Topics on Strings, Branes and Calabi-Yau Compactifications

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Basics of some topics on perturbative and non-perturbative string theory are reviewed. After a mathematical survey of the Standard Model of particle physics and GUTs, the bosonic string kinematics for the free case and with interaction is described. The effective action of the bosonic string and the spectrum is also discussed. Five perturbative superstring theories and their spectra is briefly outlined. Calabi-Yau three-fold compactifications of heterotic strings and their relation to some four-dimensional physics are given. T-duality in closed and open strings are surveyed. D-brane definition is provided and some of their properties and applications to brane boxes

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configurations, in particular to the cube model are discussed. Finally, non-perturbative issues like S-duality, M-theory, F-theory and basics of their non-perturbative Calabi-Yau compactifications are considered.

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1. Introduction

String theory is by now, beyond the Standard Model (SM) of particle physics, the best and most sensible understanding of all of the basic components of matter and their interactions in an unified scheme. There are well known the ‘aesthetic’ problems arising in the standard model of particles, they include the hierarchy problem, the abundance of free parameters and the apparent arbitrariness of the flavor and gauge groups. The SM is for this reason commonly regarded as the low energy effective description of a more fundamental theory, which solves these problems (for a nice review see [1]). It is also widely recognized that Quantum Mechanics and General Relativity cannot be reconciliated in the context of a perturbative quantum field theory of point particles. Hence the nonrenormalizability of the general relativity can be regarded (similarly to the standard model case) as a genuine evidence that it is just an effective field theory and new physics associated to some fast degrees of freedom should exist at higher energies (for a review, see for instance [2]). String theory propose that these fast degrees of freedom are precisely the strings at the perturbative level and at the non-perturbative level the relevant degrees of freedom are higher-dimensional extended objects called D-branes (dual degrees of freedom).

At the perturbative level String Theory has intriguing generic predictions such as: (i) Spacetime supersymmetry, (ii) General Relativity and (iii) Yang-Mills fields. These subjects interesting by themselves are deeply interconnected in a rich way in string theory.

The study of theories involving D-branes is just in the starting stage and many surprises surely are coming up. Thus we are still at an exploratory stage of the whole structure of the string theory. Therefore the theory is far to be completed and we cannot give yet concrete physical predictions to take contact with collider experiments and/or astrophysical observations. However many aspects of theoretic character, necessary in order to make of string theory a physical theory, are quickly in progress. The purpose of these lectures are to overview the basic ideas to understand these progresses. This paper is an extended version of the lectures presented at the Ninth Mexican School on Particles and Fields held at Metepec Puebla. México. We don’t pretend to be exhaustive and we will limit ourselves to describe the basics of string theory and some particular new developments like Calabi-Yau compactifications of the M and F theories. We apologize for omiting numerous original references and we prefer to cite review articles and some few seminal papers.

We first survey very briefly some basic concepts of the gravitational field and gauge theories pointing out the difficulties to put they together. After that we overview the string
and the superstring theories, including Calabi-Yau compactifications and the relation of strings to physics in four dimensions from the perturbative point of view. T-duality, D-branes and brane boxes configurations is also considered. After that, we devote some time to describe the string dualities and the web of string theories connected by dualities. M and F theories are also briefly described. Finally we overview an approach to non-perturbative Calabi-Yau compactifications of M and F theories.

2. Motivation for Using Strings

First we overview the basic structure of General Relativity (GR) and Yang-Mills (YM) theory in four dimensions. They are very different theories. GR for instance, is the dynamical theory of the spacetime metric while quantum YM theories and in general, Quantum Field Theory (QFT) describes the dynamical building blocks of matter in a fixed spacetime background. Here we survey basics aspects of GR and YM theory following closely Ref. [3].

The pure gravitational field is described by a pseudo-Riemannian metric $g_{\mu\nu}$ with $\mu, \nu = 0, 1, 2, 3$ (on a four-dimensional manifold $M$) satisfying the vacuum Einstein equations, $R_{\mu\nu} = 0$. Einstein equations can be derived from the Einstein-Hilbert action

$$S_{GR} = \frac{1}{16\pi G_N} \int_M d^4x \sqrt{-g} R,$$

where $G_N$ is the Newton’s constant. This constant together with $\hbar$ and $c$, determines the Planck scale where gravitational effects in the quantum theory are relevant. The mass scale termed Planck mass is $M_{Pl} = \sqrt{\frac{\hbar c}{G_N}} = 1.2 \times 10^{-5} \text{grams}$ or equivalently the Planck length $L_{Pl} = \frac{\hbar}{M_{Pl} c} \approx 10^{-33} \text{centimeters}$.

On the other hand the SM of particle physics is described by the gauge field theory which is a quantum field theory provided with the gauge symmetry structure. If one wants to formulate the gauge theory on a pseudo-Riemannian manifold spite of the metric $g_{\mu\nu}$ we require from an additional structure on the spacetime i.e. a connection $A$ on a $G$-principal bundle on $M$: $G \to E \to M$, where $G$ is the SM gauge group, $G = SU_C(3) \times SU_L(2) \times U_Y(1)$. As usual, the gauge field $A_{\mu}(x)$ given by the connection one-form has associated the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + if_{bc}^a A_\mu^a A_\nu^b$ with $f_{bc}^a$ being the structure constants of $G$. Given any representation $\mathcal{R}$ of $G$ one can construct the associated vector bundle $V_{\mathcal{R}}$. 

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The Yang-Mills action is given by

\[ S_{YM} = -\frac{1}{4g_{YM}^2} \int_M g^{\mu\mu'} g^{\nu\nu'} {\cal T}_{\cal R} F_{\mu\nu} F_{\mu'\nu'}, \]  

(2.2)

where \( {\cal T}_{\cal R} \) denotes the trace in the adjoint representation of \( G \).

Now we would like to introduce fermions. The chiral fermions are sections of the the chiral spin bundles \( \hat{S}_\pm \) over spacetime manifold with \( Spin \) structure \( M \), i.e. \( \hat{S} \to M \), where \( \hat{S} = \hat{S}_+ \oplus \hat{S}_- \). The fibers are the Clifford modules constructed with the Dirac matrices \( \Gamma^\mu \). Dirac operator is \( \partial \equiv \Gamma^\mu D_\mu : \Gamma(\hat{S}) \to \Gamma(\hat{S}) \) with \( D_\mu \) being the spacetime covariant derivative. In even dimensions Dirac operator decomposes as: \( \partial = \partial_+ \oplus \partial_- \) where \( \partial_\pm : \Gamma(\hat{S}_\pm) \to \Gamma(\hat{S}_\mp) \) with

\[ \partial_- \psi_+ = 0, \quad \partial_+ \psi_- = 0, \]  

(2.3)

and \( \psi_\pm \in \Gamma(\hat{S}_\pm) \).

The possibility to add a mass term in the above equation implies that that mass should be of order one in mass Planck unities \( M_{Pl} \). But the mass of the low energies particle \( m \) should be much lower than the Planck mass \( M_{Pl} \), i.e. \( m \ll M_{Pl} \). A very nice solution can be given by introducing gauge fields in \textit{complex} representations of the gauge group \( G \). In that case the fermions should be sections of the original spin bundle \( \hat{S} \) coupled to the associated vector bundle \( V_{\cal R} \). If \( \cal R \not\cong \tilde{\cal R} \) then the corresponding bundles are not isomorphic \( V_{\cal R} \not\cong V_{\tilde{\cal R}} \). So we have four possibilities

\[ W_+ = \hat{S}_+ \otimes V_{\cal R}, \quad W_- = \hat{S}_- \otimes V_{\cal R}, \]

\[ \tilde{W}_+ = \hat{S}_+ \otimes V_{\tilde{\cal R}}, \quad \tilde{W}_- = \hat{S}_- \otimes V_{\tilde{\cal R}}. \]

\( CPT \) theorem implies that the fermions with different chirality are given by

\[ \psi_+ \in \Gamma(\hat{S}_+ \otimes V_{\cal R}), \quad \tilde{\psi}_- \in \Gamma(\hat{S}_- \otimes V_{\tilde{\cal R}}). \]

This explains why the mass term are not allowed in the right-hand of Eq. (2.3).

Now define \( \cal R = \bigoplus_{i=1}^{15} \cal R_i \) and \( \tilde{\cal R} = \bigoplus_{i=1}^{15} \tilde{\cal R}_i \) where \( \cal R_i \) and \( \tilde{\cal R}_i \) are irreducible complex representations of the gauge group of the SM. Define the formal difference

\[ \Delta \equiv U \ominus \tilde{U} \]  

(2.4)
between the general complex representation of a particle \( U = U_0 \oplus \mathcal{R} \) and its corresponding complex conjugated \( \tilde{U} = U_0 \oplus \overline{\mathcal{R}} \) with \( U_0 \) being a real irreducible representation (irrep). Thus in the computation of \( \Delta \) only the complex representations are relevant, \( \text{i.e. } \Delta = \mathcal{R} \oplus \overline{\mathcal{R}}. \)

On the other hand, spontaneously symmetry breaking is then the important mechanism to give mass to the fermions and gauge particles in the SM. The possibility to get lower symmetries through the breaking of the gauge group lead us to consider theories with higher symmetries than the SM and recuperate it by symmetry breaking. These are the Grand Unified Theories (GUTs). The extension of the gauge group \( G \) to another of higher dimensionality \( \overline{G} \) was an exciting hope for understanding the ‘aesthetic’ problems of the SM mentioned in the introduction. One of the more successful GUT is the so called SU(5) GUT, where the gauge group \( \overline{G} \) is SU(5) and it breaks to the SM group. Computation of \( \Delta \) for this model consist in taking the formal difference between all irreducible representations of SU(5) and their complex conjugated ones, this gives

\[
\Delta = 3 \left( 5^* \oplus 10 \ominus 5 \oplus 10^* \right),
\]

where \( 5 \) and \( 5^* \) are the fundamental and anti-fundamental representations of SU(5), and \( 10 \) and \( 10^* \) are the antisymmetric part of representation \( 5 \otimes 5 \) of SU(5) and its complex conjugated. The ‘3’ in the front part stands for the mysterious number of generations of quarks and leptons. We will come back later to comment about this mysterious number.

SU(5) is by itself a non-trivial maximal subgroup of SO(10). The GUT with gauge group SO(10) is another candidate for a unified model. The decomposition of irreps of SO(10) in terms of irreps of SU(5) is as follows: the fundamental representation of SO(10) \( 10 \) decomposes under SU(5) irreps as \( 10 = 5 \oplus 5^* \). SO(10) has two complex conjugated spinor representations of 16 dimensions, they are: \( 16 \) and \( 16^* \). They can be decomposed under SU(5) irreps as, \( 16 = 1 \oplus 5^* \oplus 10 \) and \( 16^* = 1 \oplus 5 \oplus 10^* \). Then computation of \( \Delta \) yields

\[
\Delta = 3 \left( 16 \ominus 16^* \right).
\]

Higher dimensionality group \( E_6 \) is the next candidate for a GUT. This group has complex representations which are: \( 27 \) and \( 27^* \). Under SO(10) irreps, these representations decompose into the spinor, vector and identity irreps: \( \text{i.e. } 27 = 16 \oplus 10 \oplus 1 \). Vector representation is real. Thus \( \Delta \) is computed easily to get
Bigger exceptional groups like $E_8$ only has real representations and $\Delta = 0$.

