_\mu \psi^a_\beta (x) \right) - \frac{g}{2} \left( C^{abc} \overline{\psi}_\alpha^a (x) \star \{ A_\mu^b (x), \psi^c_\beta (x) \}_* - i D^{abc} \overline{\psi}_\alpha^a (x) \star [ A_\mu^b (x), \psi^c_\beta (x) ]_* \right) \right]. \quad (2.2.4)$$

Here, we have $\text{Tr} (t^a t^b) = N \delta^{ab}$. The infinitesimal chiral gauge transformations

$$\delta_\alpha \overline{\Psi} = -ig[\overline{\Psi} (x), \alpha (x)]_*, \quad \text{and} \quad \delta_\alpha \Psi = +ig[\alpha (x), \Psi (x)]_*, \quad (2.2.5a)$$

where $\alpha (x) = \alpha^a (x) t^a$ become

$$\delta \overline{\psi}^i = -\frac{ig}{2} \left( D^{abc} [\overline{\psi}^a, \alpha^b]_* + i C^{abc} [\overline{\psi}^i, \alpha^b]_* \right),$$

$$\delta \psi^c = -\frac{ig}{2} \left( D^{abc} [\psi^a, \alpha^b]_* + i C^{abc} [\psi^c, \alpha^b]_* \right). \quad (2.2.5b)$$
or

\[ \delta A^\alpha_\mu = -\partial_\mu \alpha^a + \frac{ig}{2} \left( D^{abc}_{\alpha b} A^c_\mu \right)_* + i C^{abc}_{\alpha b} \{ \alpha^b, A^c_\mu \}_* \].

(2.2.6b)

Now, consider the action from Eq. (2.2.4). Gauge invariance of this action under local infinitesimal transformations (2.2.5b) can be established after a lengthy but straightforward calculation using the Jacobi identities given in the Eq. (A.1.1) [See appendix A]. In this model the local current reads

\[ J_\mu (x) = -i \frac{1}{\mathcal{N}} \left\{ \Psi^\alpha_L (x), \overline{\Psi}^\beta_L (x) \right\}_* (\gamma_\mu)_\alpha^\beta \],

(2.2.7a)

where \( J_\mu = J^\alpha_\mu t^a \). The components of the current read

\[ J^a_\mu (x) = -i \frac{1}{2} \left( D^{abc}_{\alpha b} (\psi^b_\alpha (x), \overline{\psi}_\xi (x))_* + i C^{abc}_{\alpha b} [\psi^b_\alpha (x), \overline{\psi}_\xi (x)]_* \right) (\gamma_\mu P_+) \xi^a \].

(2.2.7b)

To keep the calculation as short as possible we have used the following form of \( J^a_\mu (x) \):

\[ J^a_\mu (x) = -i \left( K^{abc} (\psi^b_\alpha (x), \overline{\psi}_\xi (x))_* - i C^{abc} \overline{\psi}_\xi (x) \star \psi^b_\alpha (x) \right) (\gamma_\mu P_+) \xi^a \],

(2.2.8)

where \( K^{abc} = \frac{1}{2} (D^{abc} + iC^{abc}) \). Here, taking the limit \( \theta \to 0 \), where \( \theta \) is the noncommutativity parameter, in both equation (2.2.7b) and (2.2.8) and using the antisymmetry of Grassmann variables \( \overline{\psi} \) and \( \psi \), leads to the usual definition of the current:

\[ J^a_\mu = i \overline{\psi}_\xi T^a \psi_\alpha (\gamma_\mu P_+) \xi^a = - \overline{\psi}_\xi C^{abc} \psi^b_\alpha (\gamma_\mu P_+) \xi^a \],

with \( T^a, T^a_{bc} = iC^{abc} \), as the generators of U(N) in their adjoint representation.

To study local gauge invariance and to determine the local chiral anomaly for NC-U(N), we consider the three-point function

\[ \Gamma^{cfm}_{\mu\lambda\nu} (x, y, z) = \left\langle T \left( J^c_\mu (x) J^f_\lambda (y) J^m_\nu (z) \right) \right\rangle \],

(2.2.9)

where \( c, f, m = 0, 1, \cdots, N^2 - 1 \) are U(N) indices. The non-Abelian current is introduced in the Eq. (2.2.8). In the lowest order of perturbative expansion of \( \Gamma^{cfm}_{\mu\lambda\nu} \), again, two types of diagrams appear [Fig. 1.1, where \( J_\mu, J_\lambda, J_\nu \) have to be replaced by \( J^J_\mu, J^J_\lambda, J^J_\nu \), respectively], each of them is decorated by a set of planar and nonplanar phases. The contribution of diagram A and B are given by

\[ \left[ \Gamma^{cfm}_{\mu\lambda\nu} (x, y, z) \right]_{A+B} = \]

\[ = \int_{-\infty}^{+\infty} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 k_3}{(2\pi)^4} \frac{d^4 \ell}{(2\pi)^4} e^{-i(k_2+k_3)x} e^{ik_2y} e^{ik_3z} \left\{ \text{Tr} \left( D^{-1} (\ell - k_3) \gamma_\mu P_+ D^{-1} (\ell + k_2) \gamma_\lambda D^{-1} (\ell) \gamma_\nu \right) \right\} \]

\[ \times \sum_{j=1}^{8} \left[ g^{cfm}_A f^a_A (\ell) + \left[ (k_2, \lambda) \leftrightarrow (k_3, \nu) \right] \sum_{j=1}^{8} \left( g^{cfm}_B \right)^j f^a_B (\ell) \right], \]

(2.2.10)
Gauge Invariance: Using again the relations $k_2^\ell = D (\ell + k_2) - D (\ell)$ in the contribution of diagram $A$ and $k_2^\ell = D (\ell) - D (\ell - k_2)$ in the contribution of diagram $B$ to $\partial^\lambda \Gamma^{cfm}_{\mu \lambda \nu}$, we arrive at an equation similar to Eq. (2.1.12), where only the phase factors $F_{a/b} (\ell)$ have to be replaced by the product of phase and group factors $\sum_{j=1}^8 [G_{A/B}^{cfm}]_j f_{A/B}^j (\ell)$. Performing now a current algebra analysis, as in the previous case, by using the equal-time commutation relation (A.1.2b) we find, that all the terms contributing to $\partial^\lambda \Gamma^{cfm}_{\mu \lambda \nu}$ are given by its formal part. Hence the potentially anomalous part vanishes. This guarantees the local gauge invariance of noncommutative U(N) gauge theory with adjoint matter fields.

Chiral Gauge Anomaly: Take the contributions of diagrams $A$ and $B$ to $\partial^\mu \Gamma^{cfm}_{\mu \lambda \nu}$, and use the identity (2.1.16a) for the diagram $A$ and Eq. (2.1.16b) for the diagram $B$ to get a separation similar to (2.1.17a), where again the phase factors $f_{a/b} (\ell)$ in (2.1.17b) and (2.1.17c) have to be replaced by $\sum_{j=1}^8 [G_{A/B}^{cfm}]_j f_{A/B}^j (\ell)$. Going now through a current algebra analysis, as in the case of U(1) gauge theory presented in the previous section, where equal-time commutation relations similar to (A.1.2b) are used and appropriate shift of integration variables are performed, one can show, that all terms contributing to the rest terms, Eq. (2.1.17c), are fully reproduced by the formal part of $\partial^\mu \Gamma^{cfm}_{\mu \lambda \nu}$. The potentially anomalous part of $\partial^\mu \Gamma^{cfm}_{\mu \lambda \nu}$ is therefore given by $A_{\lambda \nu}^{cfm} (k_2, k_3, \theta)$, Eq. (2.1.17b), with the replacement of phase factors with the combination of phase and group factors, which will be treated next.

