MEASURING MULTIPOLE MOMENTS OF WEYL METRICS BY MEANS OF GYROSCOPES

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Abstract

Using the technique of Rindler and Perlick we calculate the total precession per revolution of a gyroscope circumventing the source of Weyl metrics. We establish thereby a link between the multipole moments of the source and an “observable” quantity. Special attention deserves the case of the $\gamma$-metric. As an extension of this result we also present the corresponding expressions for some stationary spacetimes.

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1 Introduction

Weyl exterior solutions to Einstein equations [1], represent all possible static axially symmetric space-times in the context of general relativity. They may be represented as series expansions of suitable defined relativistic multipole moments [2]. Therefore, any of the Weyl metrics are, in principle, characterized by a specific combination of such multipoles. One way to provide physical content to those solutions consists in establishing a link between their multipole moments and quantities measured from well defined and physically reasonable experiments. It is the purpose of this work to establish such a link, by calculating the total precession per revolution of gyroscopes circumventing the symmetry axis. By doing so, the multipole moments of different Weyl metrics become “measurable” in the sense that they are expressed through quantities obtained from well defined and physically reasonable experiments (we are of course not discussing about the actual technical feasibility of such experiments). These results illustrate further the usefulness of gyroscopes in the study of gravitational phenomena [3], [4].

All calculations are carried out using the method proposed by Rindler and Perlick [5], a brief resume of which is given in the next section, together with the notation and the specification of the space-time under consideration.

In section 3 we obtain, for a selection of Weyl metrics, the precession per revolution relative to the original frame of a gyroscope rotating round the axis of symmetry. For sake of generality we present in Section 4 some results concerning stationary metrics. Finally, the results are discussed in the last section.

2 The space-time and the Rindler-Perlick method

2.1 The Weyl metrics

Static axysymmetric solutions to Einstein equations are given by the Weyl metric [1]

\[ ds^2 = -e^{2\psi} dt^2 + e^{-2\psi} \left[ e^{2\Gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2 \right] \]  

(1)
where metric functions have to satisfy

\[ \Psi_{,\rho\rho} + \rho^{-1}\Psi_{,\rho} + \Psi_{,zz} = 0 \]  \hspace{1cm} (2)

and

\[ \Gamma_{,\rho} = \rho(\Psi^2_{,\rho} - \Psi^2_{,z}) \quad ; \quad \Gamma_{,z} = 2\rho\Psi_{,\rho}\Psi_{,z} \]  \hspace{1cm} (3)

Observe that (2) is just the Laplace equation for \( \Psi \) (in the Euclidean space), and furthermore it represents the integrability condition for (3), implying that for any "Newtonian" potential we have a specific Weyl metric, a well known result.

The general solution of the Laplace equation (2) for the function \( \Psi \), presenting an asymptotically flat behaviour, results to be

\[ \Psi = \sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} P_n(\cos \theta) \]  \hspace{1cm} (4)

where \( r = (\rho^2 + z^2)^{1/2} \), \( \cos \theta = z/r \) are Weyl spherical coordinates and \( P_n(\cos \theta) \) are Legendre Polynomials. The coefficients \( a_n \) are arbitrary real constants which have been named in the literature "Weyl moments", although they cannot be identified as relativistic multipole moments in spite of the formal similarity between expression (4) and the Newtonian potential. Then, equations (3) are solved to give function \( \Gamma \) in terms of Weyl moments as follow [1]

\[ \Gamma = \sum_{n,k=0}^{\infty} \frac{(n+1)(k+1)}{n+k+2} \frac{a_n a_k}{r^{n+k+2}} (P_{n+1}P_{k+1} - P_nP_k) \]  \hspace{1cm} (5)

Another interesting way of writing the solution (4) was obtained by Erez-Rosen [6] and Quevedo [7], integrating equations (2, 3) in prolate spheroidal coordinates, which are defined as follows

\[ x = \frac{r_+ + r_-}{2\sigma} \quad , \quad y = \frac{r_+ - r_-}{2\sigma} \]

\[ r_{\pm} \equiv [\rho^2 + (z \pm \sigma)^2]^{1/2} \]

\[ x \geq 1 \quad , \quad -1 \leq y \leq 1 \]  \hspace{1cm} (6)

where \( \sigma \) is an arbitrary constant which will be identified later with the Schwarzschild's mass. Inverse relation between both families of coordinates
is given by

\[ \rho^2 = \sigma^2(x^2 - 1)(1 - y^2) \]
\[ z = \sigma xy \]  

(7)

The prolate coordinate \( x \) represents a radial coordinate, whereas the other coordinate, \( y \) represents the cosine function of the polar angle.

