The Wigner functions on the one dimensional lattice are studied. Contrary to the previous claim in literature, Wigner functions exist on the lattice with any number of sites, whether it is even or odd. There are infinitely many solutions satisfying the conditions which reasonable Wigner functions should respect. After presenting a heuristic method to obtain Wigner functions, we give the general form of the solutions. Quantum mechanical expectation values in terms of Wigner functions are also discussed.

PACS: 01.55.+b; 03.65.-w; 03.65.Sq

I. INTRODUCTION

The Wigner function, the Wigner transform of the density matrix, has been recognized as a bridge between quantum and classical mechanics [1]. It is the quantum mechanical counterpart to the distribution function in the classical phase space: the Wigner function \( W(q,p) \) gives the position distribution if integrated over the momentum \( p \) and the momentum distribution if integrated over the coordinate \( q \). Though the Wigner function may be negative, the equations of motion for \( W(q,p) \), in the classical limit, are reduced to the classical Liouville equation for the distribution function. The Wigner functions have been used to analyze various phenomena where the quantum and classical correspondence is crucial [2] [3].

Some time ago Cohendet et. al. studied the Wigner function for a spin system [4] in order to use it in the Nelson’s quantization program [5]. For an integer spin \( s \) in a pure state, they found a possible Wigner function is given by

\[
W(a,b) = \frac{1}{2s+1} \sum_{h=-s}^{s} \exp\left( i \frac{4\pi}{2s+1} hb \right) \psi^*(a+h)\psi(a-h), \quad (a,b = -s,-s+1,\ldots,s),
\]

where \( \psi(\alpha) \) is the amplitude for each spin state labeled by an integer \( \alpha = -s,-s+1,\ldots,s \) and \( \psi(\alpha+2s+1) = \psi(\alpha) \) is assumed. Their results, however, can not be directly applied to half-integer spin systems [4]. For convenience, we relabel the spin state \( \alpha \) by an integer value \( a = s+\alpha \) with \( a \) ranging from 0 to \( 2s \equiv N-1 \) (the total number of sites is \( N \equiv 2s+1 \)).

The equivalence between these two labeling will be discussed in Sec. 5 in detail. The Wigner function given by Cohendet et. al. is only for the lattice with odd number sites. Thus the construction of Wigner functions on a discrete space or lattice has still been an open problem.

Labeling each site by an integer \( 0,1,\ldots,N-1 \), the Wigner function of Cohendet et. al. is given by

\[
W(a,b) = \frac{1}{N} \sum_{h=0}^{N-1} \exp\left( i \frac{4\pi}{N} hb \right) \phi^*(a+h)\phi(a-h), \quad (a,b = 0,1,\ldots,N-1),
\]

where \( \phi(a+N) = \phi(a) \) is assumed. Generally Wigner functions are written in terms of the Fano operators (matrices) \( \Delta(a,b) \) [6] as

\[
W(a,b) = \frac{1}{N} \sum_{a_1,a_2=0}^{N-1} \phi^*(a_1)\Delta_{a_1a_2}(a,b)\phi(a_2), \quad (a,b = 0,1,\ldots,N-1),
\]

or in terms of the density matrix \( \rho \) as

\[
W(a,b) = \frac{1}{N} \text{Tr}\rho\Delta(a,b).
\]

In this paper we postulate the following four conditions (A)~(D) for eligible Wigner functions and corresponding Fano operators.
(A) The Wigner functions reproduce the distribution in the configuration space:

\[
\sum_{b=0}^{N-1} W(a, b) = |\phi(a)|^2,
\]

or equivalently

\[
\sum_{b=0}^{N-1} \Delta_{a_1 a_2}(a, b) = N \delta_{a,a_1} \delta_{a_1, a_2}.
\]

(B) The Wigner functions reproduce the distribution in the momentum space:

\[
\sum_{a=0}^{N-1} W(a, b) = |\tilde{\phi}(b)|^2,
\]

where the Fourier transform \(\tilde{\phi}\) is given by

\[
\tilde{\phi}(b) = \frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} \exp \left(-\frac{2\pi}{N} ba\right) \phi(a), \quad (b = 0, 1, \ldots, N - 1),
\]

or equivalently

\[
\sum_{a=0}^{N-1} \Delta_{a_1 a_2}(a, b) = \exp \left(i \frac{2\pi}{N} b(a_1 - a_2)\right),
\]

