String Field Theory and 

Perturbative Dynamics of Noncommutative Field Theory

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Abstract

The perturbative dynamics of noncommutative field theory (NCFT) is discussed from a point view of string field theory. As in the commutative case it is inevitable to introduce a closed string, which may be described as a bound state of two open strings. We point out that the closed string, interacting nontrivially with the open string, plays an essential role in the ultraviolet region. The contribution of the closed string is responsible for the discrepancy between the NCFT and the string field theory. It clarifies the controversial issues associated with the ultraviolet/infrared (UV/IR) behaviour of the perturbative dynamics of the NCFT.
I. INTRODUCTION

As dynamics of an open string attached to a D-brane with a $B$-field is known to be noncommutative [1], the noncommutative field theory (NCFT), becomes a focal point in string theory recently. The open string theory reduces to the NCFT in a certain limit where the massive modes decouple from the string dynamics. This decoupling limit has been useful to discuss various aspects of the NCFT in the framework of string theory, such as the Seiberg-Witten map [2], which relates the noncommutative Yang-Mills theory to the commutative one. The latest development along this avenue is the perturbative dynamics of the NCFT [3–5]. The one loop amplitudes of the NCFT reveal some intriguing properties of the NCFT, namely UV/IR mixing; its infrared behaviour strongly depends upon its ultraviolet behaviour. It is not easy to understand the UV/IR mixing in the conventional renormalization scheme, which implicitly assumes the ultraviolet-infrared decoupling. The UV/IR mixing is considered as an indication that the NCFT captures some novel features of the string theory.

As pointed out in ref. [3], the one loop effective action of the NCFT shows that its Wilsonian action has an undesirable feature: The Wilsonian action with a finite cutoff $\Lambda$ differs significantly from that with $\Lambda \to \infty$. In order to improve it, one may modify the Wilsonian action by introducing a new degree of freedom $\chi$. By analogy with string theory diagrams one may interpret $\chi$ as a closed string degrees of freedom. This proposal has been examined as the corresponding nonplanar one loop amplitude of string theory is calculated. In ref. [5] it has been argued that $\chi$ may be identified with the IR closed string degree of freedom if one maps the open string UV regime to the closed string IR regime by making use of the open/closed string channel duality. However, it does not fully explain the reason for $\chi$ to exist, since energy regimes of two channels do not overlap each other. The evidence that the closed string may play an important role in the perturbative dynamics of the NCFT also can be found in the NCFT at finite temperature [6]. On the contrary Gomis et al. [7] assert that no closed string modes are relevant to the UV/IR mixing of NCFT. Therefore, it remains an open problem to identify precisely the closed string degrees of freedom in the interacting string theory and to examine their role in the NCFT.

The purpose of this paper is to clarify the issue associated with the UV/IR mixing of NCFT in the framework of the interacting string field theory. To this end we briefly discuss construction of string field theory in the presence of the D-brane with a $B$-field. Then we show that the closed string exists as a bound state of two open strings and its interaction with the open string is highly nontrivial. As in the commutative case it is inevitable to include the closed string field in the interacting string theory: The moduli region of the nonplanar one loop string diagram does not cover that of the corresponding field theory diagram [8]. In order to cover the moduli region completely we need to introduce an open-closed string diagram. Thus, contribution of the closed string to the one loop nonplanar amplitude makes discrepancy between the NCFT and the string theory, which may be effectively described by $\chi$. Since the present work does not resort to the open/closed string channel duality, the difficulty found in the previous work is avoided. It is clear that the string field theory serves as a complete framework in which the NCFT is embedded consistently. The relationship between the string theory and the NCFT also has been discussed in refs. [9].
If the D-brane carries a constant $B$-field, the open string variable $\hat{X}$

$$\hat{X}^i(\sigma) = \hat{x}^i + \theta^{ij} \hat{p}_j \left( \sigma - \frac{\pi}{2} \right) + \sqrt{2} \sum_{n=1}^{\infty} \left( \hat{Y}_n \cos n\sigma + \frac{1}{n} \theta^{ij} \hat{K}_{jn} \sin n\sigma \right)$$  \(1\)

is no longer commutative. Here $(\hat{x}, \hat{p})$ and $(\hat{Y}, \hat{K})$ are canonical conjugate pairs. It has an important consequence that one can neither construct a basis for string state with eigenstates of $\hat{X}$, $\{|X\rangle\}$ nor express the field theoretical action in terms of the string functional $\Phi[X] = \langle \Phi | X \rangle$. This difficulty may be overcome if we employ the basis, consisting of the eigenstates of the momentum operators, $\{|P\rangle; \hat{P}|P\rangle = P|P\rangle\}$. In contrast to $\hat{X}$, the momentum operators