In these prolate spheroidal coordinates the Weyl metric is given by

\[
d s^2 = -e^{2\Psi} dt^2 + \sigma^2 e^{-2\Psi} \left[ e^{2\Gamma}(x^2 - y^2) \left( \frac{d x^2}{x^2 - 1} + \frac{d y^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2) d\phi^2 \right] \]

(8)

Then, the corresponding equations that the metric functions \( \Gamma \) and \( \Psi \) have to satisfy, can be solved to obtain for \( \Psi \)

\[
\Psi = \sum_{n=0}^{\infty} (-1)^{n+1} q_n Q_n(x) P_n(y) \]

(9)

being \( Q_n(y) \) Legendre functions of second kind and \( q_n \) a set of arbitrary constants.

Both sets of coefficients, \( a_n \) and \( q_n \), characterize any Weyl metric [7]. Nevertheless these constants do not give us physical information about the metric since they do not represent the “real” multipole moments of the source. That is not the case for the relativistic multipole moments defined by Geroch [8], Hansen [9] and Thorne [10], which, as it is known, characterize completely and uniquely, at least in the neighbourhood of infinity, every asymptotically flat and stationary vacuum solution [11], [12] providing at the same time a physical description of the corresponding solution.

An algorithm to calculate the Geroch multipole moments was developed by G.Fodor, C. Hoenselaers and Z. Perjes [13] (FHP). By applying such method, the resulting multipole moments of the solution are expressed in terms of the Weyl moments. Similar results are obtained from the Thorne’s definition, using harmonic coordinates. The structure of the obtained relation between coefficients \( a_n \) and these relativistic moments allows to express the Weyl moments as a combination of the Geroch relativistic moments. For instance, the first coefficients result to be

\[
a_0 = -M_0 \]

(10)
\[
\begin{align*}
    a_2 &= -\frac{1}{3} M^3_0 - M_2 \\
    a_4 &= -\frac{1}{5} M^5_0 - \frac{8}{7} M^2_0 M_2
\end{align*}
\]

where only massive multipole moments (\(M_n\)) appear since the metrics are considered to possess equatorial symmetry. Obviously, the general relation between Weyl moments and relativistic multipole moments are not known. Nevertheless, choosing some specific multipole moments, it is possible to obtain the whole set of Weyl moments needed to define a solution containing the required multipole structure. Thus, for example, one of us [2] obtained the relativistic vacuum solution corresponding to an object consisting exclusively of mass and quadrupole moment. This metric will be treated later in our analysis.

### 2.2 The Rindler-Perlick method

This method consists in transforming the angular coordinate \(\phi\) by

\[
\phi = \phi' + \omega t,
\]

where \(\omega\) is a constant. Then the original frame is replaced by a rotating frame. The transformed metric is written in a canonical form, (we have slightly changed the original notation in [5] to avoid confusion with our notation)

\[
ds^2 = -e^{2\Phi}(dt - \omega_i dx^i)^2 + h_{ij} dx^i dx^j,
\]

with latin indexes running from 1 to 3 and \(\Phi, \omega_i\) and \(h_{ij}\) depend on the spatial coordinate \(x^i\) only (we are omitting primes). Then, it may be shown that the four acceleration \(A_\mu\) and the rotation three vector \(\Omega^i\) of the congruence of world lines \(x^i = \text{constant}\) are given by [5],

\[
\begin{align*}
    A_\mu &= (0, \Phi, \omega) \\
    \Omega^i &= \frac{1}{2} e^\Phi (det h_{mn})^{-1/2} e^{ij} \omega_{k,j},
\end{align*}
\]

where the comma denotes partial derivative. It is clear from the above that if \(\Phi, \omega = 0\), then particles at rest in the rotating frame follow a circular geodesic. On the other hand, since \(\Omega^i\) describes the rate of rotation with respect to
the proper time at any point at rest in the rotating frame, relative to the
compass of inertia, then $-\Omega^i$ describes the rotation of the compass of inertia
(the gyroscope) with respect to the rotating frame. Applying (13) to the
original frame of (1) written in Weyl’s spherical coordinates, i.e,

$$ds^2 = -f dt^2 + f^{-1}[e^{2\Gamma}(dr^2 + r''d\theta^2) + r^2\sin^2\theta d\phi^2]$$  \hspace{1cm} (17)

we cast (17) into the canonical form (14) where

$$e^{2\Phi} = f - \omega^2 f^{-1}r^2 \sin^2\theta$$  \hspace{1cm} (18)