(C) The Wigner functions are real. Therefore the Fano operators are hermitian:

\[W(a, b)^* = W(a, b), \text{ or } \Delta(a, b)^+ = \Delta(a, b).\]

(D) The Fano operators are orthonormalized and complete:

\[
\text{Tr} (\Delta^+(a, b) \Delta(a', b')) = N \delta_{aa'} \delta_{bb'}.
\]

The completeness in the last condition (D) is necessary for that the density matrix can be expressed in terms of the Wigner functions. The completeness, together with the orthonormality, enables us to obtain the simple form:

\[
\rho_{a_1, a_2} = \sum_{a, b=0}^{N-1} W(a, b) \Delta_{a_1 a_2}(a, b).
\]

We note that Eqs.(4) and (12) are valid for a mixed state as well. It is easy to see that these four conditions (A)~(D) are satisfied by the Wigner function by Cohendet et. al.

In this paper we will show that the Wigner functions with the properties (A)~(D) exist for the lattice with any number of sites, and therefore any spin system. In fact there are infinitely many solutions and we will give the general form. In Sec. 2 we will first give a heuristic way to obtain a solution for even site systems. The general form for an arbitrary number of sites will be given in Sec. 3. In Sec. 4 we will discuss expectation values of physical quantities in terms of Wigner functions and the problem on the two ways of site labeling.
In this section, we describe a heuristic method to construct a Wigner function on an even site lattice and show its existence. Let $N$ be an even integer and $\mathbb{Z}_N$ be the cyclic group whose elements are labeled from 0 to $N-1$. We construct the $N \times N$ Fano operators by a reduction method starting from $2N \times 2N$ matrices. The enlarged matrices have a similar form to a cyclic representation of the Weyl matrices on an odd site lattice.

We define phase and shift operators $P$ and $S$ as

$$P_{a_1a_2} = \omega'^a \delta_{a_1a_2}, \quad S_{a_1a_2} = \delta_{a_1,a_2-1},$$

and two operators $T$ and $B$ as

$$T_{a_1a_2} = \delta_{a_1,2N-1-a_2}, \quad B_{a_1a_2} = \delta_{a_1,a_2+(-1)^{a_2}},$$

which are $2N \times 2N$ matrices, where $a_1, a_2 \in \mathbb{Z}_{2N}$ and $\omega'$ is the primitive $2N$-th root of unity:

$$\omega' = \exp \left( \frac{2\pi i}{2N} \right).$$

These operators have the following properties:

$$P^+ = P^{-1} = P^*, \quad S^+ = S^{-1} = S^t, \quad T^+ = T^{-1} = T^*,$$

$$PS = \omega'^{-1}SP, \quad TS = S^{-1}T, \quad BT = TB.$$  \hfill (16)

We consider a set of matrices $\overline{W}(a,b)$ constructed from these matrices:

$$\overline{W}(a,b) = \omega'^{-ab} P^b S^{-(2a+1)},$$

$$\overline{W}(a+N,b) = \omega'^{-a-N} P^b S^{-(2a+1)} B,$$  \hfill (17)

where $a \in \mathbb{Z}_N, b \in \mathbb{Z}_{2N}$. They are orthogonal in the trace norm and complete in the $2N \times 2N$-dimensional operator space, but not hermitian. We define matrices $\overline{\Delta}(a,b)$ by

$$\overline{\Delta}(a,b) = \overline{W}(a,b)T,$$  \hfill (18)

where $a, b \in \mathbb{Z}_{2N}$. For these operators $\overline{\Delta}(a,b)$, it can be shown that the condition (D) in the previous section is fulfilled and the following properties (A'), (B') hold.

(A') A property which is similar to (A) in the previous section:

$$\sum_{b=0}^{2N-1} \overline{\Delta}_{a_1a_2}(a,b) = N \delta_{a_1a_2},$$

$$\sum_{b=0}^{2N-1} \overline{\Delta}_{a_1a_2}(a+N,b) = N \delta_{a_1a_2},$$  \hfill (19)

where $a \in \mathbb{Z}_N$.

(B') A property which is similar to (B) in the previous section:

$$\sum_{a=0}^{2N-1} \overline{\Delta}_{a_1a_2}(a,b) = \exp \left( \frac{2\pi i}{N} b(a_1 - a_2) \right), \quad \text{for even } b$$

$$\sum_{a=0}^{2N-1} \overline{\Delta}_{a_1a_2}(a,b) = 0, \quad \text{for odd } b$$  \hfill (20)

where $b \in \mathbb{Z}_{2N}$.