$$\hat{P}_i(\sigma) = \hat{p}_i + \sqrt{2} \sum_{n=1}^{\infty} \hat{K}_{in} \cos n\sigma$$  \(2\)

satisfy the usual commutative relation. It suggests us to construct the interacting string field theory on momentum space, where the momentum eigenstate is represented as

$$|P\rangle = \exp \left[ i \int_{-\pi}^{\pi} d\sigma P(\sigma) \cdot \hat{X}(\sigma) \right] |0\rangle, \quad P(\sigma) = \frac{1}{2\pi} \left( p + \sqrt{2} \sum_{n=1}^{\infty} K_n \cos n\sigma \right).$$  \(3\)

With this representation we find that the inner product is given as usual

$$\langle P'|P''\rangle = \prod_{n} \delta (P'_n - P''_n).$$  \(4\)

The kinetic term of the string field action can be obtained by evaluating the Polyakov string path integral over a strip

$$G(P'; P'') = \int_0^\infty ds \langle P'| \exp \left( -is\hat{L}_0 \right) |P''\rangle$$  \(5\)

$$= i \int D[\Phi] \Phi[P'] \Phi[P''] \exp \left[ -i \int D[\Phi] K \Phi \right].$$

The canonical analysis given in refs. [10] helps us to get

$$K = (2\pi\alpha') \frac{1}{2} p_i (G^{-1})^{ij} p_j + (2\pi\alpha') \sum_{n=1}^{\infty} \frac{1}{2} \left\{ K_{in}(G^{-1})^{ij} K_{jn} - \frac{n^2}{(2\pi\alpha')^2} Y_n^i G_{ij} Y_n^j \right\}.$$  \(6\)

Interaction term of the string field action is determined by an overlapping condition between three strings in momentum space. We should note here that the usual overlapping condition in terms of $\hat{X}$ cannot be adopted, since it may not be consistent due to their noncommutativity. Let us consider the process of splitting one open string into two; the string 3 splits into string 1 and string 2. The quantum state of string $I = 1, 2, 3$ is represented as
\[ |\Phi_I\rangle = \int D[P_I] \exp \left( i \int_{-\pi\alpha_I}^{\pi\alpha_I} d\sigma_I P_I(\sigma_I) \cdot \hat{X}_I(\sigma_I) \right) |0\rangle \Phi[P_I], \]  
\[ P_I(\sigma_I) = \frac{1}{2\pi\alpha_I} \left( p_I + \sqrt{2} \sum_{n=1}^{K_I} K_{I,n} \cos \frac{n\sigma_I}{\alpha_I} \right), \]

where \(\alpha_1 + \alpha_2 = \alpha_3\). The interaction term is obtained by taking the inner products between the strings. However, we should treat this splitting process with care. Before taking the inner products we need to rewrite the quantum state of the open string 3 as

\[ |\Phi_3\rangle = \int D[P_3] \exp \left( i \int_{|\sigma_3| < \pi\alpha_1} d\sigma_3 P_3(\sigma_3) \cdot \hat{X}_3 \right) \otimes \exp \left( i \int_{|\sigma_3| \geq \pi\alpha_2} d\sigma_3 P_3(\sigma_3) \cdot \hat{X}_3 \right) \]
\[ \exp \left( \frac{1}{2} \int_{|\sigma_3| < \pi\alpha_1} d\sigma_3 \int_{|\sigma_3| \geq \pi\alpha_2} d\sigma_3' \left[ P_3(\sigma_3) \cdot \hat{X}_3, P_3(\sigma_3') \cdot \hat{X}_3 \right] \right) |0\rangle. \]  

The third factor in the integrand Eq.(8) turns out to be the Moyal phase factor, \(\exp \left( i \frac{\pi}{2} p_{1i} \theta^{ij} p_{2j} \right)\). (The parameter \(\theta\) in the present work differs from that in ref. [2] by a factor of \(\pi\).) Taking the inner product we get the usual overlapping condition in the momentum space,

\[ \begin{cases} |\sigma_1| \leq \pi\alpha_1 : & P_3(\sigma_3) = P_1(\sigma_1) \\ |\sigma_2| \leq \pi\alpha_2 : & P_3(\sigma_3) = P_2(\sigma_2). \end{cases} \]  