The planar and nonplanar parts of $A_{\lambda \nu}^{cfm} (k_2, k_3, \theta)$ have to be treated separately. Using the Feynman parametrization procedure and the properties of dimensionally regularized $\gamma_5$, the planar part of the anomaly reads:

$$\left[ A_{\lambda \nu}^{cfm} (k_2, k_3; \theta) \right]_{pl.} = \frac{-i}{8\pi^2} \varepsilon_{\lambda \nu \alpha \beta} k_2 \alpha k_3 \beta \times \left\{ e^{i k_2 \times k_3} \left( [G_{A}^{cfm}]_1 + [G_{B}^{cfm}]_1 \right) + e^{-i k_2 \times k_3} \left( [G_{A}^{cfm}]_8 + [G_{B}^{cfm}]_8 \right) \right\}. \quad (2.2.11)$$

After some group algebra, it can be shown that the group factors of both diagrams $A$ and $B$ cancel, i.e. $[G_{A}^{cfm}]_1 + [G_{B}^{cfm}]_1 = 0$, and $[G_{A}^{cfm}]_8 + [G_{B}^{cfm}]_8 = 0$. Hence the planar part of the chiral anomaly vanishes.

For the nonplanar part of the anomaly, although the Feynman integrals including the nonplanar phases are naively convergent for finite values of the noncommutativity parameter $\theta$, they have to be dimensionally regularized. This is necessary in order to explore possible UV/IR mixing. To compute the nonplanar contribution to the anomaly in dimensional regularization, we use the commutation...
which lead to
\[ \text{Tr} \left( \gamma_5 (\gamma_\perp) \gamma_\lambda \gamma_\nu \gamma_\alpha \right) = 0. \]  

(2.2.13)

After some Dirac algebra we arrive at:

\[
[A^{cfm}_{\lambda\nu}(k_2, k_3; \theta)]_{n.pl.} = -i \int_{-\infty}^{+\infty} \frac{d^D\ell}{(2\pi)^D} \left\{ \text{Tr} \left[ (\ell - k_3^\perp) \gamma_5 \gamma_\perp (\ell + k_2^\perp) \gamma_\lambda \gamma_\nu \gamma_\alpha \right] \sum_{j=2}^{7} [G^{cfm}_{A}]_j f^j_\ell(\ell) \right\}.
\]

(2.2.14)

where the expression on the first (second) line is the contribution of diagram $A$ ($B$). Next, a Feynman parametrization have to be carried out, which dictates a shift $\ell \rightarrow \ell + P_a$ with $P_a \equiv -k_2\alpha_1 + k_3\alpha_2$ in the contribution of diagram $A$ and $\ell \rightarrow \ell + P_b$ with $P_b \equiv -P_a$ in the contribution of diagram $B$. Here, $\alpha_1$ and $\alpha_2$ are Feynman parameters. To add the contribution of both diagrams $A$ and $B$ a finite shift $\ell \rightarrow -\ell$ have to be also performed. Using the property of nonplanar phases of both diagrams, which satisfy the identity $f^j_\ell(\ell \rightarrow -\ell + P_a) = f^j_\ell(\ell + P_b)$, we arrive at expression of the form

\[
I_{\eta\xi} \equiv \int_{-\infty}^{+\infty} \frac{d^D\ell}{(2\pi)^D} \frac{(\ell_\perp\xi)}{(\ell^2 - \Delta)^3} e^{-2i\ell \times q},
\]

(2.2.15)

\[
I_{\xi} \equiv \int_{-\infty}^{+\infty} \frac{d^D\ell}{(2\pi)^D} \frac{(\ell_\perp\xi)}{(\ell^2 - \Delta)^3} e^{-2i\ell \times q},
\]

As we will show in Appendix B, these integrals vanish for any dimension. The main contribution comes therefore from

\[
\left[ A^{cfm}_{\lambda\nu}(k_2, k_3; \theta) \right]_{n.pl.} = 8\varepsilon_{\lambda\nu\alpha\beta} k_{2\alpha} k_{3\beta} \times \sum_{j=2}^{7} \left( [G^{cfm}_{A}]_j + [G^{cfm}_{B}]_j \right) \int_{0}^{1-\alpha_1} d\alpha_1 \int_{0}^{1-\alpha_2} d\alpha_2 \int_{-\infty}^{+\infty} \frac{d^D\ell}{(2\pi)^D} \frac{\ell^2 f^j_B(\ell + k_2\alpha_1 - k_3\alpha_2)}{(\ell^2 - \Delta)^3},
\]

(2.2.16)

where $\Delta \equiv -k_2^2\alpha_1 (1 - \alpha_1) - k_3^2\alpha_2 (1 - \alpha_2) - 2k_2k_3\alpha_1\alpha_2$. But, as it turns out, since the fermions are taken in the adjoint representaton of the U(N) gauge group, the corresponding group factors in diagrams $A$ and $B$ cancel, i.e. $[G^{cfm}_{A}]_j + [G^{cfm}_{B}]_j = 0, \ \forall j = 2, \cdots , 7$.

Combining this with the previous result, we have shown that planar as well as nonplanar part of $A^{cfm}_{\lambda\nu}$ vanish. Hence U(N) chiral gauge theory with adjoint matter fields turns out to be free of chiral gauge anomaly.
usual global symmetry transformation of the theory. It turns out, that the global anomaly receives non-vanishing contributions from planar and nonplanar part of triangle diagrams under certain circumstances.

**III. Global Anomalies of U(1)**

In this section we study the global symmetries and the nonplanar anomaly for noncommutative U(1) gauge theory with matter fields in the fundamental representation. All the results may be extended to the case where matter fields are coupled to the gauge fields in the adjoint representation.

**III.1 Classical Global Symmetries of U(1)**

To check the global symmetry and classical current conservation laws for U(1), let us briefly review the derivation of global currents in commutative gauge theories.

Consider the action
\[ S = \int d^4x \, \mathcal{L} (\psi^\ell, \partial_\mu \psi^\ell) \]
and a local transformation of fields
\[ \delta_\alpha \psi^\ell = i \alpha (x) \mathcal{F}^\ell (x). \tag{3.1.1} \]

If the action is not invariant under this local transformation, but is invariant under the corresponding global transformations, then its variation will have to be of the form
\[ \delta S = - \int d^4x \, J^\mu (x) \partial_\mu \alpha (x). \tag{3.1.2} \]

Integrating by part leads to the conservation law
\[ \partial^\mu J_\mu (x) = 0, \tag{3.1.3} \]
giving \( \frac{d}{dt} Q = 0 \) with \( Q = \int d^3x \, J_0 (x) \). In commutative field theory, there is one such conserved current and one constant of motion for each independent infinitesimal symmetry transformation. This is the content of the Noether’s first theorem. The current \( J_\mu \) can be given explicitly, if also the Lagrangian density is invariant under the global symmetry transformation corresponding to (3.1.1). Then
\[ J_\mu (x) \equiv -i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi^\ell (x))} \mathcal{F}^\ell (x). \tag{3.1.4} \]

In the noncommutative gauge theory, the action
\[ S_F [\bar{\psi}, \psi, A_\mu] = \int d^Dx \left\{ i \bar{\psi}_\alpha (x) \star \partial_\mu \psi_\beta (x) - g \bar{\psi}_\alpha (x) \star A_\mu (x) \star \psi_\beta (x) \right\} (\gamma_\mu)^{\alpha\beta}, \tag{3.1.5} \]
and the corresponding Lagrangian density are invariant under the global axial transformation
\[ \delta_\alpha \psi = i \alpha \gamma_5 \psi (x), \quad \text{and} \quad \delta_\alpha \bar{\psi} = i \alpha \bar{\psi} (x) \gamma_5. \tag{3.1.6} \]
I) \[ \delta_\alpha \overline{\psi}_\beta = i \alpha (x) \star \overline{\psi}_\xi (x) (\gamma_5)^\xi \beta, \quad \text{and} \quad \delta_\alpha \psi_\xi = i \beta (x) \star \alpha (x) (\gamma_5)^\xi \beta, \]