Note that the Weyl matrices on an odd lattice, i.e. for odd $N$, can be represented as
The hermiticity is recovered after the reduction. It can be seen that the next two identities hold for the ∆ matrix.

\[ \langle \phi | = \{ \phi \in \mathbb{C}^{2N} : B\phi = \phi \}, \]

which means \( \phi(2a) = \phi(2a + 1), a \in \mathbb{Z}_N \) if \( \phi \in V \). The matrices \( \Delta(a, b) \) take \( N \times N \)-dimensional form on this restricted space, e.g. choosing \( \phi(2a), a \in \mathbb{Z}_N \) as independent ones, and we denote the reduced matrices by \( \Delta'(a, b) \).

\[ \Delta'(a, b) = (-1)^{\phi(a + N, b)}, \]

\[ \langle \Delta'(2a, b), a \in \mathbb{Z}_N, b \in \mathbb{Z}_{2N} \rangle = \langle \Delta'(2a + 1, b), a \in \mathbb{Z}_N, b \in \mathbb{Z}_{2N} \rangle, \]

where \( \langle \rangle \) means the vector space spanned by the elements between the bra and ket. Only \( N^2 \) matrices out of \( 4N^2 \) matrices of \( \Delta' \) are independent. We define Fano operators as

\[ \Delta(a, b) = \frac{1}{2} \{ \Delta'(2a, 2b) + \Delta'(2a, 2b + 1) \}, \]

where \( a, b \in \mathbb{Z}_N \). We can see that these operators satisfy the properties (A)~(D). The explicit form of the corresponding Wigner functions is given by

\[
\begin{align*}
W(a, b) &= \frac{1}{2N} \sum_{h=0}^{N-1} (\omega^{2hb} + \omega^{2h(2b+1)})\phi^*(a + h)\varphi(a - h) \\
&+ \frac{1}{2N} \sum_{h=0}^{N-1} (\omega^{2h+1})^{2b} + \omega^{2h+1(2b+1)})\phi^*(a + h)\varphi(a - h - 1).
\end{align*}
\]

III. GENERAL FORM OF WIGNER FUNCTIONS ON A LATTICE

In this section, we will give the general form for Wigner functions on the lattice with arbitrary lattice size \( N \). As in the previous sections, we assume that the Wigner function should satisfy the four conditions (A)~(D) given in Sec. 1.

It turns out that the Fourier transforms of the Fano operators are much convenient.

\[
\Delta(n, m) = \frac{1}{N} \sum_{a,b=0}^{N-1} e^{i\frac{2\pi}{N}(na+mb)}\Delta(a, b), \quad (n, m = 0, 1, \ldots, N - 1),
\]

\[
\Delta(a, b) = \frac{1}{N} \sum_{n,m=0}^{N-1} e^{-i\frac{2\pi}{N}(na+mb)}\Delta(n, m),
\]

where the site suffixes are suppressed. In terms of these Fourier transforms \( \Delta \), the four conditions take simple forms:

\[
\begin{align*}
\hat{\Delta}_{a_1,a_2}(n, 0) &= e^{i\frac{2\pi}{N}n}a_1\delta_{a_1,a_2}, \\
\hat{\Delta}_{a_1,a_2}(0, m) &= \delta_{-a_1+a_2,m}, \\
\Delta^+(n, m) &= \Delta(N - n, N - m), \\
\text{Tr} \left( \Delta(n, m)^\dagger \Delta(n', m') \right) &= N\delta_{n,m}\delta_{m,m'}.
\end{align*}
\]

We will see there exist infinitely many solutions to the above non-linear equations Eqs.(28). We first show that general solutions can be written in terms of any special solution and an arbitrary \((N-1)^2 \times (N-1)^2\) orthogonal matrix. Construction for special solutions will be given later.
From Eqs.(28) we first notice that \( \tilde{\Delta}(n, 0) \) and \( \tilde{\Delta}(0, m) \) are unique. Denoting a special solution by \( \tilde{\Delta}_0 \) and an arbitrary solution by \( \Delta \), we have
\[
\begin{align*}
\tilde{\Delta}_{a_1 a_2}(n, 0) &= \tilde{\Delta}_{0 a_1 a_2}(n, 0) = e^{i\pi a_2/2} \delta_{a_1, a_2}, \\
\tilde{\Delta}_{a_1 a_2}(0, m) &= \tilde{\Delta}_{0 a_1 a_2}(0, m) = \delta_{-s+1, m}.
\end{align*}
\] (29)