Hence, the interaction term is determined as

\[ V = \int D[P_1] D[P_2] D[P_3] \Phi[P_1] \Phi[P_2] \Phi[P_3] \exp \left( i \frac{\pi}{2} p_{1i} \theta^{ij} p_{2j} \right) \]
\[ \prod_{\sigma_1} \delta[P_1(\sigma_1) - P_3(\sigma_3)] \prod_{\sigma_2} \delta[P_2(\sigma_2) - P_3(\sigma_3)]. \]  

We may apply the same procedure also to the Witten’s covariant open string [11]. In the low energy regime the string field theory effectively reduces to the NCFT.

### III. OPEN AND CLOSED STRING THEORY

In the last section we discussed the open string field theory in the presence of the D-brane with a \(B\)-field. If we study the interacting open string theory, we are lead to include the closed string sector. The reason is that the open string theory itself is not unitary; the closed string would appear at higher orders in perturbation and the BRST invariance requires inclusion of the closed string. This argument also applies to the noncommutative case. As often discussed, the closed string degrees of freedom may be suppressed in the decoupling limit. At tree level this is certainly the case. However, at higher loop level a careful examination is required, since the closed string state may be relevant in the ultraviolet regime of the loop amplitudes.

The open string kinetic term Eq.(6) suggests that there exists a closed string with both left and right movers, which respects the metric \(G_{ij}\). Such a closed string may be constructed as a bound state of the two noncommutative open strings.
\[ |P\rangle_{\text{closed}} = \exp \left( i \int_{-\pi}^{\pi} d\sigma_1 P_1(\sigma_1) \cdot \dot{X}_1(\sigma_1) \right) |0\rangle \otimes \exp \left( i \int_{-\pi}^{\pi} d\sigma_2 P_2(\sigma_2) \cdot \dot{X}_2(\sigma_2) \right) |0\rangle \] 

with \( P_1(0) = P_2(\pi) \), \( P_2(0) = P_1(\pi) \). Alternatively given a closed string with momentum \( P(\sigma) = \frac{1}{2\pi} \sum_n P_n e^{-in\sigma} \), we may write momenta of the two open strings which constitute the closed string as

\[
P_1(\sigma) = \begin{cases} P(\sigma) : -\pi < \sigma \leq 0 \\ P(-\sigma) : 0 < \sigma \leq \pi \end{cases} \quad \text{(12a)}
\]

\[
P_2(\sigma) = \begin{cases} P(-\sigma) : -\pi < \sigma \leq 0 \\ P(\sigma) : 0 < \sigma \leq \pi \end{cases} \quad \text{(12b)}
\]

The first equation yields

\[
p_1 = \frac{p}{2} + \frac{i}{2\pi} \sum_n' p_n \left( \frac{1 - (-1)^n}{n} \right), \quad \text{(13a)}
\]

\[
K_{1n} = \frac{1}{\sqrt{2}} (P_n + P_{-n}) - i\frac{\sqrt{2}}{\pi} \sum_m' p_m \left( 1 - (-1)^{n-m} \right) \left( \frac{m}{n^2 - m^2} \right) \quad \text{(13b)}
\]

where in Eq.(13a) the term with \( n = 0 \) is excluded and in Eq.(13b) the terms with \( m = \pm n \). And the second equation yields

\[
p_2 = \frac{p}{2} - \frac{i}{2\pi} \sum_n' p_n \left( \frac{1 - (-1)^n}{n} \right), \quad \text{(14a)}
\]

\[
K_{2n} = \frac{1}{\sqrt{2}} (P_n + P_{-n}) + i\frac{\sqrt{2}}{\pi} \sum_m' p_m \left( 1 - (-1)^{n-m} \right) \left( \frac{m}{n^2 - m^2} \right) \quad \text{(14b)}
\]

As this closed string is formed by two noncommutative open strings we expect that its interaction with the open string may be nontrivial. Since the closed string is described by two open strings, the interaction between the open string and the closed string may be obtained from the three open string interaction. It yields that the open-closed string interaction term acquires an additional phase factor, similar to the Moyal phase

\[
\exp \left( i \pi \frac{\theta^{ij} p_{1i} p_{2j}}{2} \right) = \exp \left[ p_i \theta^{ij} \sum_n' p_{jn} \frac{1}{4n} (1 - (-1)^n) \right]. \quad \text{(15)}
\]

Here we note that higher modes of the closed string participate in the phase factor in contrast to the open string interaction. This phase factor would play an important role in the open/closed string interaction in the ultraviolet regime.