The SM and GUTs are thus unable to answer the arbitrariness of the number of families of lepton and quarks (basically the ‘3’ arising in Eqs. (2.5), (2.6) and (2.7)) as well as the arbitrariness of the gauge group. The hierarchy of lepton and quarks masses, the existence of the Higgs mechanism and the abundance of free parameters are ‘aesthetic problems’ as they don’t contradict any experiment. However its is clear that the explanation of the origin has to come of somewhere beyond SM and GUTs. In the last 15 years we have learned that string theory has the necessary ingredients to solve these potential problems and it is a serious candidate to provide us with a complete unified theory of all known fundamental interactions of nature. In these lectures we attempt to give the very basic notions of some topics of perturbative and non-perturbative string theory.

3. Perturbative String and Superstring Theories

In this section we overview some basic aspects of bosonic and fermionic strings. We focus mainly in the description of the spectrum of the theory in the light-cone gauge, the effective action, the description of spectra of the five consistent superstring theories and the perturbative Calabi-Yau compactifications (for details, precisions and further developments see for instance Refs. [4,5,6,7,8,9]).

First of all consider, as usual, the action of a relativistic point particle. It is given by $S = -m \int d\tau \sqrt{-X^I X_I}$, where $X^I$ are $D$ functions representing the coordinates of the $(D - 1, 1)$-dimensional Minkowski spacetime (the target space), $\dot{X}^I \equiv \frac{dX^I}{d\tau}$ and $m$ can be identified with the mass of the point particle. This action is proportional to the length of the world-line of the relativistic particle.

In analogy with the relativistic point particle, the action describing the dynamics of a string (one-dimensional object) moving in a $(D - 1, 1)$-dimensional Minkowski spacetime (the target space) is proportional to the area $A$ of the worldsheet $\Sigma$. We know from the theory of surfaces that such an area is given by $A = \int \sqrt{det(-g)}$, where $g$ is the induced metric (with signature $(-, +)$) on the worldsheet $\Sigma$. The background metric will be denoted
by $\eta_{IJ}$ and $\sigma^a = (\tau, \sigma)$ with $a = 0, 1$ are the local coordinates on the worldsheet. $\eta_{IJ}$ and $g_{ab}$ are related by $g_{ab} = \eta_{IJ}\partial_a X^I \partial_b X^J$ with $I, J = 0, 1, \ldots, D - 1$. Thus the classical action of a relativistic string is given by the Nambu-Goto action

$$S_{NG}[X^I] = - T \int_{\Sigma} d\tau d\sigma \sqrt{-\text{det}(\partial_a X^I \partial_b X^J \eta_{IJ})}, \quad (3.1)$$

where $T = \frac{1}{2\pi a^'}$ is the string tension, $X^I$ are $D$ embedding functions of the worldsheet $\Sigma$ into the target space $X$. Now introduce a metric $h$ describing the intrinsic worldsheet geometry, we get a classically equivalent action to the Nambu-Goto action. This is the Polyakov action originally proposed by Brink, di Vecchia, Howe and Zumino

$$S_P[X^I, h_{ab}] = - \frac{1}{4\pi a^'} \int_{\Sigma} d^2\sigma \sqrt{-h} h^{ab} \partial_a X^I \partial_b X^J \eta_{IJ}, \quad (3.2)$$

where the $X^I$'s are $D$ scalar fields on the worldsheet. Such a fields can be interpreted as the coordinates of spacetime $X$ (target space), $h = \det(h_{ab})$ and $h_{ab} = \partial_a X^I \partial_b X^J \eta_{IJ}$.

Polyakov action has the following symmetries: (i) Poincaré invariance, (ii) Worldsheet diffeomorphism invariance, and (iii) Weyl invariance (rescaling invariance). The energy-momentum tensor of the two-dimensional theory is given by

$$T_{ab} := \frac{1}{\sqrt{-h}} \frac{\delta S_P}{\delta h_{ab}} = \frac{1}{4\pi a^'} \left( \partial^a X^I \partial^b X_I - \frac{1}{2} h^{ab} h^{cd} \partial_c X^I \partial_d X_I \right). \quad (3.3)$$

Invariance under worldsheet diffeomorphisms implies that it should be conserved i.e. $\nabla_a T^{ab} = 0$, while the Weyl invariance gives the traceless condition, $T_a^a = 0$. The equation of motion associated with Polyakov action is given by

$$\partial_a \left( \sqrt{-h} h^{ab} \partial_b X^I \right) = 0. \quad (3.4)$$

Whose solutions should satisfy the boundary conditions for the open string: $\partial_\sigma X^I|_{\sigma = \pi} = 0$ (Neumann) and for the closed string: $X^I(\tau, \sigma) = X^I(\tau, \sigma + 2\pi)$ (Dirichlet). Here $\ell = \pi$ is the characteristic length of the open string. The variation of $S_P$ with respect to $h_{ab}$ leads to the constraint equations: $T_{ab} = 0$. From now on we will work in the conformal gauge. In this gauge: $h_{ab} = \eta_{ab}$ and equations of motion (3.4) reduce to the Laplace equation in the flat worldsheet whose solutions can be written as linear superposition of plane waves.
The Closed String

For the closed string the boundary condition \( X^I(\tau, \sigma) = X^I(\tau, \sigma + 2\pi) \), leads to the general solution of Eq. (3.4)

\[
X^I(\tau, \sigma) = X^I_0 + \frac{1}{\pi T}P^I\tau + \frac{i}{2\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \left\{ \alpha^I_n \exp \left( -i2n(\tau - \sigma) \right) + \tilde{\alpha}^I_n \exp \left( -i2n(\tau + \sigma) \right) \right\}
\]

(3.5)

where \( X^I_0 \) and \( P^I \) are the position and momentum of the center-of-mass of the string and \( \alpha^I_n \) and \( \tilde{\alpha}^I_n \) satisfy the conditions \( \alpha^I_n^* = \alpha^I_{-n} \) (left-movers) and \( \tilde{\alpha}^I_n^* = \tilde{\alpha}^I_{-n} \) (right-movers).

The Open String

For the open string the corresponding boundary condition is \( \partial_\sigma X^I|_{\ell=\pi} = 0 \) (this is the only boundary condition which is Lorentz invariant) and the solution is given by

\[
X^I(\tau, \sigma) = X^I_0 + \frac{1}{\pi T}P^I\tau + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha^I_n \exp \left( -in\tau \right) \cos(n\sigma)
\]

(3.6)

with the matching condition \( \alpha^I_n = \tilde{\alpha}^I_{-n} \).

Quantization

The quantization of the closed bosonic string can be carried over, as usual, by using the Dirac prescription to the center-of-mass and oscillator variables in the form

\[
[X^I_0, P^J] = i\eta^{IJ},
\]

\[
[\alpha^I_m, \alpha^J_n] = [\tilde{\alpha}^I_m, \tilde{\alpha}^J_n] = m\delta_{m+n,0}\eta^{IJ},
\]

\[
[\alpha^I_m, \tilde{\alpha}^J_n] = 0.
\]

(3.7)

One can identify \((\alpha^I_n, \tilde{\alpha}^I_n)\) with the annihilation operators and the corresponding operators \((\alpha^I_{-n}, \tilde{\alpha}^I_{-n})\) with the creation ones. In order to specify the physical states we first denote the center of mass state given by \(|P^I\rangle\). The vacuum state is defined by \(\alpha^I_m|0, P^I\rangle = 0\) with \(m > 0\) and \(P^I|0, P^I\rangle = p^I | 0, P^I\rangle\) and similar for the right movings (here \(|0, P^I\rangle = |P^I\rangle \otimes |0\rangle\)). For the zero modes these states have negative norm (ghosts). However one can choice a suitable gauge where ghosts decouple from the Hilbert space when \(D = 26\).
Now we turn out to work in the so called light-cone gauge. In this gauge it is possible to solve explicitly the Virasoro constraints: $T_{ab} = 0$. This is done by removing the light-cone coordinates $X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^D)$ leaving only the transverse coordinates $X^i$ representing the physical degrees of freedom (with $i, j = 1, 2, \ldots, D-2$). In this gauge the Virasoro constraints are explicitly solved. Thus the independent variables are $(X_0^-, P^+, X_0^+, P^-, \alpha_n, \tilde{\alpha}_n)$. Operators $\alpha^-_n$ and $\tilde{\alpha}^-_n$ can be written in terms of $\alpha^+_j$ and $\tilde{\alpha}^+_j$ respectively as follows: $\alpha^-_n = \frac{1}{\sqrt{2\alpha^+_n}}(\sum_{m=-\infty}^{\infty} \alpha^+_n \alpha^-_m : -2A\delta_n)$ and $\tilde{\alpha}^-_n = \frac{1}{\sqrt{2\alpha^+_n}}(\sum_{m=-\infty}^{\infty} \tilde{\alpha}^+_n \tilde{\alpha}^-_m : -2A\delta_n)$. For the open string we get $\alpha^-_n = \frac{1}{2\sqrt{2\alpha^+_n}}(\sum_{m=-\infty}^{\infty} \alpha^+_n \alpha^-_m : -2A\delta_n)$. Here $\cdot :$ stands for the normal ordering and $A$ is its associated constant.

In this gauge the Hamiltonian is given by

$$H = \frac{1}{2}(P^i)^2 + N - A \quad \text{(open string),} \quad H = (P^i)^2 + N_L + N_R - 2A \quad \text{(closed string)} \quad (3.8)$$

where $N$ is the operator number, $N_L = \sum_{m=-\infty}^{\infty} : \alpha_- \alpha_m :$, and $N_R = \sum_{m=-\infty}^{\infty} : \tilde{\alpha}::< \alpha_- \tilde{\alpha}_m :$. The mass-shell condition is given by $\alpha'M^2 = (N - A)$ (open string) and $\alpha'M^2 = 2(N_L + N_R - 2A)$ (closed string). For the open string, Lorentz invariance implies that the first excited state is massless and therefore $A = 1$. In the light-cone gauge $A$ takes the form $A = -\frac{D-2}{2} \sum_{n=1}^{\infty} n$. From the fact $\sum_{n=1}^{\infty} n^{-s} = \zeta(s)$, where $\zeta$ is the Riemann’s zeta function (which converges for $s > 1$ and has a unique analytic continuation at $s = -1$, where it takes the value $-\frac{1}{12}$) then $A = -\frac{D-2}{24}$ and therefore $D = 26$.

3.1. Spectrum of the Bosonic String

Closed Strings

The spectrum of the closed string can be obtained from the combination of the left-moving states and the right-moving ones. The ground state ($N_L = N_R = 0$) is given by $\alpha'M^2 = -4$. That means that the ground state includes a tachyon. The first excited state ($N_L = 1 = N_R$) is massless and it is given by $\alpha^i \alpha^j |0, P\rangle$. This state can be naturally decomposed into irreducible representations of the little group $SO(24)$ as follows

$$\alpha^i \alpha^j |0, P\rangle = \alpha^i \alpha^j |0, P\rangle + \left(\alpha^i \alpha^j - \frac{1}{D-2} \delta^{ij} \alpha^k \alpha^k \right) |0, P\rangle$$
\[ + \frac{1}{D-2} \delta^{ij} \alpha_{i-1}^k \tilde{\alpha}_{-1}^k | 0, P \].

The first term of the rhs is interpreted as a spin 2 massless particle \( G_{ij} \) (graviton). The second term is a range 2 anti-symmetric tensor \( B_{ij} \). While the last term is an scalar field \( \Phi \) (dilaton). Higher excited massive states are combinations of irreducible representations of the corresponding little group \( \text{SO}(25) \).