$$\omega_i = e^{-2\Phi}(0, 0, \omega f^{-1}r^2 \sin^2\theta)$$  \hspace{1cm} (19)

$$h_{rr} = f^{-1}e^{2\Gamma}$$  \hspace{1cm} (20)

$$h_{\theta\theta} = r^2f^{-1}e^{2\Gamma}$$  \hspace{1cm} (21)

$$h_{\phi\phi} = e^{-2\Phi}r^2 \sin^2\theta$$  \hspace{1cm} (22)

whit $f \equiv e^{2\Psi}$. If we take into account the condition for circular geodesics, it
results

$$\omega^2 = \frac{f^2(f_r + f_\theta)}{2rf \sin\theta (\sin\theta + r \cos\theta) - r^2 \sin^2\theta (f_r + f_\theta)}$$  \hspace{1cm} (23)

If we consider circular geodesics in the equatorial plane, then the parameter $\omega$ turns out to be

$$\omega = \frac{f_r^{1/2}f}{(2rf - r^2 f_r)^{1/2}}$$  \hspace{1cm} (24)

2.3 Rate of precession

The expression for the rate of precession $\Omega \equiv (\Omega^i \Omega^j h_{ij})^{1/2}$ results to be

$$\Omega = \frac{f^{1/2}e^{-\Gamma}\omega}{f^2 - \omega^2 r^2 \sin^2\theta} \left[(f_\theta \sin\theta - f \cos\theta)^2 + \sin^2\theta (rf_r - f)^2\right]^{1/2}$$  \hspace{1cm} (25)

Taking into account only circular geodesics in the equatorial plane, then
we get from (24) and (25)

$$\Omega = \frac{1}{2} \left(\frac{f_r}{f}\right)^{1/2} \frac{f^{-1/2}}{r^{1/2}} \left(2f - rf_r\right)^{1/2}$$  \hspace{1cm} (26)
According to the meaning of Ω given above, it is clear that the orientation of the gyroscope, moving around the axis of symmetry, after one revolution, changes by

$$\Delta \phi' = -\Omega \Delta \tau$$

(27)

where $\Delta \tau$ is the proper time interval corresponding to one period. Then from the canonical form of the metric (14),

$$\Delta \phi' = -2\pi \frac{\Omega e^\Phi}{\omega},$$

(28)

as measured in the rotating frame. In the original system we have

$$\Delta \phi = 2\pi \left( 1 - \frac{\Omega e^\Phi}{\omega} \right).$$

(29)

This quantity $\Delta \phi$, calculated over one revolution around a circular orbit for the Weyl metrics, results to be

$$\Delta \phi = 2\pi \left( 1 - e^{-\Gamma} \left[ (f,\theta \sin \theta - f \cos \theta)^2 + \sin^2 \theta (rf_r - f)^2 \right]^{1/2} \right).$$

(30)

And considering a circular geodesic in the equatorial plane we have,

$$\Delta \phi = 2\pi \left( 1 - \frac{\sqrt{2}}{2} e^{-\Gamma} f^{-1} [(f - rf_r)(2f - rf_r)]^{1/2} \right).$$

(31)

In terms of the metric function $\Psi$ we have,

$$\Omega = \frac{1}{r^{1/2}} e^{-\Gamma} e^\Psi \Psi_{,r}^{1/2} (1 - r \Psi_{,r})^{1/2}$$

(32)

$$\Delta \phi = 2\pi \left( 1 - e^{-\Gamma} (1 - r \Psi_{,r})^{1/2} (1 - 2r \Psi_{,r})^{1/2} \right).$$

(33)

Since the metric functions $\Psi$ and $\Gamma$ are known for any Weyl solution, we can obtain both the rate of precession $\Omega$ and $\Delta \phi$, using (4) and (5), in terms of the Weyl moments of the solution. Up to order $O(r^{-5})$ we have the following result

$$\Omega = 1 + \frac{5}{2} a_0 \frac{1}{r} + \frac{19}{8} a_0^2 \frac{1}{r^2} + \left( \frac{107}{48} a_0^3 - \frac{11}{4} a_2 \right) \frac{1}{r^3}$$

$$+ \left( \frac{457}{384} a_0^4 - \frac{31}{8} a_2 a_0 - \frac{1}{4} a_1^2 \right) \frac{1}{r^4} + O(r^{-5})$$