These relations and the orthonormal completeness of \( \tilde{\Delta} \) imply, for the remaining suffixes \( n, m, n', m' = 1, 2, \ldots, N - 1 \), each \( \tilde{\Delta}(n, m) \) is expressed by a linear combination of \( \tilde{\Delta}_0(n', m') \)'s. Thus we have
\[
\tilde{\Delta}(n, m) = \sum_{n', m' = 1}^{N-1} \mathcal{U}_{n,m';n',m'} \tilde{\Delta}_0(n', m'), \quad (n, m, n', m' = 1, 2, \ldots, N - 1).
\] (30)

It can be easily shown that \( \tilde{\Delta} \) is also a solution to Eqs.(28), if and only if the complex \((N - 1)^2 \times (N - 1)^2\) matrix \( \mathcal{U}_{nm;n'm'} \) satisfies the conditions:
\[
\mathcal{U}_{nm;n'm'}^* = \mathcal{U}_{N-n,N-m;N-n',N-m'}, \quad \mathcal{U}^+ = \mathcal{U}^{-1}.
\] (31)

These unfamiliar conditions Eqs.(31) for the matrix \( \mathcal{U} \) can be expressed in a more familiar form by use of an \((N - 1)^2 \times (N - 1)^2\) orthogonal matrix. We introduce an \((N - 1)^2 \times (N - 1)^2\) matrix \( \mathcal{T} \) as
\[
\mathcal{T}_{nm;n'm'} = \delta_{n,N-n'} \delta_{m,N-m'}, \quad (n, m, n', m' = 1, 2, \ldots, N - 1)
\] (32)
with the following properties:
\[
\mathcal{T} = \mathcal{T}^* = \mathcal{T}^+ = \mathcal{T}^{-1}.
\] (33)

The conditions Eqs.(31) take compact forms as
\[
\mathcal{U}^* = \mathcal{T} \mathcal{U} \mathcal{T}, \quad \mathcal{U}^+ = \mathcal{U}^{-1}.
\] (34)

And using the unitary matrix \((1 + i\mathcal{T})/\sqrt{2}\), we write
\[
\mathcal{U} = \frac{1 + i\mathcal{T}}{\sqrt{2}} \mathcal{R} \frac{1 - i\mathcal{T}}{\sqrt{2}},
\] (35)
where \( \mathcal{R} \) is an \((N - 1)^2 \times (N - 1)^2\) matrix. Now it is not difficult to show that the matrix \( \mathcal{U} \) satisfies the conditions Eqs.(34), if and only if the matrix \( \mathcal{R} \) is real and orthogonal.

In conclusion, as for the Fourier transforms of the Fano operators with the four properties \((\text{A}) \sim (\text{D})\), \( \tilde{\Delta}(n, 0) \) and \( \tilde{\Delta}(0, m) \) are uniquely given as in Eqs.(29) and the remaining other ones are given in terms of a special solution \( \tilde{\Delta}_0 \) and an \((N - 1)^2 \times (N - 1)^2\) real orthogonal matrix \( \mathcal{R} \) as follows:
\[
\begin{align*}
\tilde{\Delta}(n, m) &= \sum_{n', m' = 1}^{N-1} \mathcal{U}_{nm;n'm'} \tilde{\Delta}_0(n', m'), \quad (n, m, n', m' = 1, 2, \ldots, N - 1), \\
\mathcal{U} &= \frac{1 + i\mathcal{T}}{\sqrt{2}} \mathcal{R} \frac{1 - i\mathcal{T}}{\sqrt{2}},
\end{align*}
\] (36)
where
\[
\begin{align*}
\mathcal{R}^T &= \mathcal{R}^{-1}, \\
\mathcal{T}_{nm;n'm'} &= \delta_{n,N-n'} \delta_{m,N-m'}, \quad (n, m, n', m' = 1, 2, \ldots, N - 1).
\end{align*}
\] (37)

Now we will construct special solutions to Eqs.(28) and discuss how they are related to solutions given before: the solution for odd \( N \) given by Cohendet et. al. and the solution for even \( N \) obtained by the reduction method in this paper.