**Open Strings**

For the open string, the ground state includes once again a tachyon since \( \alpha' M^2 = -1 \). The first exited state \( N = 1 \) is given by a massless vector field in 26 dimensions. The second excitation level is given by the massive states \( \alpha^i_{-2} | 0, P \rangle \) and \( \alpha^i_{-1} \alpha^j_{-1} | 0, P \rangle \) which are in irreducible representations of the little group \( \text{SO}(25) \).

**3.2. Interacting Strings and the Effective Action**

**Interacting Strings**

So far we have described the free propagation of a closed (or open) bosonic string. In what follows we consider the interaction of these strings. Here we focus in the closed string case, the open case requires from further definitions. The interaction of strings at the perturbative level is just the extension of the technique of Feynman diagrams for point particles to extended objets. The vacuum-vacuum amplitude \( A \) is given by

\[
A \sim \int D h_{ab} DX^I \exp \left( i S_P [X^I, h_{ab}] \right).
\]  

The interacting case requires to sum over all loop diagrams. In the closed string case it means that we have to sum over all compact orientable surfaces with non-trivial boundary \( (\partial \Sigma \neq 0) \). In two dimensions these surfaces are completely characterized by their number of holes \( g \) (the genus) and boundaries \( b \). The relevant topological invariant is the Euler number \( \chi(\Sigma) = \frac{1}{4\pi} \int_{\Sigma} d^2 \sigma \sqrt{-h} R^{(2)} \), where \( R^{(2)} \) is the scalar curvature of the worldsheet \( \Sigma \). In order to include the interaction of strings, the generalization of the Polyakov action consistent with its symmetries is given by

\[
S = S_P [X^I, h_{ab}] + \frac{\Phi(X^I)}{4\pi} \int_{\Sigma} d^2 \sigma \sqrt{-h} R^{(2)} + \frac{1}{2\pi} \int_{\partial \Sigma} ds K, 
\]
where $\Phi(X)$ is an scalar background field and represents the gravitational coupling constant of the two-dimensional Einstein-Hilbert Lagrangian. $K$ in the above equation stands for the geodesic curvature of $\Sigma$. If we define the string coupling constant by $g_S \equiv e^\Phi$, then Eq. (3.10) for the case of closed strings generalizes to

$$\mathcal{A} \sim \sum_\chi g_S^{\chi(\Sigma)} \int D h_{ab} D X^I \exp \left( i S_P[X^I, h_{ab}] \right).$$

(3.12)

The amplitude defined on-shell correspond to $g = 0$ and the rest ($g \geq 1$) corresponds to $g$-loop corrections.

The definition of correlation functions of operators requires of the idea of vertex operators $\mathcal{W}_\Lambda$. These operators are defined as

$$\mathcal{V}_\Lambda(k) = \int d^2 \sigma \sqrt{-h} \mathcal{W}_\Lambda(\sigma, \tau) \exp (i k \cdot X),$$

(3.13)

where $\mathcal{W}_\Lambda(\sigma, \tau)$ (with $\Lambda$ being a generic massless field of the bosonic spectra) is a local operator assigned to some specific state of the spectrum of the theory. For instance for the tachyon it is given by $\mathcal{W}_T(\sigma, \tau) \sim \partial_a X^I \partial^a X^I$. While that for the graviton $G$ with polarization $\zeta_{IJ}$ it is given by $\mathcal{W}_G(\sigma, \tau) = \zeta_{IJ} \partial_a X^I \partial^a X^J$. Local operators $\mathcal{V}_\Lambda$ are diffeomorphism and conformal invariant and therefore more convenient to define scattering amplitudes.

Thus one can define the scattering amplitude of the vertex field operators by their corresponding invariant operators $\mathcal{V}_\Lambda$. In perturbation theory the scattering amplitude is given by

$$\mathcal{A}(\Lambda_1, k_1; \ldots \Lambda_N, k_N) \sim \sum_\chi g_S^{\chi(\Sigma)} \int D h D X \exp \left( i S_P[X^I, h_{ab}] \right) \prod_{i=1}^N \mathcal{V}_{\Lambda_i}(k_i).$$

(3.14)

This scattering amplitude is, of course, proportional to the correlation function of the product of $N$ invariant operators $\mathcal{V}_{\Lambda_i}(k_i)$ as follows

$$\mathcal{A}(\Lambda_1, k_1; \ldots \Lambda_N, k_N) \propto \left( \prod_{i=1}^N \mathcal{V}_{\Lambda_i}(k_i) \right).$$

(3.15)
Effective String Actions

In order to make contact with the spacetime physics we now describe how the spacetime equations of motion come from conformal invariance conditions for the non-linear sigma model in curved spaces. The immediate generalization of the Polyakov action is

$$S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-h} h^{ab} \partial_a X^I \partial_b X^J G_{IJ}(X),$$

where $G_{IJ}(X)$ is an arbitrary background metric of the curved target space $X$. The perturbation of this metric $G_{IJ}(X) = \eta_{IJ} + h_{IJ}(X)$ in the partition function $Z \sim \exp(-S[X^I, \eta_{IJ} + h_{IJ}])$, leads to an expansion in powers of $h_{IJ}$. This partition function can be easily interpreted as containing the information of the interaction of the string with a coherent state of gravitons with invariant operator

$$V_G(k) = \int d^2\sigma \sqrt{-h} W_G(\sigma, \tau) \exp(ik \cdot X)$$

with $W_G = h^{ab} \partial_a X^I \partial_b X^J h_{IJ}(X)$.

On the other hand, the Polyakov action can be generalized to be consistent with all symmetries and with the massless spectrum of the bosonic closed strings in the form of a non-linear sigma model

$$\hat{S} = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-h} \left( h^{ab} G_{IJ}(X) + i \varepsilon^{ab} B_{IJ}(X) \right) \partial_a X^I \partial_b X^J + \alpha' \Phi(X) R^{(2)},$$

where $G_{IJ}(X)$ is the target space curved metric, $B_{IJ}(X)$ is an anti-symmetric field, also called the \textit{Kalb-Ramond} field, and $\Phi(X)$ is the scalar field called the \textit{dilaton} field. From the viewpoint of the two-dimensional non-linear sigma model these background fields can be regarded as coupling constants and the renormalization group techniques become applied. The computation of the quantum \textit{conformal anomaly} by using the dimensional regularization method, leads to express the energy-momentum trace as a linear combination

$$T^a_a = -\frac{1}{2\alpha'} \beta^G_{IJ} h^{ab} \partial_a X^I \partial_b X^J - \frac{i}{2\alpha'} \beta^B_{IJ} \varepsilon^{ab} \partial_a X^I \partial_b X^J - \frac{1}{2} \beta^\Phi R^{(2)},$$

where $\beta$ are the one-loop beta functions associated with each coupling constant or background field. They are explicitly computed and give

$$\beta^G_{IJ} = \alpha' \left( R_{IJ} + 2 \nabla_I \nabla_J \Phi - \frac{1}{4} H_{IKL} H^{KL}_J \right) + O(\alpha'^2),$$

$$\beta^B_{IJ} = \alpha' \left( - \frac{1}{2} \nabla^K H_{KIJ} + \nabla^K \Phi H^{K}_{IJ} \right) + O(\alpha'^2),$$

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\[
\beta^\Phi = \alpha' \left( \frac{D - 26}{6\alpha'} - \frac{1}{2} \nabla^2 \Phi + \nabla_K \Phi \nabla^K \Phi - \frac{1}{24} H_{IJK} H^{IJK} \right) + O(\alpha'^2), \quad (3.21)
\]

where \( H_{IJK} = \partial_I B_{JK} + \partial_J B_{KI} + \partial_K B_{IJ} \). Weyl invariance at the quantum level implies the vanishing of the conformal anomaly and therefore the vanishing of each beta function. This leads to three coupled field equations for the background fields. These conditions for these fields can been regarded as equations of motion derivable from the spacetime action in \( D \) dimensions

\[
S = \frac{1}{2\kappa^2} \int_X d^D x \sqrt{-G} e^{-2\Phi} \left( R + 4\nabla_I \Phi \nabla^I \Phi - \frac{1}{12} H_{IJK} H^{IJK} - \frac{2(D - 26)}{3\alpha'} + O(\alpha') \right), \quad (3.22)
\]

where \( \kappa_0 \) is a normalization constant.

It is interesting to see that a redefinition of background metric under the transformation in \( D \) dimensions \( \tilde{G}_{IJ}(X) = \exp(2\varpi(X)) G_{IJ} \) with \( \varpi = \frac{2}{D-2}(\Phi_0 - \Phi) \) leads to the background action in the ‘Einstein frame’

\[
\tilde{S} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-\tilde{G}} \left( \tilde{R} - \frac{4}{D - 2} \nabla_I \tilde{\Phi} \nabla^I \tilde{\Phi} - \frac{1}{12} e^{-\Phi/(D-2)} H_{IJK} H^{IJK} \right.
\]

\[
\left. - \frac{2(D - 26)}{3\alpha'} e^{\Phi/(D-2)} + O(\alpha') \right), \quad (3.23)
\]

where \( \tilde{R} = e^{-2\varpi} [R - 2(D - 1)\nabla^2 \varpi - (D - 2)(D - 1) \partial_I \varpi \partial^I \varpi] \) and \( \tilde{\Phi} = \Phi - \Phi_0 \). The form of this action will of extreme importance later when we describe the strong/weak coupling duality in effective supergravity actions of the different superstring theories types. In the above action \( \kappa \equiv \kappa_0 e^{\Phi_0} = \kappa_0 \cdot g_S \) is the gravitational coupling constant in \( D \) dimensions, i.e. \( \kappa = \sqrt{8\pi G_N} \).

A very close procedure can be performed for the open string and compute its effective low energy action. It was done about 15 years ago and for gauge fields with constant field strength \( F_{IJ} \) it is given by the Dirac-Born-Infeld action

\[
S_O = -T \int d^D x e^{-\Phi} \sqrt{-\det(G_{IJ} + B_{IJ} + 2\pi \alpha' F_{IJ})}. \quad (3.24)
\]

Later we shall talk about some applications of this effective action.
3.3. Superstrings

In bosonic string theory there are two bold problems. The first one is the presence of tachyons in the spectrum. The second one is that there are no spacetime fermions. Here is where superstrings come to the rescue. A superstring is described, despite of the usual bosonic fields $X^I$, by fermionic fields $\psi^L_R$ on the worldsheet $\Sigma$. Which satisfy anticommutation rules and where the $L$ and $R$ denote the left and right worldsheet chirality respectively. The action for the superstring is given by

$$L_{SS} = -\frac{1}{8\pi} \int d^2 \sigma \sqrt{-h} \left( h^{ab} \partial_a X^I \partial_b X_I + 2i \bar{\psi}^I \gamma^a \partial_a \psi_I - i \bar{\chi}_a \gamma^b \gamma^a \psi^I \left( \partial_b X_I - \frac{i}{4} \bar{\chi}_b \psi_I \right) \right),$$

where $\psi^I$ and $\chi_a$ are the superpartners of $X^I$ and the tetrad field $e^a$ respectively. In the superconformal gauge ($h_{ab} = \eta_{ab}$ and $\chi_a = 0$) and in light-cone coordinates it can be reduced to

$$L_{SS} = \frac{1}{2\pi} \int \left( \partial_L X^I \partial_R X_I + i \bar{\psi}_R^I \partial_L \psi_I + i \psi_L^I \partial_R \psi_I \right).$$

In analogy to the bosonic case, the local dynamics of the worldsheet metric is manifestly conformal anomaly free at the quantum level if the critical spacetime dimension $D$ is 10. Thus the string oscillates in the 8 transverse dimensions. The action (32) is invariant under: (i) worldsheet supersymmetry, (ii) Weyl transformations, (iii) super-Weyl transformations, (iv) Poincaré transformations and (v) Worldsheet reparametrizations. The equation of motion for the $X'$s fields is the same that in the bosonic case (Laplace equation) and whose general solution is given by Eqs. (3.5) or (3.6). Equation of motion for the fermionic field is the Dirac equation in two dimensions. Constraints here are more involved and they are called the super-Virasoro constraints. However in the light-cone gauge, everything simplifies and the transverse coordinates (eight coordinates) become the bosonic physical degrees of freedom together with their corresponding supersymmetric partners. Analogously to the bosonic case, massless states of the spectrum come into representations of the little group $SO(8)$ which is a subgroup of $SO(9,1)$, while that the massive states lie into representations of the little group $SO(9)$.