IV. UV/IR BEHAVIOURS OF STRING FIELD THEORY

In the low energy regime the noncommutative string field theory (NCSFT) action reduces to the that of NCFT at tree level. But at higher orders in perturbation theory one must examine carefully the moduli regions which both theories cover. Here we are mostly concerned with the one loop 2-point amplitudes. A recent study [8] on the one loop amplitude
in open/closed string field theory shows that the moduli region covered by the nonplanar diagram is not complete. It becomes complete only when we take into account the open-closed-open string amplitude in which an open string becomes a closed string, then turning into the open string back. On the other hand the moduli region covered by the corresponding nonplanar diagram of NCFT is complete while the NCFT agrees well with the NCSFT in the (infrared) upper half part of the moduli region. In the (ultraviolet) low half part of the moduli region the open-closed-open string amplitude is relevant. Hence, some discrepancy between the NCFT and the NCSFT is expected.

In the commutative case the 2-point one loop tachyon amplitude is given up to a numerical coefficient as [12]

\[ A_{P/NP} = g^2 \int d\tau dx \left[ \omega^{\frac{1}{24}} f(\omega) \right]^{-(d-2)} (2\pi \alpha')^{\frac{d}{2}} \left( \psi_{P/NP}(\rho, \omega) \right)^{-2}, \tag{16a} \]

\[ \psi_{P/NP}(\rho, \omega) = \frac{1}{\sqrt{\rho}} \exp \left( \frac{\ln^2 \rho}{2 \ln \omega} \right) \prod_{n=1}^{\infty} \frac{(1 \mp \omega^n \rho)(1 \mp \omega^n / \rho)}{(1 - \omega^n)^2} \tag{16b} \]

in the presence of \((n-1)\)-brane. Here \(\omega = e^{-2\pi \tau}, \rho = e^{-4\pi x}\) and \(A_P\) and \(A_{NP}\) denote the amplitudes of planar and the nonplanar diagrams respectively. Note that integration over the zero modes of momenta in the transverse directions are not performed. The open-closed-open string amplitude, \(A_{UU}\) takes the same expression as \(A_{NP}\). But \(A_{NP}\) and \(A_{UU}\) cover different regions of the moduli space.

In the noncommutative case the open string one loop amplitudes \(A_P\) and \(A_{NP}\) can be easily calculated if one observes that only the zero modes of momenta are involved in the Moyal phase and the zero momentum sectors are factorized from the rest. It implies that the additional factor due to the Moyal phase is same as that of NCFT. Thus, \(A_P\) does not receive any correction while \(A_{NP}\) is modified as

\[ A_{NP} = g^2 \int d\tau dx \left[ \omega^{\frac{1}{24}} f(\omega) \right]^{-(d-2)} (2\pi \alpha')^{\frac{d}{2}} \left( \psi_{NP}(\rho, \omega) \right)^{-2} \exp \left( -\frac{p \circ p}{2\pi \alpha' \tau} \right) \tag{17} \]

where \(p \circ p = -\frac{\pi^2}{2} p_i (\theta G \theta)^{ij} p_j\). Evaluation of \(A_{UU}\) in the noncommutative case would be more involved, since all higher modes participate in the phase interaction. However, in the decoupling limit the moduli of \(A_{UU}\) covers mostly the ultraviolet region. Thus, it suffices to calculate the additional factor due to the phase interaction Eq.(15) in the ultraviolet region. In the UV region where \(\tau \to 0\) we may ignore the harmonic potential terms in the first quantized Hamiltonian of the string, so that the propagator for the intermediate closed string may be taken as

\[ \mathcal{K}_{\text{closed}}^{-1} = \left[ (2\pi \alpha')^{\frac{1}{2}} \sum_{n \geq 0} P_{in} (G^{-1})^{ij} P_{j(-n)} \right]^{-1}. \tag{18} \]