II) \[ \delta_\alpha \overline{\psi}_\beta = i \psi_\xi (x) \star \alpha (x) (\gamma_5)^\xi \beta, \quad \text{and} \quad \delta_\alpha \psi_\xi = i \alpha (x) \star \psi_\beta (x) (\gamma_5)^\xi \beta, \]

III) \[ \delta_\alpha \overline{\psi}_\beta = i \{ \alpha (x), \overline{\psi}_\xi (x) \} (\gamma_5)^\xi \beta, \quad \text{and} \quad \delta_\alpha \psi_\xi = i \{ \alpha (x), \psi_\beta (x) \} (\gamma_5)^\xi \beta, \quad (3.1.7) \]

leading to three different axial vector currents

I) \[ J_{\mu,5}^I (x) = + i \overline{\psi}_\alpha (x) \star \psi_\beta (x) (\gamma_\mu \gamma_5)^\alpha \beta, \]

II) \[ J_{\mu,5}^{II} (x) = - i \overline{\psi}_\beta (x) \star \psi_\alpha (x) (\gamma_\mu \gamma_5)^\alpha \beta, \]

III) \[ J_{\mu,5}^{III} (x) = - \frac{i}{2} [\psi_\beta (x), \psi_\alpha (x)] (\gamma_\mu \gamma_5)^\alpha \beta, \quad (3.1.8) \]

respectively. Indeed, the action (3.1.5) is not invariant under the local version of the transformations (3.1.7). However, it is possible to derive the following equations

\[
\begin{align*}
\partial^\mu J_{\mu,5}^I (x) &= 0, \\
\partial^\mu J_{\mu,5}^{II} (x) + ig [A^\mu (x), J_{\mu,5}^{II} (x)] &= 0, \\
\partial^\mu J_{\mu,5}^{III} (x) + \frac{ig}{2} [A^\mu (x), J_{\mu,5}^{III} (x)] &= 0,
\end{align*}
\]

(3.1.9)

from the equations of motion:

\[ \partial_\mu \overline{\psi}_\gamma \gamma^\mu = ig \overline{\psi}_\gamma \gamma^\mu \star A_\mu, \quad \text{and} \quad \gamma^\mu \partial_\mu \psi = - ig A_\mu \star \gamma^\mu \psi. \]

(3.1.10)

If the fields satisfy

\[ \int d^3x \ f (x) \star g (x) = \int d^3x \ g (x) \star f (x), \]

(3.1.11)

one may show from Eq. (3.1.9) that, all the currents from Eq. (3.1.8) lead to the same conserved axial charge

\[ Q_5 \equiv \int d^3x \ J_0^\sigma (x). \quad \text{for} \quad \sigma = I, II, III. \]

(3.1.12)

Besides, integrating equations (3.1.9) over all space-time coordinates, where the noncommutativity parameter \( \theta \) does not vanish, i.e. by definition over \( dx_\theta \), and using the definition of the \( \star \)-product, one may prove the continuity equation

\[ \int \partial^\mu J_\sigma^\mu (x) \ dx_\theta = 0 \quad \text{for} \quad \sigma = I, II, III. \]

(3.1.13)
which result from the invariance of the action with respect to global transformation
\[ \delta_\alpha \psi = ig_\alpha \psi (x), \quad \text{and} \quad \delta_\alpha \overline{\psi} = -ig_\alpha \overline{\psi} (x), \] (3.1.15)
and are associated with
\[ I) \quad \delta_\alpha \overline{\psi} = -ig_\alpha (x) \star \overline{\psi} (x), \quad \text{and} \quad \delta_\alpha \psi = +ig_\alpha (x) \star \alpha (x), \]
\[ II) \quad \delta_\alpha \overline{\psi} = -ig \overline{\psi} (x) \star \alpha (x), \quad \text{and} \quad \delta_\alpha \psi = +ig_\alpha (x) \star \psi (x), \]
\[ III) \quad \delta_\alpha \overline{\psi} = -ig \{ \alpha (x), \overline{\psi} (x) \}_\star, \quad \text{and} \quad \delta_\alpha \psi = +ig \{ \alpha (x), \psi (x) \}_\star, \] (3.1.16)
respectively. Similar conservation equation hold for these vector currents.

These arguments are not only true for the action (3.1.5), where the matter field are coupled to the gauge field in the fundamental representation, but also where this coupling is in the bi-fundamental and adjoint representation, where the covariant derivatives are given by
\[ D_\mu \psi (x) \equiv \partial_\mu \psi (x) - ig \psi (x) \star A_\mu (x), \quad \text{and} \quad D_\mu \psi (x) \equiv \partial_\mu \psi (x) + ig[A_\mu (x), \psi (x)]_\star, \]
respectively. In [10], we have calculated the global anomaly and checked the Ward identities corresponding to the axial vector current \( J_{\mu,5}^I \) from Eq. (3.1.8) and the vector current \( J_{\mu}^I \) from Eq. (3.1.14). There, we showed that, only planar triangle diagrams appear and the anomaly could be given by the usual commutative anomaly, where the ordinary products are replaced by noncommutative \( \star \)-products.

In the following section, we calculate the anomaly corresponding to the axial vector current \( J_{\mu,5}^I \) from Eq. (3.1.8), which will involve nonplanar diagrams.

### III.2 Ward Identities and the Global Anomaly

Consider again the three-point function
\[ \Gamma_{\mu\lambda\nu} (x, y, z) \equiv \left\langle T \left( J_{\mu,5}^{III} (x) J_{\lambda}^{III} (y) J_{\nu}^{III} (z) \right) \right\rangle. \] (3.2.1)
The Feynman integrals corresponding to the diagrams of Fig. 1.1 are given again by (2.1.10), where now \( P_+ \) has to be replaced by \( \gamma_5 \) and the phase factors are given by
\[ F_a (\ell) = \frac{1}{8} e^{i k_2 \times k_3} \left[ 1 + e^{2i \ell \times k_3} + e^{2i \ell \times k_2} + e^{2i \ell \times (k_3+k_2)} \right] \]
\[ + \frac{1}{8} e^{-i k_2 \times k_3} \left[ 1 + e^{-2i \ell \times k_3} + e^{-2i \ell \times k_2} + e^{-2i \ell \times (k_2+k_3)} \right], \] (3.2.2a)
Axial Anomaly: Using the identities \((k_2 + k_3) γ_5 = -γ_5 D(ℓ + k_2) - D(ℓ - k_3) γ_5 + 2γ_5 f_⊥\), and \((k_2 + k_3) γ_5 = -γ_5 D(ℓ + k_3) - D(ℓ - k_2) γ_5 + 2γ_5 f_⊥\) in the dimensionally regulated integrals for diagrams A and B, respectively, we obtain

\[
\partial^μΓ_{μλν} = \int_{-∞}^{+∞} \frac{d^4k_2 \ d^4k_3}{(2π)^4 (2π)^4} e^{-i(k_2+k_3)z} e^{ik_2y} e^{ik_3z} \left[ A_{λν}(k_2, k_3; \theta) + R_{λν}(k_2, k_3; \theta) \right],
\]

where

\[
A_{λν}(k_2, k_3; \theta) \equiv -2i \int_{-∞}^{+∞} \frac{dDℓ}{(2π)^D} \left[ \text{Tr} \left(D^{-1}(ℓ - k_3) γ_5 γ_λ D^{-1}(ℓ + k_2) γ_ν D^{-1}(ℓ) γ_µ \right) F_a(ℓ) \right] + \left[(k_2, λ) \leftrightarrow (k_3, ν)\right] F_b(ℓ),
\]

and

\[
R_{λν}(k_2, k_3; \theta) \equiv +i \int_{-∞}^{+∞} \frac{dDℓ}{(2π)^D} \left\{ \left[ \text{Tr} \left(D^{-1}(ℓ - k_3) γ_5 γ_λ D^{-1}(ℓ) γ_ν \right) + \text{Tr} \left(γ_5 D^{-1}(ℓ + k_2) γ_λ D^{-1}(ℓ) γ_ν \right) \right] F_a(ℓ) \right. \\
+ \left. \left[ \text{Tr} \left(D^{-1}(ℓ - k_2) γ_5 γ_ν D^{-1}(ℓ) γ_λ \right) + \text{Tr} \left(γ_5 D^{-1}(ℓ + k_3) γ_ν D^{-1}(ℓ) γ_λ \right) \right] F_b(ℓ) \right\}.
\]