For the closed string there are two possibilities for the boundary conditions of fermions: (i) periodic boundary conditions (Ramond (R) sector) $\psi^I_{L,R} (\sigma) = + \psi^I_{L,R} (\sigma + 2\pi)$ and (ii)
anti-periodic boundary conditions (Neveu-Schwarz (NS) sector) \( \psi_{L,R}^I(\sigma) = -\psi_{L,R}^I(\sigma + 2\pi) \).

Solutions of Dirac equation satisfying these boundary conditions are

\[
\psi_L^I(\sigma, \tau) = \sum_n \bar{\psi}_{-n}^I \exp\left(-in(\tau + \sigma)\right), \quad \psi_R^I(\sigma, \tau) = \sum_n \psi_n^I \exp\left(-in(\tau - \sigma)\right), \quad (3.27)
\]

where \( \bar{\psi}_{-n}^I \) and \( \psi_n^I \) are fermionic modes of left and right movers respectively.

In the case of the fermions in the R sector \( n \) is integer and it is semi-integer in the NS sector.

The quantization of the superstring come from the promotion of the fields \( X^I \) and \( \psi^I \) to operators whose oscillator variables are operators satisfying the relations \([\alpha_n^I, \alpha_m^I]_- = n\delta_{m+n,0}\eta^{IJ}\) and \([\psi_n^I, \psi_m^J]_+ = \eta^{IJ}\delta_{m+n,0}\), where \([,]_-\) and \([,]_+\) stand for commutator and anti-commutator respectively.

The zero modes of \( \alpha \) are diagonal in the Fock space and its eigenvalue can be identified with its momentum. For the NS sector there is no fermionic zero modes but they can exist for the R sector and they satisfy a Clifford algebra \([\psi_0^I, \psi_0^J]_+ = \eta^{IJ}\). The Hamiltonian for the closed superstring is given by \( H_{L,R} = N_{L,R} + \frac{1}{2}P^2_{L,R} - A_{L,R} \). For the NS sector \( A = \frac{1}{2} \), while for the R sector \( A = 0 \). The mass is given by \( M^2 = M^2_L + M^2_R \) with \( \frac{1}{2}M^2_{L,R} = N_{L,R} - A_{L,R} \).

There are five consistent superstring theories: Type IIA, IIB, Type I, SO(32) and \( E_8 \times E_8 \) heterotic strings, represented by HO and HE respectively. In what follows of this section we briefly describe the spectrum in each one of them.

**Type II Superstring Theories**

In this case the theory consist of closed strings only. They are theories with \( \mathcal{N} = 2 \) spacetime supersymmetry. There are 8 scalar fields (representing the 8 transverse coordinates to the string) and one Weyl-Majorana spinor. There are 8 left-moving and 8 right-moving fermions.

In the NS sector there is still a tachyon in the ground state. But in the supersymmetric case this problem can be solved through the introduction of the called GSO projection. This projection eliminates the tachyon in the NS sector and it acts in the R sector as a ten-dimensional spacetime chirality operator. That means that the application of the GSO projection operator defines the chirality of a massless spinor in the R sector. Thus
from the left and right moving sectors, one can construct states in four different sectors: 
(i) \textbf{NS-NS}, (ii) \textbf{NS-R}, (iii) \textbf{R-NS} and (iv) \textbf{R-R}. Taking into account the two types of chirality \(L\) and \(R\) one has two possibilities:

\begin{itemize}
  \item \textbf{a)}– The \textbf{GSO} projections on the left and right fermions produce different chirality in the ground state of the \(R\) sector (\textit{Type IIA}).
  \item \textbf{b)}– \textbf{GSO} projection are equal in left and right sectors and the ground states in the \(R\) sector, have the same chirality (\textit{Type IIB}). Thus the spectrum for the Type IIA and IIB superstring theories is:
\end{itemize}

- \textbf{Type IIA}

The \textbf{NS-NS} sector has a symmetric tensor field \(G_{IJ}\) (spacetime metric), an antisymmetric tensor field \(B_{IJ}\) and a scalar field \(\Phi\) (dilaton). In the \textbf{R-R} sector there is a vector field \(A_I\) associated with a 1-form \(A_{(1)}\) \((A_I \leftrightarrow A_{(1)})\) and a rank 3 totally antisymmetric tensor \(A_{IJK} \leftrightarrow A_{(3)}\) and by Hodge duality in ten dimensions also we have \(A_{(5)}, A_{(7)}\) and \(A_{(9)}\). In general the \textbf{R-R} sector consist of \(p\)-forms \(F_p = dA_{(p-1)}\) (where \(A_{(p)}\) are called RR fields) on the ten-dimensional spacetime \(X\) with \(p\) even i.e. \(F_2, F_4, \ldots, F_{10}\). In the \textbf{NS-R} and \textbf{R-NS} sectors we have two gravitinos with opposite chirality and the supersymmetric partners of the mentioned bosonic fields.

- \textbf{Type IIB}

In the \textbf{NS-NS} sector Type IIB theory has exactly the same spectrum that of Type IIA theory. On the \textbf{R-R} sector it has a scalar field \(a \leftrightarrow A_{(0)}\) (the axion field), an antisymmetric tensor field \(B'_{IJ} \leftrightarrow A_{(2)}\) and a rank 4 totally antisymmetric tensor \(D_{IJKL} \leftrightarrow A_{(4)}\) whose field strength is self-dual i.e., \(F_{(5)} = dA_{(4)}\) with \(\ast F_{(5)} = +F_{(5)}\). Similar than for the case of Type IIA theory one has also the Hodge dual fields \(A_{(6)}, A_{(8)}, A_{(10)}\). In general, RR fields in Type IIB theory are given by \(p\)-forms \(F_p = dA_{(p-1)}\) on the spacetime \(X\) with \(p\) odd i.e. \(F_1, F_3, \ldots, F_{11}\). The \textbf{NS-R} and \textbf{R-NS} sectors do contain two gravitinos with the same chirality and the corresponding fermionic matter.

\subsection*{3.4. Type I Superstrings}

In this case the \(L\) and \(R\) degrees of freedom are identified. Type I and Type IIB theories have the same spectrum, except that in the former one the states which are not invariant under the change of orientation of the worldsheet, are projected out. This worldsheet parity \(\Omega\) interchanges the left and right modes. Type I superstring theory is a
theory of breakable closed strings, thus it incorporates also open strings. The $\Omega$ operation leave invariant only one half of the spacetime supersymmetry, thus the theory is $\mathcal{N} = 1$.

The spectrum of bosonic massless states in the NS-NS sector is: $G_{IJ}$ (spacetime metric) and $\Phi$ (dilaton) from the closed sector and $B_{IJ}$ is projected out. On the R-R sector there is an antisymmetric field $B_{IJ}$ of the closed sector. The open string sector is necessary in order to cancel tadpole diagrams. A contribution to the spectrum come from this sector. Chan-Paton factors can be added at the boundaries of open strings. Hence the cancellation of the tadpole are needed 32 labels at each end. Therefore in the NS-NS sector there are 496 gauge fields in the adjoint representation of SO(32).

### 3.5. Heterotic Superstrings

This kind of theory involves only closed strings. Thus there are left and right sectors. The left-moving sector contains a bosonic string theory and the right-moving sector contains superstrings. This theory is supersymmetric on the right sector only, thus the theory contains $\mathcal{N} = 1$ spacetime supersymmetry. The momentum at the left sector $P_L$ lives in 26 dimensions, while $P_R$ lives in 10 dimensions. It is natural to identify the first ten components of $P_L$ with $P_R$. Consistency of the theory tell us that the extra 16 dimensions should belong to the root lattice $E_8 \times E_8$ or a $\mathbb{Z}_2$-sublattice of the SO(32) weight lattice.

The spectrum consists of a tachyon in the ground state of the left-moving sector. In both sectors we have the spacetime metric $G_{IJ}$, the antisymmetric tensor $B_{IJ}$, the dilaton $\phi$ and finally there are 496 gauge fields $A_I$ in the adjoint representation of the gauge group $E_8 \times E_8$ or SO(32).

All these Types of superstring theories do admit a low energy effective description in terms of a supergravity theory. These theories involves the corresponding background fields of their spectra. Supergravity actions of these diverse types will be constructed later to study strong/weak coupling duality in string theory.

### 3.6. Calabi-Yau Compactifications in Perturbative String Theory

In order to connect superstring theories to the observed 4-dimensional spacetime physics, we have to reduce the critical dimension $D = 10$ to four dimensions. To preserve certain supersymmetry consistent with chirality in four dimensions it is necessary to require some properties to the ten dimensional spacetime $X$. Perhaps the simplest ansatz
is to assume that the four-dimensional Minkowski spacetime $M$ and a six-dimensional internal space $K$ factorizes as $X \cong M \times K$, where $K$ has tiny dimensions and unobservable in our present experiments. It is worth to say that this factorization ansatz is not unique and other possibility is the warped compactification of the celebrated Randall-Sundrum scenarios, which are nicely reviewed in Ref. [10].

It is useful to classify the compactifications according to how much supersymmetries is broken, because this number is related with the quantum corrections that we shall consider. We choose $K$ to be a manifold with the property that a certain number of supersymmetries are preserved$^4$.

We are now looking for conditions in the background which leave some supersymmetry unbroken. These conditions are given by null variations of the Fermi fields.

Consider the diagonal metric for ten-dimensional spacetime $X$ given by $G_{IJ} = f(y)g_{\mu\nu} + G_{mn}(y)$ where $y$ denotes the compactified coordinates and $I, J = 0, \ldots, 9, \mu, \nu = 0, \ldots, 3, m, n = 4, \ldots, 9$. For $D = 10, \mathcal{N} = 1$ heterotic string theory the Fermi fields variations are:

- **gravitino**: $\nabla \psi_\mu = \Delta_\mu \epsilon, \ \delta \psi_m = (\partial_m + \frac{1}{4} \Omega_{mnp} \Gamma^{np}) \epsilon$,
- **dilatino**: $\delta \xi = (\Gamma^m \partial_m \phi - \frac{1}{12} \Gamma^{mnp} H_{mnp}) \epsilon$,
- **gaugino**: $\delta \lambda = F_{mn} \Gamma^{mn} \epsilon$,

where $\epsilon$ is a Weyl-Majorana spinor in ten dimensions, $\Omega_{mnp}$ is the internal component of $\Omega_{MNP} = \omega_{MNP} - \frac{1}{2} H_{MNP}$, $\Gamma$ are the Dirac matrices.