It greatly simplifies calculation of the additional factor due to the noncommutative interaction Eq.(15) for \(A_{UU}\)

\[ \frac{\int D'[P] \exp \left( -\mathcal{K}_{\text{closed}} \tau + p_0 \theta^{ij} \sum_n \left( \frac{1}{2n+1} \left( 1 - (-1)^n \right) \right) P_{jn} \right) \exp \left( -\mathcal{K}_{\text{closed}} \tau \right)}{\int D'[P] \exp \left( -\mathcal{K}_{\text{closed}} \tau \right)} = \exp \left( -\frac{p \circ p}{2\pi \alpha' \tau} \right) \tag{19} \]
where the zero mode is excluded in the integration measure $D'[P]$ and we make use of 
$\sum_{n=1} 1/(2n - 1)^2 = \pi^2/8$. Therefore, the nontrivial closed/open string phase interaction at
tree level produces the same additional factor obtained in the one loop nonplanar open string
amplitude. Taking advantage of the result obtained in the commutative case we conclude
that the open-closed-open string amplitude $A_{UU}$ has the same expression as the nonplanar
open string one loop amplitude Eq.(17) in the decoupling limit. Of course, the moduli
regions covered by $A_{NP}$ and $A_{UU}$ differ from each other. Now let us turn our attention to the
relationship between the UV/IR behaviours of the NCFT and the NCSFT. In the decoupling
limit, the moduli region is mostly covered by that of the nonplanar diagram. However, we
may not completely ignore the open-closed-open string diagram. Let us divide the moduli
region into two; region 1 where, $2\pi\alpha'\tau \geq 1/\Lambda^2$ and region 2 where, $0 \leq 2\pi\alpha'\tau < 1/\Lambda^2$.
In the decoupling limit, $A_{NP}$ dominates in the region 1, while $A_{UU}$ in the region 2. So the
contribution from the region 1 may be described by $A_{NP}$ as the one-loop 2-point function
of the NCFT with a cutoff $1/\Lambda^2$ in the decoupling limit. The contribution from the region
2 may well be approximated by $A_{UU}$, the integral over the moduli region 2, where $s = 1/\tau$,
$s/(2\pi\alpha') > \Lambda^2$, which may be written in the decoupling limit as

$$A_{UU} \simeq \int_0^\infty \frac{ds}{s} \frac{d^{26}p}{(2\pi\alpha')^d} \left( \exp \left(-\frac{p \cdot p \cdot s}{2\pi\alpha'}\right) - \exp \left(-\frac{(p \cdot p + 1/\Lambda^2) \cdot s}{2\pi\alpha'}\right) \right).$$

where we take $d = 26$. It can be reproduced as the one-loop amplitude by the following field
theoretical effective action [3] which has been introduced to improve the Wilsonian action
for the NCFT

$$\int L_{eff} = \int d^{26}x \chi \phi + \int d^{26}x \left( \partial(\partial \phi - \partial \chi + \Lambda^2(\partial \phi - \partial \chi)^2) \right),$$

where $\phi$ is the noncommutative scalar field, describing the open string degrees of freedom.
It has been suggested that the extra degrees of freedom, $\chi$ field denotes the closed string
degree of freedom at the low energy regime and the metric respected by the closed string is
$\theta G \theta$. However, the present analysis reveals that the $\chi$ field itself is not simply related to the
closed string field and the closed string does not respect the metric $\theta G \theta$, but respects the
metric $G$ like the open string.

V. CONCLUSIONS

In the present paper we construct an interacting NCSFT and compare its perturbative
dynamics with that of the NCFT. It is inevitable to introduce the closed string field in the
interacting open string theory as in the case of the commutative case and its contributions
to the string amplitudes do not vanish even in the decoupling limit. The closed string is
formed as a bound state of two open strings and its interaction with the open string is highly
nontrivial. As discussed in the literature the open string field theory reduces to the NCFT
in the low energy regime or in the decoupling limit. However, the moduli regions covered
by two theories differ from each other. The moduli of the open string one loop 2-point
nonplanar amplitude only covers part of the moduli region of the corresponding amplitude
in the NCFT and the moduli of the open-closed-open string amplitude $A_{UU}$ covers the rest.
Thus, the contribution of $A_{UU}$ should be taken into account in order to get the complete
string amplitude. In the decoupling limit the contribution of $A_{UU}$ may be approximated by that of the $\chi$ field which has been introduced to improve the Wilsonian action for the NCFT. However, the $\chi$ field itself does not have a simple relationship with the closed string field.

The closed string is found to contribute to the perturbative dynamics of the NCFT in a more intriguing way than we expect. It respects the metric $G$ like the open string, but its interaction with the open string is nontrivial. The string field theory serves as a proper framework where the perturbative dynamics of the NCFT can be discussed consistently. It is expected that the NCSFT also greatly improve our understanding of the renormalizability of the NCFT and have important applications in other interesting subjects of the NCFT.

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