Let us consider first the rest terms from Eq. (3.2.3c). After appropriate shifts \(ℓ \rightarrow ℓ + k_3 (ℓ \rightarrow ℓ - k_2)\) in the first (second) integral of the second line, it cancels the corresponding contribution of the second (first) integral on the third line only partly. The remaining contributions are from the nonplanar phases

\[
f_a^{(1)}(ℓ) = \frac{1}{8} e^{+ik_2\times k_3+2iℓ\times k_3} + \frac{1}{8} e^{-ik_2\times k_3-2iℓ\times k_3}, \quad \text{and} \\
f_a^{(2)}(ℓ) = \frac{1}{8} e^{+ik_2\times k_3+2iℓ\times k_2} + \frac{1}{8} e^{-ik_2\times k_3-2iℓ\times k_2},
\]

in the first and the second integrals of the second line, respectively, and the contributions of \(f_b^{(1)}(ℓ) = f_a^{(2)}(ℓ \rightarrow -ℓ)\), and \(f_b^{(2)}(ℓ) = f_a^{(1)}(ℓ \rightarrow -ℓ)\) in the first and second integral of the third line. But, using again a current algebra analysis shows that these remaining terms arise from the formal part of \(\partial^μΓ_{μλν}\) and do not express any anomaly (for the decomposition of \(\partial^μΓ_{μλν}\) in formal and potentially anomalous part see [10]). Hence, we have shown that the anomalous part of \(\partial^μΓ_{μλν}\) is given only by \(A_{λν}\) from Eq. (3.2.3b), which will be treated as next.
Since the contribution of the triangle diagrams to the axial vector current in the presence of vector gauge fields is
\[ \langle \mathcal{J}_{\mu 5}^{III}(x) \rangle \equiv \frac{1}{2} \int d^4y \, d^4z \, \Gamma_{\mu \lambda \nu}(x,y,z) \, A^\lambda(y) \, A^\nu(z), \] (3.2.6)
the planar part of divergence of the axial vector current is given by:
\[ \left[ \left\langle \partial^\mu \mathcal{J}_{\mu 5}^{III}(x) \right\rangle \right]_{\text{pl.}} = -\frac{i}{16\pi^2} \varepsilon_{\lambda \nu \alpha \beta} \partial^\lambda A^\nu(x) \star \partial^\alpha A^\beta(x), \] (3.2.7)
where the result from Eq. (3.2.5) is used. Here, the \( \star \)-product is defined in Eq. (2.1.3). To obtain the expected result,
\[ \left[ \left\langle \partial^\mu \mathcal{J}_{\mu 5}^{III}(x) \right\rangle \right]_{\text{pl.}} = -\frac{i}{64\pi^2} \varepsilon_{\lambda \nu \alpha \beta} F^{\lambda \nu}(x) \star F^{\alpha \beta}(x), \] (3.2.8)
with \( F_{\lambda \nu} = \partial_\lambda A_\nu - \partial_\nu A_\lambda + ig[A_\lambda, A_\nu] \). we have also to consider the planar contribution from higher loop orders [see the diagrams in Fig. 1.2]. Note that integration over space-time coordinates \( x \) guarantees the \( \star \)-gauge invariance of the axial anomaly and this integration need of course to be performed only over those space-time directions, where the noncommutativity parameter \( \theta \) does not vanish.

For the nonplanar contribution to the global anomaly we go through the same dimensional regularization procedure and get
\[ [A_{\lambda \nu}]_{\text{n.-pl.}} = +4\varepsilon_{\lambda \nu \alpha \beta} k_2^\alpha k_3^\beta \int_0^{1/\alpha_1} d\alpha_1 \int_0^{1/\alpha_2} d\alpha_2 \int_{-\infty}^{+\infty} \frac{d^D \ell}{(2\pi)^D} \frac{\ell^4 \ell^2 e^{\pm 2i \ell \times q}}{(\ell^2 + \Delta)^3}, \] (3.2.9)
where \([F_b(\ell)]_{\text{n.pl.}}\) denotes the nonplanar part of the phase factor of diagram B [see Eqs. (3.2.2a-b)] and \( \Delta = -k_2^2 \alpha_1 (1 - \alpha_1) - k_3^2 \alpha_2 (1 - \alpha_2) - 2k_2 k_3 \alpha_1 \alpha_2 \).

To do the \( \ell \)-integration we use Eqs. (A.2.8) and (A.2.18) [appendix B],
\[ I_1 = -i \int_{-\infty}^{+\infty} \frac{d^D \ell}{(2\pi)^D} \frac{\ell^4 e^{\pm 2i \ell \times q}}{(\ell^2 + \Delta)^3} = -\frac{i}{16\pi^2} \int_{-\infty}^{+\infty} \frac{d^D \ell}{(2\pi)^D} \frac{\ell^2 e^{\pm 2i \ell \times q}}{(\ell^2 + \Delta)^3} \]
\[ = -\frac{i}{\ln \Lambda^2} \left[ \frac{1}{16\pi^2} E_1(q, \Delta) - \frac{q \circ q}{128\pi^2} E_2(q, \Delta) \right], \] (3.2.10a)
where
\[ E_n(q, \Delta) \equiv \int_0^{\infty} \frac{d\rho}{\rho^n} \exp \left[ -\rho - \frac{q \circ q}{4} \frac{1}{\rho} \right]. \] (3.2.10b)
The integrals with the phases $e^{+2iℓ×q}$ or $e^{-2iℓ×q}$ yield the same result, because $I_1$ is even under $θ → -θ$.

The functions $E_n(q, ∆), n = 1, 2$ can be approximated by \[7, 14\]

\[
E_1(q, ∆) = \int_0^{∞} \frac{dρ}{ρ} \exp \left[ -ρ \Delta - \frac{1}{2} \frac{1}{Λ_{\text{eff.}}} ρ \right] = + \ln \frac{Λ_{\text{eff.}}^2}{Δ} + O(1).
\]

\[
E_2(q, ∆) = \int_0^{∞} \frac{dρ}{ρ^2} \exp \left[ -ρ \Delta - \frac{1}{2} \frac{1}{Λ_{\text{eff.}}} ρ \right] = Λ_{\text{eff.}}^2 - Δ \ln Λ_{\text{eff.}}^2 + O(1),
\]

(3.2.11)

with

\[
\frac{1}{Λ_{\text{eff.}}^2} \equiv \frac{1}{Λ^2} + q \circ q.
\]

(3.2.12)

Now putting the expression from Eq. (3.2.10a) in the Eq. (3.2.9) we arrive at:

\[
[A_λν(k_2, k_3; θ, Λ)]_{n.-\text{pl.}} = -\frac{i}{4π^2} ε_{λναβ}k_2^α k_3^β \int_0^{1-α_1} dα_1 \int_0^{1-α_2} dα_2 \mathcal{P}(α_1, α_2),
\]