(34)
\[ \Delta \phi = 2\pi \left[ -\frac{3}{2} a_0 \frac{1}{r} - \frac{3}{8} a_0^2 \frac{1}{r^2} - \left( \frac{15}{16} a_0^3 - \frac{9}{4} a_2 \right) \frac{1}{r^3} \right. \]
\[ \left. - \left( \frac{3}{8} a_0 a_2 + \frac{29}{128} a_0^4 \right) \frac{1}{r^4} + O(r^{-5}) \right]. \] 

(35)

Now, making use of the fact that the Weyl moments are some combination of the multipole moments (12) we can obtain the above quantities in term of the relativistic massive moments \( M_n \) as follows,

\[ \Omega = 1 - \frac{5}{2} M_0 \frac{1}{r} + \frac{19}{8} M_0^2 \frac{1}{r^2} + \left( \frac{11}{4} M_2 - \frac{151}{48} M_0^3 \right) \frac{1}{r^3} \]
\[ - \left( \frac{3}{128} M_0^4 + \frac{31}{8} M_2 M_0 + \frac{1}{4} M_1 \right) \frac{1}{r^4} + O(r^{-5}) \]  

(36)

\[ \Delta \phi = 2\pi \left[ \frac{3}{2} M_0 \frac{1}{r} - \frac{3}{8} M_0^2 \frac{1}{r^2} - \left( \frac{9}{4} M_2 - \frac{3}{16} M_0^3 \right) \frac{1}{r^3} \right. \]
\[ \left. + \left( \frac{3}{8} M_0 M_2 + \frac{45}{128} M_0^4 \right) \frac{1}{r^4} \right]. \]  

(37)

In the next section we shall specialize eqs. (36) and (37) to some specific solutions.

3 Some examples of Weyl solutions

3.1 Curzon metric

This solution of the Weyl’s family [14] corresponds to a function \( \Psi \) with only the first Weyl moment. Taking all the coefficients \( a_n \) equal to zero, except \( a_0 \), then we have the \( \Delta \phi \) in terms of the unique parameter of this metric

\[ \Delta \phi = 2\pi \left[ -\frac{3}{2} a_0 \frac{1}{r} - \frac{3}{8} a_0^2 \frac{1}{r^2} - \frac{15}{16} a_0^3 \frac{1}{r^3} - \left( \frac{3}{8} a_0 a_2 + \frac{29}{128} a_0^4 \right) \frac{1}{r^4} \right]. \]  

(38)

This expression is exactly the expansion in power series of the inverse radial coordinate of the quantity \( \Delta \phi \), for this solution, with metric functions \( \Psi = \)
\[
\frac{a_0}{r} \text{ and } \Gamma = -\frac{a_0^2}{2r^2}
\]

\[
\Delta \phi = 2\pi \left[ 1 - \frac{1}{r} e^{\frac{a_0^2}{2r^2}} \sqrt{(r + a_0)(r + 2a_0)} \right]
\]  

(39)

### 3.2 Erez-Rosen solution

In prolate spheroidal coordinates, this metric is given by the metric function \(\Psi\) of the form

\[
\Psi = -q_0 Q_0(x) P_0(y) + q_2 Q_2(x) P_2(y),
\]

(40)

with \(q_0 = 1\). The first term corresponds to the Schwarzschild metric, and so, this metric possesses two parameters which represent the mass and the quadrupole moment. The relation between Weyl moments and the coefficients \(q_n\) of the function \(\Psi\) is known \[15\], and therefore it is possible to use expression (35) to obtain an expansion of \(\Delta \phi\). Another way to proceed is to use the expression (37) with the relativistic multipole moments involved, knowing that the mass and quadrupole moment of this metric are respectively

\[
M_0 = \sigma
\]

(41)

\[
M_2 = \frac{2}{15}\sigma^3 q_2
\]

(42)

where \(\sigma\) is the Schwarzschild’s mass. The result is the following

\[
\Delta \phi = 2\pi \left[ -\frac{3}{2} \lambda + \frac{3}{8} \lambda^2 + \left( \frac{3}{10} q_2 - \frac{3}{16} \right) \lambda^3 - \left( \frac{1}{20} q_2 + \frac{45}{128} \right) \lambda^4 
\]

\[
+ \left( \frac{51}{560} q_2 - \frac{69}{256} \right) \lambda^5 + \mathcal{O}(\lambda^6) \right]
\]

(43)

where \(\lambda \equiv \frac{\sigma}{r}\).