The compactification ansatz $X = M \times K$ breaks the Lorentz group $SO(9,1)$ into $SO(3,1) \times SO(6)$. In the spinor representation $\mathbf{16}$ the Weyl-Majorana supersymmetry parameter $\epsilon_{\alpha \beta}$ decomposes as $\epsilon(y) \rightarrow \epsilon_{\alpha \beta}(y) + \epsilon_{\alpha \beta}^*(y)$ under $\mathbf{16} \rightarrow (\mathbf{2}, \mathbf{4}) \oplus (\mathbf{2}^*, \mathbf{4}^*)$. The general form of $\epsilon_{\alpha \beta}$ is $\epsilon_{\alpha \beta} = u_\alpha \zeta_\beta(y)$ with $u_\alpha$ an arbitrary Weyl spinor. When we put the condition that Fermi fields variations vanish, then each internal spinor $\zeta_\beta(y)$ gives the minimal ($\mathcal{N} = 1$) $D = 4$ supersymmetric algebra.

Now, by the null Fermi fields variations we can find conditions in the background fields assuming that $H_{mnp} = 0$. These are:

- $\delta \zeta = 0 \Rightarrow \partial_m \Phi = 0$,
- $\delta \psi = 0 \Rightarrow G_{\mu\nu} = \eta_{\mu\nu}$,
- $\delta \psi_m = 0 \Rightarrow \nabla_m \zeta = 0$.

---

$^4$ Cosmological constant is generated by perturbation theory. Strings propagating in a $K$ manifold in which all supersymmetries are broken distabilizes the Minkowski vacuum.
The last equation tell us that $\zeta_\beta$ is covariantly constant on the internal space $K$, and implies that $K$ is Ricci-flat. This is because $[\nabla_m, \nabla_n]z = \frac{1}{4}R_{mnpq}\Gamma^{pq}\zeta = 0$. For this reason, in general $\Gamma^{pq}$ do not belong to SO(6) but to SU(3), which is a subgroup that leaves one component of the spinor $\zeta$ invariant. Thus the compact manifold $K$ must have SU(3) holonomy. The second unbroken susy condition implies that the warped factor $f(y)$ in metric is 1 and the metric $G_{IJ}$ is unwarped. Finally, the first condition implies that the dilation is constant. This is a *Calabi-Yau* three-fold. A Calabi-Yau three-fold is also a Kähler manifold in which the first Chern class zero *i.e.* $c_1(TK) = 0$. Any Calabi-Yau manifold possesses a unique Ricci-flat metric. When we consider $N = 1$ heterotic string theory on Calabi-Yau three-fold we obtain a four-dimensional chiral theory with spacetime supersymmetry $N = 1$. In fact, compactification on manifolds of SU(3) holonomy preserves 1/4 of supersymmetry. If we consider $N = 2$ theories (for example, type II superstrings) in $D = 10$ dimensions, after compactification on a Calabi-Yau three-fold we obtain $N = 2$ theories in $D = 4$.

In addition to the CY-threefold structure for $K$ the unbroken susy condition $\delta \lambda^a = 0 = F^a_{mn}\Gamma^{mn}\epsilon$, leads to the equations in complex coordinates

$$F_{I,J} = F_{T,I,J} = 0, \quad G^{I\overline{J}}F_{I,J} = 0.$$  \hspace{1cm} (3.28)

These equations require to specify a gauge subbundle $V$ of a $E_8 \times E_8$ gauge bundle over $K$ and a gauge connection $A$ on $V$ with curvature $F$. The condition $F_{I,J} = F_{T,I,J} = 0$ tell us that the subbundle $V$ as well as the corresponding connection should be holomorphic. The second condition $G^{I\overline{J}}F_{I,J} = 0$ is the celebrated Donaldson-Uhlenbeck-Yau equation for $A$. This equation has a unique solution if the bundle $V$ is stable and if it is satisfied the integrability condition $\int_K \Omega^{n-1} \wedge c_1(V) = 0$, where $\Omega$ is the Kähler form of $K$. There is a further condition to be satisfied by the connection $A$, the Bianchi identity for $H$ and $F$, it is given by

$$dH = trR \wedge R - \frac{1}{30}trF \wedge F.$$  \hspace{1cm} (3.29)

The only solution is $trR \wedge R \propto trF \wedge F$ which implies that $c_2(TK) = c_2(V)$. This situation is usually known as the *standard embedding* of the spin connection in the gauge connection and it is a method to determine the connection $A$ on $V$.

Thus in the compactification of phenomenological interest of the heterotic theory with the ansatz $X = M \times K$, the internal space has to be a Calabi-Yau three-fold and one has
to specify a stable, holomorphic vector bundle \( V \) over \( X \) (or \( \mathcal{K} \)) satisfying \( c_1(V) = 0 \) and \( c_2(V) = c_2(TX) \).

If \( V \) is a SU\((n)\) vector bundle over \( X \) the subgroups of \( E_8 \times E_8 \) that commutes are \( E_6, \text{SO}(10) \) and \( \text{SU}(5) \) for \( n = 3, 4, 5 \) respectively. This leads to GUTs in four dimensions justly with the gauge groups \( E_6, \text{SO}(10) \) or \( \text{SU}(5) \).

**Massless Spectrum**

In order to describe the impact of the characteristics of \( \mathcal{K} \) and \( V \) on the properties of the spectrum of the four dimensional theory we start by decomposing the ten-dimensional Dirac operator under \( M \times \mathcal{K} \) into

\[
\mathcal{D}^{(10)} = \sum_{I=0}^{9} \Gamma^I D_I = \mathcal{D}^{(4)} + \mathcal{D}_K,
\]

where \( \mathcal{D}^{(4)} = \sum_{I=0}^{3} \Gamma^I D_I \) and \( \mathcal{D}_K = \sum_{J=4}^{9} \Gamma^J D_J \). Dirac equation in ten dimensions is

\[
\mathcal{D}^{(10)} \Psi(x^I, y^J) = (\mathcal{D}^{(4)} + \mathcal{D}_K) \Psi(x^I, y^J).
\]

Thus the spectrum of the Dirac operator \( \mathcal{D}_K \) on \( \mathcal{K} \) determines the massive spectrum of fermions in four dimensions.

In ten dimensions the Lorentz group only has real spinor representations and the Clifford modules decomposes as: \( S^{(10)} = S_+^{(10)} \oplus S_-^{(10)} \). Positive and negative chirality are distinguished by \( \Gamma^{(10)} = \Gamma^0 \Gamma^1 \ldots \Gamma^9 \). CPT theorem implies that we must take only one chirality

\[
\Gamma^{(10)} \Psi = +\Psi.
\]

Decompose the spinor representation of SO\((1,9)\) under SO\((1,3) \times \text{SO}(6)\) with \( \Gamma^{(10)} = \Gamma^{(4)} \cdot \Gamma^{(6)} \) where \( \Gamma^{(4)} = i \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \) and \( \Gamma^{(6)} = -i \Gamma^4 \Gamma^5 \ldots \Gamma^9 \). One solution with \( \Gamma^{(10)} = +1 \) is given by \( \Gamma^{(4)} = \Gamma^{(6)} \) and then the spin bundle decomposes under \( M \times \mathcal{K} \) as

\[
\hat{S}^{(10)} = \left( \hat{S}_+^{(4)} \otimes \hat{S}_+^{(6)} \right) \oplus \left( \hat{S}_-^{(4)} \otimes \hat{S}_-^{(6)} \right).
\]

Now solve the Dirac equation with the ansatz \( \Psi(x^I, y^J) = \sum_m \phi_m(x^I) \otimes \chi_m(y^J) = \sum_m \psi_m \) and \( \mathcal{D}_K \chi_m = \lambda_m \chi_m \). It leads to
\[ (D^{(4)} + \lambda_m) \psi_m = 0, \]

where \( D^{(4)} = \Gamma^{(4)} D^{(4)} \) and \( D_K = \Gamma^{(4)} D_K \).

\( D_K \) is an elliptic operator on the compact manifold \( K \), this implies that that operator has a \textit{finite} number of fermion zero modes. Massless fermions in four dimensions originate as zero modes of the Dirac operators \( D^K \) of the internal manifold \( K \). By the Atiyah-Singer theorem, a topological invariant of \( K \) containing the information of the chiral fermions on \( K \) is given by the index of the Dirac operator

\[ \text{Index}(D_K) = N^\lambda=0_+ - N^\lambda=0_-, \]  

for chiral fermions on \( K \) with \( \Gamma^{(6)} = \pm 1 \). Here \( N^\lambda=0_\pm \) are the number of positive or negative chiral zero modes. In \( 2k + 2 \) dimensions this index is vanishing. We need to couple gauge fields coming from the heterotic string theory. Recall that they are \( E_8 \times E_8 \) valued gauge fields.

The \textit{standard embedding} of the spin connection in the gauge connection leads to the chain of maximal subgroups: \( SO(6) \times SO(10) \subset SO(16) \subset E_8 \). This breaks \( SO(16) \) to \( SO(10) \). The computation of the \( \Delta \) for this case yields \( \Delta = \bigoplus \delta_i L_i \otimes R_i \) where \( L_i \) are irreps of \( SO(6) \) and \( R_i \) are complex irreps of \( SO(10) \). These latter determine the irreps where are distributed the massless fermions of the four-dimensional theory. The former irreps \( L_i \) determine the number of fermionic chiral zero modes described by the topological index \( \delta = \text{Index}(D_K) \). This is given by

\[ \delta = N^{\lambda=0}_{\Gamma^{(6)}=+1} - N^{\lambda=0}_{\Gamma^{(6)}=-1} = \int_K \text{ch}(V) \text{td}(K) = \frac{1}{2} \int_K c_3(V) \]

and from the solution \( \Gamma^{(6)} = \Gamma^{(4)} \) it determines the chiral fermion families in four dimensions

\[ \delta = N^{\lambda=0}_{\Gamma^{(4)}=+1} - N^{\lambda=0}_{\Gamma^{(4)}=-1}. \]  

Thus the theory in four dimensions has \( \Delta = \bigoplus \delta_i R_i \) where

\[ \Delta = \delta \left( 16 \oplus 16^* \right), \]  

where \( \delta = \chi(K)/2 \) with \( \chi(K) \) is the Euler number of \( K \).
In this section we intend to make contact with some four-dimensional physics. The development of this line of work is known as string phenomenology. Recent reviews of this topic at the light of string dualities is given in Ref. [11]. In the present short review we follows Refs. [5,12].

Continuous and Discrete Symmetries

In building models coming from string theory, there are no global internal symmetries in spacetime (there are no continuous global symmetries in all string theories). This is because if there is an internal symmetry, there should be a vector field in the spectrum because the properties of SCFT and it has the same properties of the gauge field of that symmetry.

Take for instance Type I or Type II superstring theory. We know from Noether’s theorem of the two-dimensional theory that associated with the Type I or II superconformal symmetry there is a worldsheet conserved supercharge, \( Q = \frac{1}{2\pi i} \int (dzd\theta J - d\bar{z}d\bar{\theta} \overline{J}) \), where, by uses of this symmetry, \( J \) should be a \((\frac{1}{2}, 0)\) tensor superfield and \( \overline{J} \) is a \((0, \frac{1}{2})\) tensor superfield. The associated bosonic vertex operators (when we combine these tensors with the fermionic fields \( \tilde{\psi}^I \) and \( \psi^I \) respectively) have the property to couple with left and (or) right-moving parts of \( Q \), giving rise to a spacetime gauge symmetry.

The absence of internal global symmetries in spacetime physics coming from string theory may help to understand some exciting problems of particle physics like the existence of the non-zero neutrino masses, which lie in the violation of the leptonic number. This was argued recently by Witten in Ref. [13].