(3.2.13a)

with

\[
\mathcal{P}(α_1, α_2) \equiv \frac{1}{\ln Λ^2} \left\{ [E_1(k_1, ∆) - k_1 \circ k_1 E_2(k_1, ∆)] \cos [k_2 × k_3 (1 - 2α_1 - 2α_2)] 
+ [E_1(k_2, ∆) - k_2 \circ k_2 E_2(k_2, ∆)] \cos [k_2 × k_3 (1 - 2α_2)] 
+ [E_1(k_3, ∆) - k_3 \circ k_3 E_2(k_3, ∆)] \cos [k_2 × k_3 (1 - 2α_1)] \right\},
\]

(3.2.13b)

where $k_1 = -(k_2 + k_3)$. In the following we will show, that for light-like $θ_{μν}$ a finite contribution will arise from the first term on the r.h.s. of Eq. (3.2.13b), which survives the limit $Λ → ∞$ (or $D → 4$):

Taking the noncommutativity tensor as

\[
θ_{μν} = \begin{pmatrix} 0 & 0 & +θ & 0 \\ 0 & 0 & +θ & 0 \\ -θ & -θ & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

(3.2.14)

we get

\[
\int dx^2 dx^+ dx^- \left< \partial^μ J_{μ,5}^{III}(x) \right>_{n.-\text{pl.}} \equiv \frac{1}{2} \int dx^2 dx^+ dx^- d^4y d^4z \left[ \partial^μ Γ_{μλν}(x, y, z) \right]_{n.-\text{pl.}} A_λ(y) A_ν(z)
= -\frac{i}{8π^2} ε_{λναβ} \int dx^2 dx^+ dx^- d^4y d^4z \int A_λ(y) A_ν(z) \int \frac{d^4k_2}{(2π)^4} \frac{d^4k_3}{(2π)^4} e^{-ik_1x} e^{ik_2y} e^{ik_3z} k_2^α k_3^β 
\times \int_0^{1-α_1} dα_1 \int_0^{1-α_2} dα_2 \mathcal{P}(α_1, α_2).
\]

(3.2.15)
commutative. Here, using the identity
\[ \int dx^2 dx^+ dx^- f(x) \ast g(x) = \int dx^2 dx^+ dx^- g(x) \ast f(x), \]
the continuity equation for \( \mathcal{J}^{III}_{\mu,5} \) [Eq. (3.1.13)] then reads
\[ \int \! dx^2 dx^+ \partial^\mu \mathcal{J}^{III}_{\mu,5} (x) = 0. \] (3.2.16)

We will next show, that it is broken by quantum effects. To do this let us look at the contribution of the first term of \( \mathcal{P} (\alpha_1, \alpha_2) \), Eq. (3.2.13b), to the nonplanar part of the axial anomaly [Eq. (3.2.15)]:
\[
\left[ \int \! dx^2 dx^+ dx^- \left\langle \partial^\mu \mathcal{J}^{III}_{\mu,5} (x) \right\rangle \right]_{\text{n.pl.}} \left. 1. \right. \text{Term} \equiv \nonumber \\
\equiv - \frac{i}{8\pi^2} \int \! dy \, d^4 z \int \frac{dk^+_2}{(2\pi)^4} \frac{dk^-_3}{(2\pi)^4} \frac{dk^+_2 dk^-_3}{(2\pi)^4} \int \! dx^- e^{ik^+_1 x^-} \int \! dx^+ e^{ik^-_1 x^+} \nonumber \\
\times \int \! dx^2 e^{-ik^+_1 y^+} e^{-ik^+_2 y^+} e^{ik^+_2 y^2} e^{ik^+_3 (y-x)} e^{-ik^-_3 z^-} e^{-ik^-_3 z^-} e^{ik^+_3 (z-x)^3} \nonumber \\
\times \left[ \varepsilon^+ m (A^+_m (y) A^- (z) - A^- (y) A^+_m (z)) k^m_2 k^m_3 + \varepsilon^+ m A^+_m (y) A^+_n (z) \left( k^+_2 k^-_3 - k^+_3 k^-_2 \right) \right] \nonumber \\
\times \frac{1}{\alpha_1} \int \! d\alpha_2 \frac{1}{\ln \Lambda^2} \left[ \ln \frac{1}{\xi^2 + \left( \theta k^+_1 \right)^2} - \frac{1}{\Lambda^2} \left( \frac{1}{\xi^2 + \left( \theta k^+_1 \right)^2} - \Delta \ln \frac{1}{\Lambda^2 + \left( \theta k^+_1 \right)^2} \right) \right] \nonumber \\
\times \cos \left( \frac{\theta}{\sqrt{2}} \left( k^+_2 k^-_3 + k^+_3 k^-_2 \right) \right) (1 - 2\alpha_1 - 2\alpha_2), \] (3.2.17)

where we have used light-cone variables \( k_i^\pm \equiv \frac{1}{\sqrt{2}} (k_i^0 \mp k_i^1) \), and Eq. (A.1.3). On the fifth line we have used Eqs. (3.2.11). Integration over \( x^- \) and \( x^+ \) lead to \( \delta \)-functions \( \delta \left( k^+_2 + k^+_3 \right) \) and \( \delta \left( k^-_2 + k^-_3 \right) \), respectively. Performing the integration over \( k^+_3 \) will then lead to a finite contribution from the first term on the fifth line, which survives the limit \( \Lambda \to \infty \) (or \( D \to 4 \)), i.e.
\[
\int \! dk^+_3 \delta \left( k^+_2 + k^+_3 \right) \frac{1}{\ln \Lambda^2} \ln \frac{1}{\xi^2 + \left( \theta^2 (k^+_2 + k^+_3) \right)^2} = 1 \nonumber 
\]
The second term on the same line, proportional to \( \theta k^+_1 \), vanishes, however. The second term on the fourth line vanishes after integrating over \( k^-_3 \). Further, Integrating over \( x^2 \) leads to \( 2\pi \delta (k^2_2 + k^2_3) \), which after integrating over \( k^+_3 \) and the parameters \( \alpha_1 \) and \( \alpha_2 \) yields then:
\[
\left[ \int \! dx^2 dx^+ dx^- \left\langle \partial^\mu \mathcal{J}^{III}_{\mu,5} (x) \right\rangle \right]_{\text{n.pl.}} \left. 1. \right. \text{Term} \equiv - \frac{i}{16\pi^2} \int \! dy \, d^4 z \int \! \frac{dk^+_2}{2\pi} e^{ik^+_2 (y-z)} \nonumber \\
\times \int \! \frac{dk^+_3}{2\pi} e^{ik^+_3 (y-z)} \int \! \frac{dk^-_2}{2\pi} e^{-ik^-_2 (y-z)} \int \! \frac{dk^-_3}{2\pi} e^{-ik^-_3 (y-z)} \int \! \frac{dk^+_3}{2\pi} e^{ik^+_3 (z-x)^3} \nonumber \\
\times \varepsilon^+ m (A^+_m (y) A^- (z) - A^- (y) A^+_m (z)) k^m_2 k^m_3 \sin \left( \frac{\theta}{\sqrt{2}} \left( k^+_2 + k^+_3 \right) k^2_2 \right) \nonumber \\
\times \frac{6}{\sqrt{2}} \left( k^+_2 + k^-_2 \right) k^2_2. \] (3.2.18)
Defining now a new $\star$-product

\[ f(x) \star g(x) \equiv f(x) \frac{\sin(\frac{\theta^{\mu\nu}}{2} \partial_\mu \partial_\nu)}{\frac{\theta^{\mu\nu}}{2} \partial_\mu \partial_\nu} g(x) \]

\[ = f(x) g(x) - \frac{1}{3!} 4 \theta^{\mu\nu} \theta^{\rho\sigma} \partial_\rho f(x) \partial_\sigma g(x) + \cdots, \quad (3.2.19) \]
equation (3.2.18) may be written in the compact form

\[ \int dx^2 dx^+ dx^- \left< \partial^\mu J^I_{\mu,5} (x) \right>_{\text{n.pl.}} = -\frac{i}{16\pi^2} \varepsilon_{\lambda\nu\alpha\beta} \int dx^2 dx^+ dx^- \partial^\lambda A^\nu (x) \star' \partial^\alpha A^\beta (x). \quad (3.2.20) \]

A product of this form has appeared in the literature [15, 16].