As we have shown in Eqs.(28), \( \tilde{\Delta}(n, 0) \) and \( \tilde{\Delta}(0, m) \) are unique. They are most conveniently expressed as
where $P$ is the phase operator and $S$ is the shift operator, which were introduced in the previous section:

$$P_{a_1a_2} = e^{i\frac{2\pi}{N}a_1\delta_{a_1,a_2}},$$
$$S_{a_1a_2} = \delta_{a_1,a_2-1}.$$  

(39)

We can show that the set of operators $P^nS^m(n,m = 0,1,\ldots,N-1)$ forms an orthonormal complete set as

$$\text{Tr}(P^nS^m)(P^{n'}S^{m'}) = N\delta_{n'n}\delta_{m'm}.  \tag{40}$$

with the property under hermitian conjugation:

$$(P^nS^m)^+ = e^{i\frac{2\pi}{N}nm}p^{N-n}S^{N-m}.  \tag{41}$$

The relations Eqs.(39), (40) and (41) imply that a special solution may be given by the form:

$$\Delta_0(n,m) = e^{i\theta(n,m)}P^nS^m,  \tag{42}$$

if the phase $\theta(n,m)$ can be properly chosen so that the condition $\Delta_0(n,m) = \Delta(N - n, N - m)$ holds. The condition for the phase $\theta$ turns out to be

$$\theta(n,m) = -\theta(N - n, N - m) + \frac{2\pi}{N}nm, \quad (\text{mod } 2\pi).  \tag{43}$$

We will give some examples. The solution for odd $N$ given by Cohendet et. al. is reproduced by the choice:

$$\theta(n,m) = \begin{cases} \frac{\pi}{2}nm & (m = \text{even}) \\ \frac{\pi}{2}(N - n)(N - m) & (m = \text{odd}) \end{cases}.  \tag{44}$$

After the Fourier inverse transformation, this gives the Fano operator for odd $N$ as

$$\Delta_{a_1a_2}(a,b) = e^{i\frac{2\pi}{N}(a_1a_2)}\delta_{2a,a_1+a_2},  \tag{45}$$

and the Wigner function given in Eq.(2).

The solution by the reduction method for even $N$ in Sec. 2 corresponds to the choice:

$$\theta(n,m) = \begin{cases} \frac{\pi}{N}nm & (n = \text{even, } m = \text{even}) \\ \frac{\pi}{N}(nm + n) & (n = \text{even, } m = \text{odd}) \\ \frac{\pi}{N}(nm - m) & (n = \text{odd}, m = \text{even}) \\ \frac{\pi}{N}(nm + n - m) & (n = \text{odd}, m = \text{odd}) \end{cases},  \tag{46}$$

which gives the Wigner function given by Eq.(26).

**IV. DISCUSSIONS**

We have postulated the four conditions (A)~(D) on the Wigner functions on the lattice. After presenting a heuristic reduction method to obtain a special solution on an even site lattice, we found the general form of the solutions for any size $N$, which is parameterized by an arbitrary orthogonal matrix.

Now we will discuss how the quantum mechanical expectation value of a physical quantity is related to the Wigner functions. Let us introduce the position, site, operator $\hat{q}$ and the momentum operator $\hat{p}$ on the lattice as follows:

$$\hat{q}_{a_1a_2} = a_1\delta_{a_1,a_2},$$
$$\hat{p}_{a_1a_2} = \frac{1}{N} \sum_{b=0}^{N-1} b e^{i\frac{2\pi}{N}b(a_1-a_2)}.  \tag{47}$$

From the conditions (A) and (B), it is evident that the expectation value of the physical quantity like $\hat{q}^n$ or $\hat{p}^m$ are expressed as
\[ \text{Tr} \rho \hat{q}^n = \sum_{a,b=0}^{N-1} a^n \mathcal{W}(a,b), \]
\[ \text{Tr} \rho \hat{p}^m = \sum_{a,b=0}^{N-1} b^m \mathcal{W}(a,b). \]  

(48)

Differences in many solutions arise, when we evaluate the expectation value for a more general quantity involving both \( \hat{q} \) and \( \hat{p} \). For a general physical quantity, however, it is more convenient to use the phase and shift operators \( P \) and \( S \) instead of \( \hat{q} \) and \( \hat{p} \):

\[ P = e^{i \frac{2\pi}{N} q}, \]
\[ S = e^{i \frac{2\pi}{N} p}, \]

(49)

whose matrix elements are given in the previous section. In view of the finiteness of the lattice, the use of \( P \) and \( S \) is more natural than \( \hat{q} \) and \( \hat{p} \). Now we calculate the average of the classical quantity \( P^n S^m \) with the Wigner function being the weight, the distribution function, as

\[ \langle P^n S^m \rangle_{\text{Wigner}} = \sum_{a,b=0}^{N-1} \left( e^{i \frac{2\pi}{N} a} \right)^n \left( e^{i \frac{2\pi}{N} b} \right)^m \mathcal{W}(a,b). \]  