\(\Delta \phi\) for a circular geodesic orbit in the equatorial plane expressed in prolate spheroidal coordinates results to be

\[
\Delta \phi = 2\pi \left[ 1 - e^{-\Gamma} \left( \frac{D_x(\Psi) - x}{2D_x(\Psi) - x} \right)^{1/2} \left( 1 + 4 \frac{D_x(\Psi)}{x^2}(D_x(\Psi) - x)^{1/2} \right) \right]
\]

(44)

with \(D_x(\Psi) \equiv (x^2 - 1)\Psi_{,x} \).
So, using (40) and the corresponding function $\Gamma$, it turns out to be

$$\Delta \phi = 2\pi \left[ 1 - e^{-r \frac{\sqrt{2}}{4x}} A^{1/2}(A - 2x)^{1/2} \right]$$  \hspace{1cm} (45)$$

where

$$A \equiv 4 - 3q_2x^3 \ln \left( \frac{x - 1}{x + 1} \right) - 6q_2x^2 - 2x + 3q_2x \ln \left( \frac{x - 1}{x + 1} \right) + 4q_2$$  \hspace{1cm} (46)$$

If we take $q_2 = 0$ then the Schwarzschild expression is obtained for $\Delta \phi$, i.e.,

$$\Delta \phi = 2\pi \left[ 1 - \sqrt{\frac{x - 2}{x + 1}} \right]$$  \hspace{1cm} (47)$$

or using the radial Schwarzschild coordinate $\hat{r} = \sigma(x + 1) = \sigma + \sqrt{r^2 + \sigma^2}$, it takes the well known form

$$\Delta \phi = 2\pi \left[ 1 - \sqrt{1 - \frac{3\sigma}{\hat{r}}} \right]$$  \hspace{1cm} (48)$$

The expression (45) can be written in Weyl coordinates as a power series in radial coordinate. Doing so one obtains the same result that putting the multipole moments (42) into the general expression (37) for $\Delta \phi$.

Another useful expression, for the discussion below, results if one expands (45) in power series of the parameter $q_2$. The order zero of the expansion corresponds to the Schwarzschild term

$$\Delta \phi = 2\pi \left[ 1 - \sqrt{\left( \sqrt{1 + \lambda^2} - \lambda \right) \left( \sqrt{1 + \lambda^2} - 2\lambda \right)} \right].$$ \hspace{1cm} (49)$$

and the order $q_2$ gives the next contribution to $\Delta \phi$, which reads

$$\frac{1}{8} \frac{1}{\sqrt{(x^2 - 1)(1 - x)(2 - x)}} \left[ \ln \left( \frac{x - 1}{x + 1} \right) (9x^4 + 18x^3 + 36x - 45x^2) \\
+ 64 - 84x + \ln \left( \frac{x^2 - 1}{x^2} \right) (24x - 8x^2 - 16) \right]$$  \hspace{1cm} (50)$$
3.3 Monopole-Quadrupole solution

This is an exact solution of the static and axysymmetric Einstein vacuum equations which is written as a series in a parameter $q$ representing the dimensionless quadrupolar moment of the solution. The first term in the expansion (order zero in the parameter $q$) corresponds to Schwarzschild, whereas the whole series describes a solution which only possesses mass and quadrupole moment [2]. The expression for the metric function $\Psi$ is the following

$$\Psi = \sum_{\alpha=0}^{\infty} q^\alpha \Psi_{q^\alpha}$$  \hspace{1cm} (51)$$

$$\Psi_{q^\alpha} = - \sum_{k=0}^{\alpha-1} b_k(\alpha) \left[ \frac{P_k^+}{(x+y)^{k+1}} + (-1)^k \frac{P_k^-}{(x-y)^{k+1}} \right]$$

$$- \sum_{k=0}^{\alpha} q_k(\alpha) Q_{2k}(x) P_{2k}(y) \hspace{1cm} (52)$$

where the $b_k(\alpha)$ and $q_k(\alpha)$ are well defined coefficients [2], and $P^\pm$ are Legendre polynomials with arguments $P_n \left( \frac{xy \pm 1}{x \pm y} \right)$.

Considering the first two terms in the expansion, one obtains a new exact solution with two parameters representing the mass and quadrupole moment. The extent to which this metric describes the field of a compact body very close to an spherical mass is discussed in [2]. Here, we want to calculate $\Delta \phi$ corresponding to this metric and to compare the result with the obtained for the Erez-Rosen metric.