Adding the contributions of the planar part of the triangle integrals and invoking gauge invariance, we get:

\[ \int dx^2 dx^+ dx^- \left< \partial^\mu J^I_{\mu,5} (x) \right>_{\text{total}} = -\frac{i}{64\pi^2} \varepsilon_{\lambda\nu\alpha\beta} \int dx^2 dx^+ dx^- F^\lambda (x) \star'' F^\alpha (x), \quad (3.2.21a) \]

with

\[ f(x) \star'' g(x) = f(x) (\star + \star') g(x). \quad (3.2.21b) \]

The $\star$-product coming from the contribution of planar diagrams and $\star'$-product from nonplanar diagrams.

It may appear that the $\star'$ and for that matter $\star$ products appearing in the integrals above may be removed. But, we know that nontrivial instanton solutions involve nontrivial boundary conditions and that under certain circumstances (in particular for non spatial $\theta$) these nontrivial gauge field contributions may prevent removing the star-operation under the integral sign.

The Limit $\theta \to 0$: We have to note that $\theta \to 0$ limit is singular in the sense that first taking limit $\theta \to 0$ and then integrating over $x^\pm$ gives different result from when we change the order of these two operations. To see this, let us take the limit $\theta \to 0$ in both planar and nonplanar part of the axial anomaly, before integrating over $x^\pm$. Taking the limit $\theta \to 0$ in (3.2.5) we get

\[ \left< \partial^\mu J^I_{\mu,5} (x) \right>_{\text{pl.}} = -\frac{i}{64\pi^2} \varepsilon_{\lambda\nu\alpha\beta} F^\lambda (x) F^\alpha (x), \quad (3.2.22) \]

for the planar part, and

\[ [A_{\lambda\nu} (k_2, k_3; \theta \to 0)]_{\text{n.pl.}} = -\frac{3i}{8\pi^2} \varepsilon_{\lambda\nu\alpha\beta} k_2^\alpha k_3^\beta. \quad (3.2.23) \]

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\[
\lim_{\Lambda \to 0} \ln \Lambda^2 \int_0^\infty \frac{e^{-\rho}}{\rho} \Delta = 1,
\]
which can be derived from Eq. (3.2.11) by taking \( \theta \to 0 \). Using the relation (3.2.6) we have
\[
\langle \partial^\mu J_{\mu,5}^{III}(x) \rangle_{\text{n.-pl.}} = -\frac{3i}{64\pi^2} \epsilon_{\lambda\nu\alpha\beta} F^{\lambda\nu}(x) F^{\alpha\beta}(x). \tag{3.2.24}
\]
Adding these two contributions we get
\[
\langle \partial^\mu J_{\mu,5}^{III}(x) \rangle_{\text{total}} = -\frac{i}{16\pi^2} \epsilon_{\lambda\nu\alpha\beta} F^{\lambda\nu}(x) F^{\alpha\beta}(x), \tag{3.2.25}
\]
which is the same result as in the commutative case.

Now we turn to the gauge invariant result Eq. (3.2.21a-b), which was obtained for the special case of light-like \( \theta_{\mu\nu} \) and after integrating \( x^\pm \). Taking the limit \( \theta \to 0 \) there, yields
\[
\lim_{\theta \to 0} \int dx^2 dx^+ dx^- \langle \partial^\mu J_{\mu,5}^{III}(x) \rangle_{\text{total}} = -\frac{i}{32\pi^2} \int dx^2 dx^+ dx^- \epsilon_{\lambda\nu\alpha\beta} F^{\lambda\nu}(x) F^{\alpha\beta}(x), \tag{3.2.26}
\]
which differs by a factor of two with Eq. (3.2.25). This means that limit \( \theta \to 0 \) and integrating over \( x^\pm \) do not commute here.

**IV. Conclusion**

In this paper we have studied the effect of nonplanar diagrams on the gauge and global anomalies of noncommutative gauge theories. In the first part we studied the chiral gauge anomaly of the chiral U(1) and U(N) theories, with adjoint matter fields and found out, that both theories are free of such anomalies.

In the second part, we then studied the global symmetries of the U(1) theory with fundamental matter field. Here we found a novel result. Three different currents were derived from the same global symmetry of the theory with the same classically conserved charge. All of these currents were shown to be anomalous, but only one involved nonplanar diagrams. In Section III.2, we calculated the divergence of the three-point function of this special current and showed, that for light-like noncommutativity tensor \( \theta_{\mu\nu} \) a very unusual form of the anomaly appears. We showed, that in the planar contribution the ordinary \( \ast \)-product replaces the usual product of the commutative field theory, whereas in the nonplanar contribution a new product, a \( \ast' \)-product appears [see Eq. (3.2.19)]. This "new" product was first introduced by Garousi in [15] and was also found by Liu and Michelson in [16].

It is interesting to note that the contribution to the current which is responsible for the nonplanar contributions to the global anomaly, and for the emergence of the \( \ast' \)-product is due to a neutral current,
Although the final result is only gauge invariant after integrating over those space-time coordinates, where the noncommutativity parameter does not vanish, but this integration will not remove the $\star$ and $\star'$ as long as nontrivial topological gauge field contributions are involved. It was also noted, that taking the limit $\theta \to 0$ and integrating over noncommutative coordinates, do not commute.

V. Acknowledgments

We would like to thank H. Arfaei and H. Yavartanoo for discussions.
In Section II.2, the Jacobi Identities between the structure constants of \( U(N) \) are:

\[
\begin{align*}
C^{abcd}D_{dcm} + C^{cad}D_{dbm} + C^{bcd}D_{dam} &= 0, \\
D^{abcd}D_{dcm} - D^{cad}D_{dbm} - C^{bcd}D_{dam} &= 0, \\
D^{abcd}C_{dcm} + C^{cad}D_{dbm} - D^{bcd}D_{dam} &= 0, \\
D^{abcd}C_{dcm} - D^{cad}D_{dbm} + D^{bcd}D_{dam} &= 0, \\
C^{abcd}D_{dcm} - D^{cad}D_{dbm} + D^{bcd}D_{dam} &= 0, \\
D^{abcd}C_{dcm} + C^{cad}D_{dbm} - D^{bcd}D_{dam} &= 0,
\end{align*}
\]  

(A.1.1)

In Section II.2, the equal-time commutation relation necessary to calculate the formal and the potentially anomalous part of \( \partial^\lambda \Gamma_{\mu\lambda
u}^{efm} \) and \( \partial^\mu \Gamma_{\mu\lambda
u}^{efm} \) is given by:

\[
[J_0^f (y), J_\nu^m (z)]_{y_0 = z_0} = -(\gamma_\nu P_+)_{\beta\gamma} \int_{-\infty}^{+\infty} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} e^{ik_2 y} e^{ik_3 z}
\times \int_{-\infty}^{+\infty} \frac{d^D \ell}{(2\pi)^D} \psi_\gamma^d (\ell) \bar{\psi_\beta} (\ell + k_2 + k_3) G^{abdfm} (\ell, k_2, k_3),
\]