(50)

From Eq. (4) and Eqs. (27), we find that the right hand of this equation is nothing but the expectation value of the Fourier transform of the Fano operator, and thus we have

\[ \langle P^n S^m \rangle_{\text{Wigner}} = \text{Tr} \rho \hat{\Delta}(n,m). \]  

(51)

As we have seen in the previous section, there are infinitely many solutions for \( \hat{\Delta} \)'s. If we take the form as in Eq. (42):

\[ \hat{\Delta}(n,m) = e^{i \theta(n,m)} P^n S^m, \]

(52)

then we have

\[ \langle P^n S^m \rangle_{\text{Wigner}} = e^{i \theta(n,m)} \text{Tr} \rho P^n S^m. \]  

(53)

With other solutions for \( \hat{\Delta} \), we would obtain different operators on the right-hand side. Detailed general discussions on this matter, including the continuum limit of the lattice Wigner functions, will be given elsewhere [7].

Next we clarify the equivalence between the two ways of site labeling mentioned in Sec. 1. In the labeling adopted in this paper, each site is specified by an integer \( a = 0, 1, \ldots, N - 1 \) with the amplitude \( \phi(a) \). In another way of labeling, which is more appropriate for a system of spin \( s \) (integer or half-integer), we label each site by \( \alpha = -s, -s + 1, \ldots, s \) with the amplitude \( \psi(\alpha) \). The total number of sites is \( N = 2s + 1 \). The correspondence satisfying \( \psi(\alpha) \sim \phi(a) \) and \( \tilde{\psi}(\beta) \sim \tilde{\phi}(b) \) is, up to a constant phase factor, given by

\[ \psi(\alpha) = e^{-i \frac{2\pi}{N} \alpha s} \phi(a), \quad \alpha = -s + a, \]
\[ \tilde{\psi}(\beta) = e^{i \frac{2\pi}{N} \beta s} \tilde{\phi}(b), \quad \beta = -s + b, \]
\[ (a,b = 0, 1, \ldots, N - 1; \quad \alpha, \beta = -s, -s + 1, \ldots, s), \]  

(54)

provided that the Fourier transforms \( \tilde{\phi} \) and \( \tilde{\psi} \) are most naturally defined as

\[ \tilde{\phi}(b) = \frac{1}{\sqrt{N}} \sum_{a=0}^{N-1} e^{-i \frac{2\pi}{N} ba} \phi(a), \quad (b = 0, 1, \ldots, N), \]
\[ \tilde{\psi}(\beta) = \frac{1}{\sqrt{2s+1}} \sum_{\alpha=-s}^{s} e^{-i \frac{2\pi}{2s+1} \beta \alpha} \psi(\alpha), \quad (\beta = -s, -s + 1, \ldots, s). \]  

(55)

Thus it is clear that the Fano operator \( \Delta^S(\alpha, \beta) \) in the spin labeling is related to our \( \Delta(a,b) \) by the unitary transformation as
\[
\Delta^S_{\alpha_1,\alpha_2}(\alpha,\beta) = e^{-i\frac{2\pi}{N}(\alpha_1+s)s}\Delta_{\alpha_1+s,\alpha_2+s}(\alpha+s,\beta+s)e^{i\frac{2\pi}{N}(\alpha_2+s)s}, \\
(\alpha,\beta,\alpha_1,\alpha_2 = -s,-s+1,\ldots,s). \quad (56)
\]

In fact we can explicitly verify that if \(\Delta(a,b)\) satisfies the four conditions, \(\Delta^S(\alpha,\beta)\) is also a solution to the four conditions in the spin labeling version, and vice versa.

However, it should be noted that the Fourier transformation given in Eqs.(55), especially in the spin labeling, is not unique. An interesting point in the spin labeling is that the Fourier transform Eqs.(55) and its inverse transformation naturally lead to the anti-periodic boundary conditions for a half-integer spin: \(\psi(\alpha+2s+1) = -\psi(\alpha)\) and \(\tilde{\psi}(\beta+2s+1) = -\tilde{\psi}(\beta)\). This reminds us of some flavor of the spin statistics theorem as mentioned in [4]. Discussions on this issue, in conjunction with the reduction method introduced in Sec. 2, will be presented elsewhere [8].

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