The parameter $q$ gives directly the value of the quadrupole moment and, therefore, we obtain $\Delta \phi$ from (37) with the values $M_0 = \sigma$ and $M_2 = q\sigma^3$. The expression for $\Delta \phi$ in prolate coordinates results to be

$$\Delta \phi = 2\pi \left( 1 - e^{-\Gamma} \frac{\sqrt{2}}{24x^5} B^{1/2}(B - 12x^5)^{1/2} \right) \hspace{1cm} (53)$$

where

$$B \equiv Ax^4 + 2x^2(10x^2 - 5x^3 - 15q) \hspace{1cm} (54)$$

and $q_2 = \frac{15}{2} q$.  

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Expanding in power series of $q$, in order to compare with Erez-Rosen, we obtain again for the order zero of the expansion the Schwarzschild term, and the order $q$ gives the next contribution to the $\Delta \phi$, which is

$$
\frac{5}{32} \frac{1}{\sqrt{(x-2)(x+1)x^2(x-1)}} \left[ \ln \left( \frac{x-1}{x+1} \right) (78x^5 + 9x^6 + 192x^3 - 279x^4)
+ 18x^5 + 156x^4 + 436x^2 - 552x^3 + 104 - 168x \right]
$$

(55)

### 3.4 $\gamma$-metric or Zipoy-Vorhees metric

This metric deserves a special attention, since the singular structure of its “Newtonian” potential $\Psi$, is the same as that of the Schwarzschild solution (a line segment). The metric functions defining this solution of Weyl’s family are the following [16], [17], [18]

$$
\Psi = \frac{\gamma}{2} \ln \left( \frac{x-1}{x+1} \right)
$$

(56)

$$
\Gamma = \frac{\gamma^2}{2} \ln \left( \frac{x^2-1}{x^2-y^2} \right)
$$

(57)

As it is known the value $\gamma = 1$ yields the Schwarzschild solution. The two first massive multipole moments of this metric, mass and quadrupole moment, are

$$
M_0 = \gamma \sigma 
$$

(58)

$$
M_2 = \gamma(1-\gamma^2) \frac{\sigma^3}{3}
$$

(59)

From (59) it follows that $\gamma > 1$ ($\gamma < 1$) correspond to oblate (prolate) sources. Putting these values into expression (37), we obtain

$$
\Delta \phi = 2\pi \left[ \frac{3}{2} \gamma \lambda - \frac{3}{8} \gamma^2 \lambda^2 - \frac{3}{4} \gamma(1 - \frac{5}{4} \gamma^2) \lambda^3 + \frac{1}{8} \gamma^2(1 + \frac{29}{16} \gamma^2) \lambda^4 \right]
$$

(60)

Equivalently, this last result for $\Delta \phi$ can be obtained by expanding in power series of $\lambda$ the expression derived from (44) using (57), which is

$$
\Delta \phi = 2\pi \left( 1 - x^2 - 1 \sqrt{\frac{(x-\gamma)(x-2\gamma)}{(x^2-1)^2}} \right)
$$

(61)
or in terms of $\lambda$

$$\Delta \phi = 2\pi \left[ 1 - (\sqrt{1 + \lambda^2})^{\gamma - 1} \sqrt{(\sqrt{1 + \lambda^2} - \gamma \lambda)(\sqrt{1 + \lambda^2} - 2\gamma \lambda)} \right]$$  \hspace{1cm} (62)

this expression can be compared with the more familiar one written in terms of the Schwarzschild radial coordinate $\hat{r}$

$$\Delta \phi = 2\pi \left[ 1 - (1 - \Lambda)^{\gamma - 1} \sqrt{(1 - \Lambda - \gamma \Lambda)(1 - \Lambda - 2\gamma \Lambda)} \right]$$  \hspace{1cm} (63)

where $\Lambda \equiv \frac{\sigma}{\hat{r}}$.

3.5 Two stationary examples

An approximate solution of stationary Einstein vacuum field equations was obtained in [19] by means of an expansion of the Ernst potential in power series of the parameter $\mathcal{J} \equiv i \frac{J}{M^2}$, where $J$ is the total angular momentum and $M$ is the mass. This metric with two parameters was constructed in order to obtain a solution which possesses, up to the considered order in $\mathcal{J}$, mass and angular momentum in its multipolar structure.