However there are generically some discrete symmetries in string models. For example, T-duality which is an infinite dimensional one, or some models inherited from the point group of orbifold constructions which are finite dimensional, are in fact regarded as discrete symmetries. The importance of this kind of symmetries relay in the fact that they are useful for model building, hierarchy of masses and other related problems.

\( P, C, T \) Symmetries

We will see how discrete spacetime symmetries \( P, C \) and \( T \) are broken in string theory. If string theory is correct, when we compactify (for example on Calabi-Yau manifolds,
orbifolds, tori, etc.) one must obtain the same symmetries (or broken symmetries) as these of the SM.

- **P-symmetry**

  Parity symmetry is violated by gauge interactions in SM. In string theory there are an analogous situation. Take for instance the heterotic string. The massless states in ten dimension are labeled by irreps of the little group $\text{SO}(8)$. The action of parity symmetry reverses the spinor representations $\overline{8}_s$ and $\overline{8}_s^{t'}$ of the left and right-moving sectors. The symmetry is realized if the corresponding gauge representations $\mathcal{R}$ and $\mathcal{R}'^*$ are equal.

  However, $\mathcal{R}$ is the adjoint representation while that $\mathcal{R}'^*$ is empty. This tell us that parity symmetry is broken and the gauge couplings are chiral. But although in ten dimensions the spectrum is chiral, when we compactify to four dimensions, the spectrum could be turned out into a non-chiral one.

  For example, for toroidal compactifications, the spectrum is no-chiral, but for $Z_3$ orbifold (compactification) the spectrum it is. Other kinds of compactifications produce chiral gauge couplings.

  The chirality of the spectrum can be expressed by a topological quantity called *Index* as we saw in the last subsection. Since the index is a topological invariant quantity, it does not suffers any change under continuous transformations of the CFT.

- **C and CP Symmetry**

  Charge conjugation symmetry is also broken in SM. This is because $C$ leaves spacetime invariant, but conjugates the gauge generators. As in SM, in string theories we require that conjugate representations (for example in SM, the fermions) satisfy $\mathcal{R}_\pm = \mathcal{R}_{\pm}^\ast$. From this we can see that chiral gauge couplings do not satisfy $C$ symmetry. For the orbifold example this is also true.

  Consider now the $CP$ symmetry. This symmetry takes $\mathcal{R}_+ \to \mathcal{R}_-^\ast$. Thus, any gauge coupling satisfies it as a consequence of $CPT$ invariance. In the case of the orbifold there is a symmetry of the action which reverses $X^k$ into $\psi^k$ with $k = 3, 5, 7, 9$, and all the $\lambda^I$ ($I$ odd). This is a $CP$ symmetry in 4-dimensions. So, $Z_3$ orbifold is $CP$ symmetric.

- **CPT Symmetry**

  In string theory, as in local Lorentz-invariant quantum field theory, $CPT$ symmetry is preserved. In string perturbation theory we use the $\theta$-operation\(^5\). This is defined as $\theta(X^{0,3}) = \psi^{0,3}$ and vice versa. In Euclidean time this can be represented by a $\pi$-rotation

\(^5\) This is basically, the same argument used in field theory to prove $CPT$ symmetry.
in the plane \((iX^0, X^3)\). In this context is clear that the action of \(\theta\) is a symmetry, that reverses time and includes parity \((in X^3)\). In order to show that this action also includes charge conjugation consider the \(S\)-matrix,

\[
\langle \alpha, \text{out} | \beta, \text{in} \rangle = \langle \overline{V}_\alpha V_\beta \rangle, \tag{3.38}
\]

where \(V\) is the vertex operator and we are only considering vertex operators to the initial and final states. The action of \(\theta\) is

\[
\langle \alpha, \text{out} | \beta, \text{in} \rangle = \langle \theta \cdot \overline{V}_\alpha \theta \cdot V_\beta \rangle = \langle \overline{\theta \beta}, \text{out} | \theta \alpha, \text{in} \rangle. \tag{3.39}
\]

When we apply CPT operation, we can see that it is antiunitary and it is \(\theta\) combined with the conjugation of the vertex operator

\[
\langle CPT \cdot \beta, \text{out} | CPT \cdot \alpha, \text{in} \rangle = \langle \alpha, \text{out} | \beta, \text{in} \rangle. \tag{3.40}
\]

The manner in what we saw that \(CPT\) is an exact symmetry in string theory is only applicable to the perturbative sector. However for the non-perturbative sector, we can argue that SM, or field theory is the low energy limit of the string theory, so we take \(CPT\) symmetry for this low energy limit, and then we put it in 10 dimensions.

**Effective Actions in Four Dimensions**

First of all, it is important to emphasize that consistent four-dimensional superstring models which are chiral lead to \(\mathcal{N} = 1\) supersymmetric theories. At \(\mathcal{N} \geq 2\) supersymmetry spoils chirality. Thus in order to consider phenomenologically string models in four dimensions we restrict ourselves to construct \(\mathcal{N} = 1\) supersymmetric actions. Mostly of the material we describe here is at the review by F. Quevedo [12] and we recommend it for checking details.

The corresponding spectrum of massless particles are composed by graviton-gravitino multiplet \((G_{IJ}, \psi^I)\) and the gauge-gaugino multiplet \((A_\alpha^I, \lambda^\alpha)\). Also there are matter and moduli fields, in the form of chiral multiplets \((Z, \chi)\). In the case of the dilaton field \(\Phi\), it couples to the antisymmetric tensor \(B_{IJ}\) to form the linear \(\mathcal{N} = 1\) multiplet \((\Phi, B_{IJ}, \rho)\). We can construct a \(\mathcal{N} = 1\) chiral multiplet \((S, \chi_S)\) from the linear multiplet, where \(S\) is obtained by a duality transformation of the dilaton field. This transformation is given by \(S = a + ie^\Phi\) and \(\nabla_I a \equiv \varepsilon_{IJKL} \nabla^J B^{KL}\).
Thus the whole theory can be described as a $\mathcal{N} = 1$ supergravity theory coupled only to gauge and chiral multiplets. The most general Lagrangian which describes these fields depends on three arbitrary functions of the chiral multiplets. These fields are: 

(i) $K(Z, Z^*)$, the Kähler potential which is a real function which determines the kinetic terms of the chiral fields. The corresponding Lagrangian is given by $L_{\text{kin}} = K_{ZZ} \partial_I Z \partial^I Z^*$. 

(ii) $W(Z)$, the superpotential, which is a holomorphic function of the chiral multiplets. 

(iii) $f_{ab}(Z)$, the gauge kinetic function (holomorphic), and determines the gauge kinetic couplings in the corresponding Lagrangian $L_{\text{gauge}} = \text{Re} f_{ab}(Z) F^a_{IJ} F^b_{IJ} + \text{Im} f_{ab}(Z) F^a_{IJ} \tilde{F}^b_{IJ}$. This function contributes to gaugino masses.

There are another quantity called the scalar potential $V = V_F + V_D$, where $V_F(Z, Z^*) = \exp\left(\frac{K}{M_{Pl}}\right) \{ D_Z W K_{ZZ}^{-1} D_Z W^* - 3 |W|^2 M_{Pl}^2 \}$, $D_Z W = W_Z + W K_{ZZ}^* M_{Pl}^2$ and $V_D = (\text{Re} f^{-1})_{ab}(K_{ZT}, T^a Z^*) (K_{ZT}^*, T^b Z^*)$.

Thus, the problem to find an effective four-dimensional action, is to calculate the functions $K, W$ and $f_{ab}$ when we are giving a specific string model. To do this, we have to use all the symmetries we have and taking into account that four dimensional string models are governed by two perturbation expansions. That is, an expansion in the sigma model (controlled by the size of the extra dimensions) and the proper string perturbation (in terms of the dilaton field) of the string coupling constant $g_s$.

First of all we consider only couplings generated at string tree-level. For the sigma model, we also take only the tree-level expansion. Using symmetries as the four dimensional Poincaré symmetry, supersymmetry, gauge symmetries and the axionic symmetry, we can extract the dependence of the effective action on the dilaton field $S$. Then, at tree-level, the functions $K, W, f$ are given by $K = -\log(S + S^*) + \tilde{K}(T, U, Q)$, $W = Y_{IJK} Q^I Q^J Q^K$ and $f_{ab} = S \delta_{ab}$ with $\tilde{K}$ an undetermined function. Our purpose is to find approximated expressions to this functions in the tree-level of the string perturbation theory, but otherwise exact in the CFT.

It can be showed that by using the axionic symmetries that at all orders in sigma-model expansion, superpotential $W$ does not depend on $T$ and $U$, so it is just a function of the matter fields $Q^I$. Thus $W$ does not admit any kind of corrections in the sigma model$^6$.

---

$^6$ The reason for this, is that the field $T$, related with the size of the extra dimensions, comes from the internal components of the metric and determines the form that the loop expansion of the worldsheet action takes.
The superpotential $W$ does not depend on $S$ as well. We know that $S$ is the string loop-counting parameter and this implies that $W$ is also an exact expression at tree-level string perturbation theory, i.e., does not admit any radiative corrections.

Now, we are interested in finding a useful expression for $K$. This is more difficult, because we only can calculate it for some simple cases. Take for example a Calabi-Yau compactification with $h_{1,1} = 1$ and $h_{2,1} = 0$. This gives us that $K = -\log(S + S^*) - 3\log(T + T^* + QQ^*)$. When we write the Kähler potential as an expansion in matter fields, it is possible to extract an exact tree-level expression. The expansion is given by

$$K = -\log(S + S^*) + K^M(T, T^*, U, U^*) + K^Q(T, T^*, U, U^*)QQ^*$$

$$+ \hat{Z}(T, T^*, U, U^*)(QQ + Q^*Q^*) + O(Q^3).$$

For some $(2, 2)$ orbifold and Calabi-Yau models, it has been computed the quantities $K^M, K^Q$ and $\hat{Z}$.

Consider now the loop corrections. We have seen that the superpotential (which is an holomorphic function) does not admit radiative corrections. However, for the Kähler potential this is different. We have just to calculate order by order in the loop expansions the corresponding expression for $K_{\text{loop}}$. On the other hand, the gauge kinetic function $f_{ab}$ is also holomorphic and we know the expression in an exact manner for the tree level. Loop corrections to this function have a great importance due to this function determines the gauge coupling. Here we do not get expressions for this corrections, but it is important to say that there are no further corrections to $f_{ab}$ beyond one loop, as in the standard supersymmetric theories.

In general, we have problems to determine how supersymmetry is broken at low energies. We can not solve this problem within perturbative string theory. We need work in the non-perturbative sector of the theory. But this sector, despite of many efforts and excellent results, we do not yet have a complete non-perturbative version of string theory. However there are some interesting non-perturbative mechanisms to break supersymmetry as gaugino condensation, composite goldstinos and instantons. The reader interested in these and another issues of non-perturbative string phenomenology is encouraged to consult Refs. [11,14].
4. T-duality, D-branes and Brane Configurations

This section has the purpose of introducing basic ideas about T-duality in closed and open string theory. The open string case leads in a natural way to the definition of D-branes (for reviews of D-branes see Refs. [5,15,16]). These objects are of extreme importance since they are precisely the solitonic degrees of freedom which realize the strong-weak coupling duality in superstring theory. This duality is also known as string S-duality. T and S dualities relate the five perturbative superstring theories discussed previously and their compactifications in diverse dimensions. Moreover, the strong coupling limit of HE and Type II string theory (and their compactifications) suggest that there is an eleven-dimensional theory which has the eleven-dimensional supergravity as low energy limit. This prospect of theory is widely known as M-theory. The name come from the words: mystery, magic, mother, etc. Compactifications to diverse lower dimensions than ten gives more evidence of the existence of this theory. The fundamental degrees of freedom of this unified theory are unknown, but macroscopically they include membranes and fivebranes. ‘Matrix Theory’ is an attempt to give the dof’s of M-theory. The proposal is that these degrees of freedom are the D0-branes. The worldvolume effective theory of a gas of $N$ D0-branes is a SU($N$) quantum mechanics. Large $N$-limit reproduces the description of membranes and fivebranes and some other results of eleven dimensional supergravity (for some reviews the reader can consult Refs. [17,18]).