(A.1.2a)

where

\[
G^{abdfm} (\ell, k_2, k_3) \equiv e^{+i \ell \times (k_2 + k_3)} - i k_2 \times k_3 \left( -K^{abf} K_{dam} \right) + e^{+i \ell \times (k_2 + k_3)} + i k_2 \times k_3 \left( K^{daf} K_{abm} \right)
\]

\[
+ e^{-i \ell \times (k_2 + k_3)} + i k_2 \times k_3 \left( -K^{abf} K_{dam} + K^{abf} iC_{dam} + iC^{abf} K_{dam} + C^{abf} C_{dam} \right)
\]

\[
+ e^{-i \ell \times (k_2 + k_3)} - i k_2 \times k_3 \left( K^{daf} K_{abm} - K^{daf} iC_{abm} - iC^{daf} K_{abm} - C^{daf} C_{abm} \right)
\]

\[
+ e^{+i \ell \times (k_2 + k_3)} - i k_2 \times k_3 \left( K^{abf} K_{dam} - K^{abf} iC_{dam} + iC^{abf} K_{dam} + C^{abf} C_{dam} \right).
\]

(A.1.2b)

In Section III.2, following relation is used to derive Eq. (3.2.17) in light cone coordinates:

\[
\varepsilon_{\lambda\rho\alpha\beta} A_\lambda (y) A_\nu (z) k_2^\alpha k_3^\beta
\]

\[
= \varepsilon_{01mn} (A_0 (y) A_1 (z) - A_1 (y) A_0 (z)) k_2^m k_3^n + \varepsilon_{mn01} A_m (y) A_n (z) \left( k_2^0 k_3^1 - k_2^1 k_3^0 \right)
\]

\[
= \varepsilon_{+-mn} (A_+ (y) A_- (z) - A_- (y) A_+ (z)) k_2^m k_3^n + \varepsilon_{mn+-} A_m (y) A_n (z) \left( k_2^- k_3^+ - k_2^+ k_3^- \right),
\]

(A.1.3)

In Section II.2, the U(N) phases and group factor are:
<table>
<thead>
<tr>
<th>Phases</th>
<th>Group factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_A^1$</td>
<td>$e^{i k_2 \times k_3}$</td>
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<tr>
<td>$f_A^2(\ell)$</td>
<td>$e^{-i k_2 \times k_3 - 2i\ell \times k_3}$</td>
</tr>
<tr>
<td>$f_A^3(\ell)$</td>
<td>$e^{-i k_2 \times k_3 - 2i\ell \times (k_2 + k_3)}$</td>
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</tr>
<tr>
<td>$f_A^4(\ell)$</td>
<td>$e^{-i k_2 \times k_3 - 2i\ell \times k_2}$</td>
</tr>
<tr>
<td>$f_A^5(\ell)$</td>
<td>$e^{-i k_2 \times k_3 + 2i\ell \times k_2}$</td>
</tr>
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<td></td>
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<tr>
<td>$f_A^6(\ell)$</td>
<td>$e^{-i k_2 \times k_3 + 2i\ell \times (k_2 + k_3)}$</td>
</tr>
<tr>
<td>$f_A^7(\ell)$</td>
<td>$e^{-i k_2 \times k_3 + 2i\ell \times (k_2 + k_3)}$</td>
</tr>
<tr>
<td>$f_A^8(\ell)$</td>
<td>$e^{-i k_2 \times k_3}$</td>
</tr>
<tr>
<td>$f_B^1$</td>
<td>$e^{i k_2 \times k_3}$</td>
</tr>
<tr>
<td>$f_B^2(\ell)$</td>
<td>$e^{-i k_2 \times k_3 + 2i\ell \times k_3}$</td>
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<tr>
<td>$f_B^3(\ell)$</td>
<td>$e^{-i k_2 \times k_3 + 2i\ell \times (k_2 + k_3)}$</td>
</tr>
<tr>
<td>$f_B^4(\ell)$</td>
<td>$e^{-i k_2 \times k_3 + 2i\ell \times k_2}$</td>
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<tr>
<td>$f_B^5(\ell)$</td>
<td>$e^{-i k_2 \times k_3 - 2i\ell \times k_2}$</td>
</tr>
<tr>
<td>$f_B^6(\ell)$</td>
<td>$e^{-i k_2 \times k_3 - 2i\ell \times (k_2 + k_3)}$</td>
</tr>
<tr>
<td>$f_B^7(\ell)$</td>
<td>$e^{-i k_2 \times k_3 - 2i\ell \times (k_2 + k_3)}$</td>
</tr>
<tr>
<td>$f_B^8(\ell)$</td>
<td>$e^{-i k_2 \times k_3}$</td>
</tr>
</tbody>
</table>
Here we want to evaluate the integrals of the generic forms:

\[ I_1 \equiv \int_{-\infty}^{+\infty} \frac{d\ell}{(2\pi)^D} \frac{\ell^2_1 e^{-2i\ell \cdot q}}{(\ell^2 + \Delta)^3}, \]

\[ I_\xi \equiv \int_{-\infty}^{+\infty} \frac{d\ell}{(2\pi)^D} \frac{(\ell_\xi)_\xi e^{-2i\ell \cdot q}}{(\ell^2 + \Delta)^3}, \]

\[ I_{\eta\xi} \equiv \int_{-\infty}^{+\infty} \frac{d\ell}{(2\pi)^D} \frac{(\ell_\eta)_\xi (\ell_\xi)_\eta e^{-2i\ell \cdot q}}{(\ell^2 + \Delta)^3}. \]  

(A.2.1)

Here \( q \) is an arbitrary external momentum and the integration are over Euclidean integration variables.

To evaluate the above integrals let us introduce the polar coordinates in \( D \) dimensions with the following components:

\[ \ell_1 \equiv \ell \prod_{k=1}^{D-2} \sin \theta_k \cos \varphi, \quad \ell_2 \equiv \ell \prod_{k=1}^{D-2} \sin \theta_k \sin \varphi, \quad \ell_3 \equiv \ell \prod_{k=2}^{D-2} \sin \theta_k \cos \theta_1, \quad \ell_4 \equiv \ell \prod_{k=3}^{D-2} \sin \theta_k \cos \theta_2, \]

\[ \cdots, \quad \ell_{D-2} \equiv \ell \sin \theta_{D-3} \sin \theta_{D-2} \cos \theta_{D-4}, \quad \ell_{D-1} \equiv \ell \sin \theta_{D-2} \cos \theta_{D-3} \quad \ell_D \equiv \ell \cos \theta_{D-4}. \]  

(A.2.2)

Here \( 0 \leq \ell \leq \infty, 0 < \varphi < 2\pi, \) and \( 0 < \theta_i < \pi \) with \( i = 1, \cdots, D-2 \). The integration measure in \( D \) dimensions is then given by:

\[ d^D \ell = \ell^{D-1} d\ell \prod_{k=1}^{D-2} \sin^k \theta_k d\theta_k. \]  

(A.2.3)

To follow 't Hooft’s notation the first four components are denoted by \( \ell_\| \) and the other \( D-4 \) components by \( \ell_\perp \). We therefore have:

\[ \ell^2_\perp \equiv \ell^2 - \sum_{i=1}^{4} \ell^2_i = \ell^2 - \ell^2 \prod_{k=3}^{D-2} \sin^2 \theta_k. \]  

(A.2.4)

Putting this result in the integral \( I_1 \) from Eq. (A.2.1) we arrive first at:

\[ I_1 = \int_0^\infty d^4 \ell \frac{\ell^{D-2} e^{-2i\ell \cdot q}}{(\ell^2 + \Delta)^3} \int_0^\pi \left( \prod_{k=3}^{D-2} \sin^k \theta_k d\theta_k \right) - \prod_{k=3}^{D-2} \sin^{k+2} \theta_k d\theta_k \). \]  