Let us now calculate $\Delta \phi$ for this metric and contrast the result with the expression obtained from the Kerr metric. The metric functions of the former (hereafter referred to as MJ) are

$$f = \frac{x - 1}{x + 1} + \frac{2xj^2}{(x + 1)^2} \left[ \frac{43x^2 - 15x^4 - 12}{16x^3} - \frac{15}{32} \frac{(x^4 - 1)}{x^2} \ln \left( \frac{x - 1}{x + 1} \right) \right]$$  \hspace{1cm} (64)

$$W = -2 \frac{y}{x(x + 1)^2}$$  \hspace{1cm} (65)

where $E \equiv f + iW$ is the Ernst potential and $j \equiv \frac{J}{M^2}$. The other metric functions, in the equatorial plane, are

$$\omega_{\phi} = 2\sigma \left[ \left( \frac{1}{x + 1} \right) + \ln \left( \frac{x - 1}{x} \right) \right] + \mathcal{O}(j^3)$$  \hspace{1cm} (66)

$$\Gamma = \frac{1}{2} \ln \left( \frac{x^2 - 1}{x^2} \right) + \mathcal{O}(j^2)$$  \hspace{1cm} (67)
As can be seen from above, the first correction to Schwarzschild appears at order $j$ in $W$, and so, since we want to compare with Kerr, we will calculate $\Delta \phi$ up to order $j$.

Since in both, Kerr and MJ, the original lattice rotates with respect to a compass of inertia, then a gyroscope fixed at radius $R$ in the original lattice precesses with respect to neighboring points, in each case, at a rate given by

$$\Omega_{Kerr} = \frac{a\sigma}{R^3 \left(1 - \frac{2\sigma}{R}\right)} \quad (68)$$

$$\Omega_{MJ} = \frac{j}{\sigma x(x - 1)(x + 1)^2} \quad (69)$$

being $R$ the Boyer-Lindquist radial coordinate. Then, within proper time $\Delta \tau = \sqrt{-g_{00}} \Delta t$ the original lattice changes its orientation with respect to neighboring points by

$$\hat{\Delta} \phi_{Kerr} = -\frac{a\sigma}{R^3 \left(\sqrt{1 - \frac{2\sigma}{R}}\right)} \Delta t \quad (70)$$

$$\hat{\Delta} \phi_{MJ} = -\frac{j}{\sigma x(x - 1)^{1/2}(x + 1)^{3/2}} \Delta t \quad (71)$$

Considering now a gyroscope orbiting in a circular geodesic, then from the condition $\Phi_{,R} = 0$ we obtain for the angular velocity

$$\omega_{Kerr} = \left(a + \sqrt{\frac{R^3}{\sigma}}\right)^{-1} \quad (72)$$

$$\omega_{MJ} = \frac{1}{\sigma(x + 1)^{3/2}} + \frac{j}{x(x + 1)^2} \left[-1 + 2x \ln \left(\frac{x}{x - 1}\right)\right] \quad (73)$$

Making use of the Rindler-Perlick method one can easily obtain the total precession per revolution $\Delta \phi$, with respect to the original lattice, of a gyroscope carried along a circular geodesic in the equatorial plane. For MJ the result is

$$\Delta \phi = 2\pi \left[1 - \sqrt{\frac{x - 2}{x + 1}} + \frac{1}{x(x + 1)(x - 2)^{1/2}j}\right] \quad (74)$$
whereas for Kerr, we recover the well known expression

$$\Delta \phi = 2\pi \left[ 1 - \sqrt{1 - \frac{3\sigma}{R} + 2a\sqrt{\frac{\sigma}{R^3}}} \right]$$

(75)

As we are calculating $\Delta \phi$ for MJ solution up to order $j$ we will handle last expression up to order $a$ (which is equivalent to consider small values of $a$), and so we obtain the Schiff precession term

$$\Delta \phi = 2\pi \left[ 1 - \sqrt{1 - \frac{3\sigma}{R} - \frac{a}{R}\sqrt{\frac{\sigma}{R}}\left(1 - \frac{3\sigma}{R}\right)^{-1/2}} \right]$$

(76)

And the expression (74) for MJ turns out to be in Boyer-Lindquist coordinates

$$\Delta \phi = 2\pi \left[ 1 - \sqrt{1 - \frac{3\sigma}{R}} + \frac{\sigma a}{R\sqrt{R}}\sqrt{\frac{\sigma}{R}}\left(1 - \frac{\sigma}{R}\right)\left(1 - \frac{3\sigma}{R}\right)^{1/2} \right]$$

(77)

(where the definition of $j$ and the fact that $a$ is the angular momentum per mass unit, have been used). As can be seen from expressions (76) and (77) the contribution to $\Delta \phi$ from the angular momentum for MJ is less than the Schiff precession term for Kerr, since $\frac{\sigma}{R}$ is eventually small.