D-branes also, are very important tools to study the strong coupling of supersymmetric theories in various dimensions. Different properties as chirality, dualities etc. are encoded in the engineering of brane configurations. The moduli space of these susy gauge theories is described by the Higgs and the Coulomb branches of the corresponding brane configuration. Many field theory results are understanding in terms of a geometrical language and many generalizations have been established motivated by the brane engineering (more about this topic can be found in Ref. [19]).

Finally, the presence of branes leads to modify the prescription of Calabi-Yau or orbifold compactifications and new non-perturbative are possible. In these sections we will discuss some of these interesting topics.

4.1. Toroidal Compactification, T-duality and D-branes

D-branes are, despite of the dual fundamental degrees of freedom in string theory, extremely interesting and useful tools to study nonperturbative properties of string and
field theories (for some reviews see [15,16]). Non-perturbative properties of supersymmetric gauge theories can be better understanding as the world-volume effective theory of some configurations of intersecting D-branes (for a review see [19]). D-branes also are very important to connect gauge theories with gravity. This is the starting point of the AdS/CFT correspondence or Maldacena’s conjecture. We don’t review this interesting subject in this paper, however the reader can consult the excellent review [20]. Roughly speaking D-branes are static solutions of string equations which satisfy Dirichlet boundary conditions. That means that open strings can end on them. To explain these objects we follow the traditional way, by using T-duality on open strings we will see that Neumann conditions are turned out into the Dirichlet ones. To motivate the subject we first consider T-duality in closed bosonic string theory.

**T-duality in Closed Strings**

The general solution of Eq. (3.5) in the conformal gauge can be written as \( X^I(\sigma, \tau) = X^I_R(\sigma^-) + X^I_L(\sigma^+) \), where \( \sigma^\pm = \sigma \pm \tau \). Now, take one coordinate, say \( X^{25} \) and compactify it on a circle of radius \( R \). Thus we have that \( X^{25} \) can be identified with \( X^{25} + 2\pi R m \) where \( m \) is called the *winding number*. The general solution for \( X^{25} \) with the above compactification condition is

\[
X^{25}_R(\sigma^-) = X^{25}_{0R} + \sqrt{\frac{\alpha'}{2}} P^{25}_R (\tau - \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{l \neq 0} \frac{1}{l} a^{25}_{R,l} \exp \left(-i l (\tau - \sigma)\right)
\]

\[
X^{25}_L(\sigma^+) = X^{25}_{0L} + \sqrt{\frac{\alpha'}{2}} P^{25}_L (\tau + \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{l} a^{25}_{L,l} \exp \left(-i l (\tau + \sigma)\right),
\]  

(4.1)

where

\[
P^{25}_{R,L} = \frac{1}{\sqrt{2}} \left( \frac{\sqrt{\alpha'}}{R} n + \frac{R}{\sqrt{\alpha'}} m \right). 
\]  

(4.2)

Here \( n \) and \( m \) are integers representing the discrete momentum and the winding number, respectively. The latter has not analogous in field theory. While the canonical momentum is given by \( P^{25} = \frac{1}{\sqrt{2\alpha'}} (P^{25}_L + P^{25}_R) \). Now, by the mass shell condition, the mass of the perturbative states is given by \( M^2 = M^2_L + M^2_R \), with

\[
M^2_{L,R} = -\frac{1}{2} P^I P_I = \frac{1}{2} (P^{25}_{L,R})^2 + \frac{2}{\alpha'} (N_{L,R} - 1). 
\]  

(4.3)
We can see that for all states with $m \neq 0$, as $R \to \infty$ the mass become infinity, while $m = 0$ implies that the states take all values for $n$ and form a continuum. At the case when $R \to 0$, for states with $n \neq 0$, mass become infinity. However in the limit $R \to 0$ for $n = 0$ states with all $m$ values produce a continuum in the spectrum. So, in this limit the compactified dimension disappears. For this reason, we can say that the mass spectrum of the theories at radius $R$ and $\frac{\alpha'}{R}$ are identical when we interchange $n \leftrightarrow m$. This symmetry is known as $T$-duality.

The importance of $T$-duality lies in the fact that the $T$-duality transformation is a parity transformation acting on the left and right moving degrees of freedom. It leaves invariant the left movers and changes the sign of the right movers (see Eq. (4.2))

$$P^25_L \to P^25_L, \quad P^25_R \to -P^25_R. \quad (4.4)$$

The action of $T$-duality transformation must leave invariant the whole theory (at all order in perturbation theory). Thus, all kind of interacting states in certain theory should correspond to those states belonging to the dual theory. In this context, also the vertex operators are invariant. For instance the tachyonic vertex operators are

$$V_L = \exp(iP^25_L X^25_L), \quad V_R = \exp(iP^25_R X^25_R). \quad (4.5)$$

Under $T$-duality, $X^25_L \to X^25_L$ and $X^25_R \to -X^25_R$; and from the general solution Eq. (13), $\alpha^{25}_{R,i} \to -\alpha^{25}_{R,i}$, $X^25_{0R} \to -X^25_{0R}$. Thus, $T$-duality interchanges $n \leftrightarrow m$ (Kaluza-Klein modes $\leftrightarrow$ winding number) and $R \leftrightarrow \frac{\alpha'}{R}$ in closed string theory.

$T$-duality in Open Strings

Now, consider open strings with Neumann boundary conditions. Take again the 25th coordinate and compactify it on a circle of radius $R$, but keeping Neumann conditions. As in the case of closed string, center of mass momentum takes only discrete values $P^25 = \frac{n}{R}$. While there is not analogous for the winding number. So, when $R \to 0$ all states with nonzero momentum go to infinity mass, and do not form a continuum. This behavior is similar as in field theory, but now there is something new. The general solutions are

$$X^25_R = \frac{X^25_0}{2} - \frac{a}{2} + \alpha' P^25(\tau - \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{l \neq 0} \alpha^25_l exp(-i2l(\tau - \sigma)).$$
\[ X_{L}^{25} = \frac{X_{0}^{25}}{2} + \frac{a}{2} + \alpha' P^{25}(\tau + \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{l \neq 0} \frac{1}{\ell} a_{l}^{25} \exp \left( -i2l(\tau + \sigma) \right) \] (4.6)

where \( a \) is a constant. Thus, \( X^{25}(\sigma, \tau) = X_{R}^{25}(\sigma^{-}) + X_{L}^{25}(\sigma^{+}) = X_{0}^{25} + \frac{2\alpha' n}{R} \tau + \text{oscillator terms}. \) Taking the limit \( R \to 0 \), only the \( n = 0 \) mode survives. Because of this, the string seems to move in 25 spacetime dimensions. In other words, the strings vibrate in 24 transversal directions. T-duality provides a new T-dual coordinate defined by \( \tilde{X}^{25}(\sigma, \tau) = X_{L}^{25}(\sigma) - X_{R}^{25}(\sigma) \). Now, taking \( \tilde{R} = \frac{\alpha'}{R} \) we have \( \tilde{X}^{25}(\sigma, \tau) = a + 2\tilde{R}\sigma n + \text{oscillator terms}. \) Using the boundary conditions at \( \sigma = 0, \pi \) one has \( \tilde{X}^{25}(\sigma, \tau) \mid_{\sigma=0} = a \) and \( \tilde{X}^{25}(\sigma, \tau) \mid_{\sigma=\pi} = a + 2\pi \tilde{R}n \). Thus, we started with an open bosonic string theory with Neumann boundary conditions, and T-duality and a compactification on a circle in the \( 25^{th} \) dimension, give us Dirichlet boundary conditions in such a coordinate. We can visualize this saying that an open string has its endpoints fixed at a hyperplane with 24 dimensions.

Strings with \( n = 0 \) lie on a 24 dimensional plane space (D24-brane). Strings with \( n = 1 \) has one endpoint at a hyperplane and the other at a different hyperplane which is separated from the first one by a factor equal to \( 2\pi \tilde{R} \), and so on. But if we compactify \( p \) of the \( X^{i} \) directions over a \( T^{p} \) torus \( (i = 1, \ldots, p) \). Thus, after T-dualizing them we have strings with endpoints fixed at hyperplane with \( 25 - p \) dimensions, the D\((25 - p)\)-brane.

Summarizing: the system of open strings moving freely in spacetime with \( p \) compactified dimensions on \( T^{p} \) is equivalent, under T-duality, to strings whose endpoints are fixed at a D\((25 - p)\)-brane i.e. obeying Neumann boundary conditions in the \( X^{i} \) longitudinal directions \( (i = 1, \ldots, p) \) and Dirichlet ones in the transverse coordinates \( X^{m} \) \( (m = p + 1, \ldots, 25) \).

The effect of T-dualizing a coordinate is to change the nature of the boundary conditions, from Neumann to Dirichlet and vice versa. If one dualize a longitudinal coordinate this coordinate will satisfies the Dirichlet condition and a D\(p\)-brane becomes a D\((p - 1)\)-brane. But if the dualized coordinate is one of the transverse coordinates the D\(p\)-brane becomes a D\((p + 1)\)-brane.

T-duality also acts conversely. We can think to begin with a closed string theory, and compactify it on to a circle in the \( 25^{th} \) coordinate, and then by imposing Dirichlet conditions, obtain a D-brane. This is precisely what occurs in Type II theory, a theory of closed strings.
Now, we will see how does emerges a gauge field on the Dp-brane world-volume. Again, for the mass shell condition for open bosonic strings and because T-duality \( M^2 = \left( \frac{2\pi R}{\alpha'} \right)^2 + \frac{1}{\alpha'}(N - 1) \). The massless state \( (N = 1, n = 0) \) implies that the gauge boson \( \alpha_{-1}^I | 0 \rangle \) (\( U(1) \) gauge boson) lies on the D24-brane world-volume. On the other hand, \( \alpha_{-25}^2 | 0 \rangle \) has a vev (vacuum expectation value) which describes the position \( \tilde{X}^{25} \) of the D-brane after T-dualizing. Thus, we can say in general, there is a gauge theory \( U(1) \) over the world volume of the Dp-brane.

Consider now an orientable open string. The endpoints of the string carry charge under a non-Abelian gauge group. For Type II theories the gauge group is \( U(N) \). One endpoint transforms under the fundamental representation \( N \) of \( U(N) \) and the other one, under its complex conjugate representation (the anti-fundamental one) \( N^* \).

The ground state wave function is specified by the center of mass momentum and by the charges of the endpoints. Thus implies the existence of a basis \( | k; ij \rangle \) called Chan-Paton basis. States \( | k; ij \rangle \) of the Chan-Paton basis are those states which carry charge 1 under the \( i^{th} \) \( U(1) \) generator and \(-1\) under the \( j^{th} \) \( U(1) \) generator. So, we can decompose the wave function for ground state as \( | k; a \rangle = \sum_{i,j=1}^{N} | k; ij \rangle \lambda_{ij}^{a} \) where \( \lambda_{ij}^{a} \) are called Chan-Paton factors. From this, we see that it is possible to add degrees of freedom to endpoints of the string, that are precisely the Chan-Paton factors.