(A.2.5)

Here we have used the definition \( d^4 \ell \equiv \ell^3 d\ell d\varphi \sin \theta_1 d\theta_1 \sin^2 \theta_2 d\theta_2 \). We use the relations

\[ \int_0^{D-2} \prod_{k=3}^{D-2} \sin^k \theta_k d\theta_k = \prod_{k=3}^{D-2} \frac{\sqrt{\pi} \Gamma(k+1)}{\Gamma(k+\frac{D}{2})} = \frac{\pi^{D-2}}{\Gamma(D + \frac{D}{2})}. \]

\[ \int_0^{D-2} \prod_{k=3}^{D-2} \sin^{k+2} \theta_k d\theta_k = \prod_{k=3}^{D-2} \frac{\sqrt{\pi} \Gamma(k+3)}{\Gamma(k+\frac{D}{2})} = \frac{2\pi^{D-2}}{\Gamma(D + \frac{D}{2} + 1)}. \]  

(A.2.6a)
Replacing this result in (A.2.5) we arrive at:

\[
I_1 = \frac{\pi^{D-2}}{\Gamma(D/2)} \frac{D-4}{D} \int_0^{\infty} d^4 \ell \frac{\ell^2 e^{-2i\ell \times q}}{(\ell^2 + \Delta)^{D/2}}.
\] (A.2.8)

This is a nonplanar integral which we want to evaluate later.

- First let us calculate \(I_\xi\) and \(I_{\eta \xi}\) from Eq. (A.2.1). We use the following relation:

\[
(\ell_\perp)_\xi \equiv \ell \prod_{k=\xi-1}^{\xi-2} \sin \theta_k \cos \theta_{\xi-2}, \quad \text{for} \quad 5 \leq \xi \leq D.
\] (A.2.9)

which arises from the definition of the polar coordinates (A.2.2). We have therefore:

\[
I_\xi = \int d^4 \ell \frac{\ell^D e^{-2i\ell \times q}}{(\ell^2 + \Delta)^{D/2}} \prod_{m=\alpha-1}^{\alpha-3} \sin \theta_m \cos \theta_{\alpha-2} \prod_{k=3}^{\alpha-2} \sin k \theta_k d\theta_k
\]

\[
= \int d^4 \ell \frac{\ell^D e^{-2i\ell \times q}}{(\ell^2 + \Delta)^{D/2}} \prod_{k=3}^{\alpha-3} \sin k \theta_k \int_0^{\pi} \sin^2 \theta_{\alpha-2} \cos \theta_{\alpha-2} d\theta_{\alpha-2} \int_0^{1} \prod_{k=\alpha-1}^{D-2} \sin^k \theta_k d\theta_k.
\] (A.2.10)

Now using \(\int_0^\pi \sin \alpha \cos \alpha \sin \beta = 0\), for \(\alpha = 5, \ldots, D\) we have \(I_\xi = 0\).

- To compute \(I_{\eta \xi}\) we have to use the relation (A.2.9) and calculate the integral for \(\eta = 1, 2, 3, 4\) and a generic index \(\xi\) separately. For \(\eta = 1, 2\) the integration over \(0 \leq \varphi \leq 2\pi\) yields zero due to \(\int_0^{2\pi} \cos \varphi d\varphi = 0\) and for \(\eta = 3, 4\) the relations:

\[
\int_0^\pi d\theta_1 \sin \theta_1 \cos \theta_1 = 0, \quad \int_0^\pi d\theta_2 \sin^2 \theta_2 \cos \theta_2 = 0,
\] (A.2.11)

lead to \(I_{\eta \xi} = 0\) for all \(1 \leq \eta \leq 4\) and \(5 \leq \xi \leq D\).

- As promised, we will determine the value of the integral

\[
I_2 \equiv \int_{-\infty}^{\infty} \frac{d^D \ell}{(2\pi)^D} \frac{\ell^2 e^{-2i\ell \times q}}{(\ell^2 + \Delta)^3}.
\] (A.2.12)

Using the replacement

\[
\frac{1}{\ell^2 + \Delta} = \int_0^{\infty} e^{-\alpha (\ell^2 + \Delta)} d\alpha,
\] (A.2.13)

we have

\[
I_2 = \sum_\xi \frac{\partial^2}{\partial z_\xi^2} \int_0^{\infty} \exp\left[z_\eta \ell_\eta - \ell^2 (\alpha_1 + \alpha_2 + \alpha_3) - \Delta (\alpha_1 + \alpha_2 + \alpha_3) - i\tilde{q}_\eta \ell_\eta\right] d\alpha_1 d\alpha_2 d\alpha_3.
\] (A.2.14)
we obtain

\[ I_2 = \frac{D}{2(2\pi)^{D/2}} \int_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(\alpha_1 + \alpha_2 + \alpha_3)^{D/2+1}} \exp \left[ \frac{(i\tilde{q}_\eta)^2}{4(\alpha_1 + \alpha_2 + \alpha_3)} - \Delta(\alpha_1 + \alpha_2 + \alpha_3) \right] \]

\[ + \frac{(i\tilde{q}_\eta)^2}{4(4\pi)^{D/2}} \int_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(\alpha_1 + \alpha_2 + \alpha_3)^{D/2+2}} \exp \left[ \frac{(i\tilde{q}_\eta)^2}{4(\alpha_1 + \alpha_2 + \alpha_3)} - \Delta(\alpha_1 + \alpha_2 + \alpha_3) \right]. \]  

(A.2.16)

Following the usual parametric representation procedure, we have to introduce

\[ 1 = \int_0^\infty d\rho \, \delta(\rho - \sum_{\ell=1}^3 \alpha_\ell), \]  

(A.2.17)

on the r.h.s. of (A.2.16). Rescaling then the Feynman parameters \( \alpha_\ell \rightarrow \rho \alpha_\ell \) for \( \ell = 1, 2, 3 \) and then integrating over Feynman parameters, we arrive at:

\[ I_2 = \frac{D}{2(4\pi)^{D/2}} \int_0^\infty \frac{d\rho}{\rho^{D/2-1}} \exp \left[ - \frac{q \cdot q}{4\rho} - \Delta \rho \right] - \frac{q^2}{4(4\pi)^{D/2}} \int_0^\infty \frac{d\rho}{\rho^{D/2}} \exp \left[ - \frac{q \cdot q}{4\rho} - \Delta \rho \right]. \]  

(A.2.18)

where \( q^2 \equiv q \cdot q \equiv -q_\sigma \theta_\sigma \theta_\eta \xi q_\xi \). This is just the result used in Eqs. (3.2.10a)-(3.2.10b).
References


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Figures

Diagram A $+$ Diagram B

$J_\mu(x)$ $\phantom{\text{Diagram A}}$ $\phantom{\text{Diagram B}}$ $J_\nu(z)$ $\phantom{\text{Diagram A}}$ $\phantom{\text{Diagram B}}$ $J_\lambda(y)$

Figure 1.1: Triangle Diagrams for the gauge anomaly in the current $J_\mu(x)$ indicated by the dashed line. Each diagram is decorated by a set of planar and nonplanar phases. For U(N) chiral gauge theory the currents are to be replaced by $J_\mu^c(x), J_\lambda^f(y),$ and $J_\nu^m(z)$. For non-chiral U(1) gauge theory only $J_\mu(x)$ is to be replaced by the anomalous $J_{\mu,5}(x)$.

$\phantom{\text{Diagram A}}$ $\phantom{\text{Diagram B}}$ $\phantom{\text{Diagram A}}$ $\phantom{\text{Diagram B}}$

Figure 1.2: Higher loop diagrams contributing to anomaly.