If that is so, we can expand both expressions in power series of $\frac{\sigma}{R}$ and keeping only the order $O\left(\frac{\sigma}{R}\right)^{3/2}$, to obtain

$$\Delta \phi \approx \frac{3\pi \sigma}{R} - 2\pi \frac{a}{R} \sqrt{\frac{\sigma}{R}} - 3\pi \frac{a\sigma}{R^2} \sqrt{\frac{\sigma}{R}}$$

(78)

for the Kerr metric. And for MJ,

$$\Delta \phi \approx \frac{3\pi \sigma}{R} + 2\pi \frac{a\sigma}{R^2} \sqrt{\frac{\sigma}{R}}$$

(79)

Observe that $\Delta \phi$ as given by (75) and (74) represents the precession of the gyroscope with respect to the neighboring points of the original lattice and not with respect to a compass of inertia. If the precession of the orbiting
gyroscope is wanted with respect to a fixed gyroscope, then (75) and (74) have to be reduced by the values (71, 70) respectively.

For a coordinate time $\Delta t = 2\pi/\omega$, the $\Delta \phi$ of the original lattice itself results from (71) and (73), to be

$$\hat{\Delta} \phi_{Kerr} = -2\pi \frac{a}{R} \sqrt{\frac{\sigma}{R}} \left(1 + \frac{\sigma}{R}\right)$$

(80)

$$\hat{\Delta} \phi_{MJ} = -2\pi \frac{a}{R} \sqrt{\frac{\sigma}{R}} \left(1 + 2\frac{\sigma}{R}\right)$$

(81)

As can be seen, the order of magnitud of these quantities is the same, and so, one observes from (78) and (79) that, whether one compares the precession with respect to a fixed point or to neighboring points of the original lattice, $\Delta \phi$ is larger for Kerr than for MJ solution. As it follows from (78) and (79) the ratio of nonspherical contributions to $\Delta \phi$, of both metrics, is of the order $\frac{\sigma}{R}$. More important is the fact that they have different signs.

4 Conclusions

We have established a relationship between the total precession per revolution of a gyroscope circumventing the source and the relativistic multipole moments describing the space-time. This result allows for “measuring” (in principle) the multipole moments (at least the first ones) of a given source. Indeed, by displaying an array of gyroscopes along the radial coordinate, circumventing the source, we obtain the curve $\Delta \phi = \Delta \phi(r)$, which leads to the values of the coefficients in (37) by adjusting parameters.

Secondly, our results open the possibility to compare different axysymmetric solutions in terms of an “observable” quantity ($\Delta \phi$), and thereby to decide what space-time is actually “in place”.

Thus for example, considering a neutron star (N-S) as a non-rotating source of the Erez-Rosen metric, we can use expressions (50) and (55) to evaluate the first contribution (and dominant, since $q \sim 1.8 \times 10^{-4}$, assuming for the N-S the same eccentricity of the sun ) to the $\Delta \phi$ due to the quadrupole moment. Typical values for a N-S are a mass like the sun and a radius of $10^4 m$. So, we obtain for the contribution of the quadrupole moment $\Delta \phi \sim 9 \times 10^{-3} q \times 2\pi$ for Erez-Rosen, and $\Delta \phi \sim -7 \times 10^{-3} q \times 2\pi$ for MQ solution, i.e, they are of the same order but different sign.
In the case of the earth, the large value of $q \sim -3 \times 10^{15}$ renders the expansion in power series of $q$ useless. Instead of that we may expand in power series of $\lambda$ which is now very small. We obtain in this case that the contribution to $\Delta \phi$ of the quadrupole moment is $\sim -2.5 \times 10^{-12}$ for both metrics. If we consider a N-S as a non-rotating source of the $\gamma$-metric, then, with the values given above and being $\gamma = 0.9997$, the $\Delta \phi$ is $\sim \frac{\pi}{2}$.

The different behaviour of Weyl metrics with respect to the $\Delta \phi$, is also applicable to stationary metrics. In the two examples examined above we see how differently, both contributions from the rotation of the source, are.

The relevance of the conclusions above becomes intelligible if we recall the absence of a Birkhoff-type theorem for axysymmetric solutions to Einstein equations.

Finally we would like to make some additional comments on the $\gamma$-metric. Because of (59) it results that

$$\gamma = \frac{1}{\sqrt{1 + 3q}}$$

and therefore any possible source for the $\gamma$-metric should satisfy the constraint $q > -1/3$. Thus for example, neither the earth nor the sun (considered as non-rotating axysymmetric bodies) could serve as sources of the $\gamma$-metric, since the corresponding values of $q$ are, approximately, $-3 \times 10^{15}$ for the Earth and $-9 \times 10^5$ for the Sun [20].

References