This is consistent with the theory, because the Chan-Paton factors have a Hamiltonian which do not posses dynamical structure. So, if one endpoint to the string is prepared in a certain state, it always will remains the same. It can be deduced from this, that \( \lambda^{a} \rightarrow U\lambda^{a}U^{-1} \) with \( U \in U(N) \). Thus, the worldsheet theory is symmetric under \( U(N) \), and this global symmetry is a gauge symmetry in spacetime. So the vector state at massless level \( \alpha_{-1}^I | k; a \rangle \) is a \( U(N) \) gauge boson.

When we have a gauge configuration with non trivial line integral around a compactified dimension (i.e a circle), we said there is a Wilson line. In case of open strings with gauge group \( U(N) \), a toroidal compactification of the 25th dimension on a circle of radius \( R \). If we choose a background field \( A^{25} \) given by \( A^{25} = \frac{1}{2\pi R} diag(\theta_1, ..., \theta_N) \) a Wilson line appears. Moreover, if \( \theta_i = 0, i = 1, ..., l \) and \( \theta_j \neq 0, j = l + 1, ..., N \) then gauge group is broken: \( U(N) \rightarrow U(l) \times U(1)^{N-l} \). It is possible to deduce that \( \theta_i \) plays the role of a Higgs field. Because string states with Chan-Paton quantum numbers \( | ij \rangle \) have charges 1 under \( i^{th} U(1) \) factor (and \(-1\) under \( j^{th} U(1) \) factor) and neutral with all others; canonical
momentum is given now by \( P_{ij}^{25} \Rightarrow \frac{n}{R} + \frac{\theta_j - \theta_i}{2\pi R} \). Returning to the mass shell condition it results,

\[
M_{ij}^2 = \left( \frac{n}{R} + \frac{\theta_j - \theta_i}{2\pi R} \right)^2 + \frac{1}{\alpha'}(N - 1). \tag{4.7}
\]

Massless states \((N = 1, n = 0)\) are those in where \( i = j \) (diagonal terms) or for which \( \theta_j = \theta_i \) \((i \neq j)\). Now, T-dualizing we have \( \tilde{X}_{ij}^{25}(\sigma, \tau) = a + (2n + \frac{\theta_j - \theta_i}{\pi})\tilde{R}\sigma + \text{oscillator terms.} \) Taking \( a = \theta_i\tilde{R}, \tilde{X}_{ij}^{25}(0, \tau) = \theta_i\tilde{R} \) and \( \tilde{X}_{ij}^{25}(\pi, \tau) = 2\pi n\tilde{R} + \theta_j\tilde{R} \). This give us a set of \( N \) D-branes whose positions are given by \( \theta_j\tilde{R} \), and each set is separated from its initial positions \((\theta_j = 0)\) by a factor equal to \( 2\pi\tilde{R} \). Open strings with both endpoints on the same D-brane gives massless gauge bosons. The set of \( N \) D-branes give us \( U(1)^N \) gauge group. An open string with one endpoint in one D-brane, and the other endpoint in a different D-brane, yields a massive state with \( M \sim (\theta_j - \theta_i)\tilde{R} \). Mass decreases when two different D-branes approximate to each other, and are null when become the same. When all D-branes take up the same position, the gauge group is enhanced from \( U(1)^N \) to \( U(N) \). On the D-brane world-volume there are also scalar fields in the adjoint representation of the gauge group \( U(N) \). The scalars parametrize the transverse positions of the D-brane in the target space \( X \).

**D-Brane Action**

With the massless spectrum on the D-brane world-volume it is possible to construct a low energy effective action. Open strings massless fields are interacting with the closed strings massless spectrum from the NS-NS sector. Let \( \xi^a \) (with \( a = 0, \ldots, p \)) be the world-volume coordinates on \( W \). The effective action is the gauge invariant action well known as the Dirac-Born-Infeld (DBI)-action

\[
S_D = -T_p \int_W d^{p+1}\xi e^{-\Phi} \sqrt{\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}, \tag{4.8}
\]

where \( T_p \) is the tension of the D-brane, \( G_{ab} \) is the world-volume induced metric, \( B_{ab} \) is the induced antisymmetric field, \( F_{ab} \) is the Abelian field strength on \( W \) and \( \Phi \) is the dilaton field.

For \( N \) D-branes the massless fields turns out to be \( N \times N \) matrices and the action turns out to be non-Abelian DBI-action (for a nice review about the Born-Infeld action in string theory see [21]).
\[ S_D = -T_p \int_W d^{p+1} \xi e^{-\Phi} Tr \left( \sqrt{\text{det}(G_{ab} + B_{ab} + 2\pi \alpha' F_{ab})} + O([X^m, X^n]^2) \right) \] (4.9)

where \( m, n = p + 1, \ldots, 9 \). The scalar fields \( X^m \) representing the transverse positions become \( N \times N \) matrices and so, the spacetime become a noncommutative spacetime. We will come back later to this interesting point.

**Ramond-Ramond Charges**

D-branes are coupled to Ramond-Ramond (RR) fields \( G_p \) [15]. The complete effective action on the D-brane world-volume \( W \) which taking into account this coupling is

\[ S_D = -T_p \int_W d^{p+1} \xi \left\{ e^{-\Phi} \sqrt{\text{det}(G_{ab} + B_{ab} + 2\pi \alpha' F_{ab})} + i \mu_p \int_W \sum_p C_{(p+1)} Tr \left( e^{2\pi \alpha'(F + B)} \right) \right\} \] (4.10)

where \( \mu_p \) is the RR charge. RR charges can be computed by considering the anomalous behavior of the action at intersections of D-branes. Thus RR charge is given by

\[ Q_{RR} = \text{ch}(j! E) \sqrt{\hat{A}(TX)}, \] (4.11)

where \( j : W \hookrightarrow X \). Here \( E \) is the Chan-Paton bundle over \( X \), \( \hat{A}(TX) \) is the genus of the spacetime manifold \( X \). This gives an ample evidence that the RR charges take values not in a cohomology theory, but in fact, in a K-Theory. This result was developed by Witten in Ref. [22] in the context of non-BPS brane configurations worked out by Sen [23].

Finally, RR charges and RR fields do admit a classification in terms of topological K-theory. The inclusion of a \( B \)-field turns out the effective theory non-commutative and a suitable generalization of the topological K-theory is needed. The right generalization seems to be the K-Homology and the K-theory of \( C^* \) algebras [24]. This subject is right now under intensive investigation.

### 4.2. D-brane Configurations and Susy Gauge Theories

As an application of the D-brane theory we consider in this section the brane box models, in particular we focus on the cube model discussed in [25].
The dynamics of D-branes in certain configurations of intersecting branes encodes many field-theoretical facts about supersymmetric theories in several dimensions (for a review, see [19]). Gauge theories in \((p + 1)\) dimensions with sixteen supercharges can be obtained as the world-volume theories of flat infinite Dp-branes. In the context of theories with eight supersymmetries in \(p\) dimensions, it was shown in [26] that such theories can be realized by considering Dp-branes with a world-volume which is finite in one direction, in which the D-brane ends on NS fivebranes. The brane is suspended between NS fivebranes spanning 012345. The low energy theory in the non-compact dimensions of the D-brane is \(p\)-dimensional. It is still a gauge theory, but the presence of the NS branes breaks half of the supersymmetries, so eight supercharges remain. This construction has been generalized in several directions, and has yielded the realization of a large family of models in several dimensions. This setup has also been exploited to compute different exact quantum results in these theories. For a review of such achievements, see [19].

A nice property of the interplay of field theories and configurations of branes is that the intersections of branes can sometimes support chiral zero modes. This opens the possibility of studying chiral gauge theories using branes. The simplest such example is provided by the realization of six-dimensional theories with eight supersymmetries, which are chiral. These can be realized in the setup described above by taking \(p = 6\), i.e. one considers D6 branes extending along 0123456, and which are bounded in 6 by NS branes with world-volume along 012345.

Chirality is a fragile property, in the sense that toroidal compactifications or too much supersymmetry spoil it. Thus, in order to obtain chiral theories in four dimensions one has to consider theories with only four supercharges. Their realization in terms of branes requires new ingredients. A fairly general family of brane configurations realizing generically chiral gauge theories in four dimensions was introduced in Ref. [27].

The idea is a clever extension of the philosophy in [26]. It consists in realizing first a five-dimensional theory with eight supercharges, by using D5 branes along 012346, suspended between NS branes with world-volume along 012345. Then, the D5 brane is bounded in the direction 4, by using a new set of NS branes oriented along 012367 (denoted NS' branes). The low energy theory is four-dimensional, since the world-volume of the D5 brane along 46 is a finite rectangle. Such configurations are known as brane box models. The presence of the new kind of branes breaks a further half of the supersymmetries, and so the theory has only four supercharges. Furthermore, the intersections of NS, NS'
and D5 branes introduce chirality in the four dimensional theory. There is no complete understanding of the quantum effects of these gauge theories in terms of branes.

*The Cube Brane Box Configurations and Susy Theories in Two Dimensions*

Here we introduce certain supersymmetric configurations of NS, NS', and NS'' branes, and D4 branes in Type IIA superstring theory. They give rise to two-dimensional (0,2) field theories. These configurations are obtained in the spirit of the brane box configurations in [27], by considering D-branes which are finite in several directions. As explained before, they belong to a natural sequence of brane box models yielding chiral theories in six, four and two dimensions (taking D branes compact in one, two and three directions, respectively).

Let us consider the ingredients of the brane configurations which we will use in this paper. Brane configurations consist of:

- NS fivebranes located along (012345).
- NS' fivebranes located along (012367).
- NS'' fivebranes located along (014567).
- D4 branes located along (01246).

In this configuration the D4 branes are finite in the directions 246. They are bounded in the direction 2 by the NS'' branes, in the direction 4 by the NS' branes, and in the direction 6 by the NS branes. For the D4 branes to be suspended in this way, it is necessary that the coordinates of all branes in 89 should be equal. It is also required that two NS branes joined by a D4 brane should have the same position in 7, and analogously that two NS' branes joined by a D4 brane should have the same position in 5, and that two NS'' branes should have the same position in 3.

The low-energy effective field theory on the D4 branes is two-dimensional, since 01 are the only non-compact directions in their world-volume. The presence of each kind of NS fivebrane breaks one half of the supersymmetries, and altogether they break to 1/8 of the original supersymmetry. A further half is broken by the D4 branes, and the world-volume theory has (0,2) supersymmetry in two dimensions. Since the D brane is bounded by NS fivebranes, the world-volume gauge bosons will not be projected out and there will be a gauge group for each box in the model. The $U(1)_R$ R-symmetry of the field theory is manifest as the rotational symmetry in the directions 89.

We note that there are a variety of other objects that can be introduced in the configuration without breaking the supersymmetry. For instance, there are three kinds of
D6 branes that can be introduced, namely D6 branes along 0124789, D6′ branes along 0125689, and D6″ branes along 0134689. They provide vector-like flavors for the gauge groups. These extensions are quite well-known from other contexts, and we will not study them in the present paper.

There is a first rough classification we can make in these brane configurations, according to whether the directions 246 are taken compact or not. If some of these directions are non-compact, then there will be some semi-infinite box, which will represent some global symmetry. For definiteness we will center on the case in which all three directions are compact, with lengths $R_2$, $R_4$ and $R_6$. Extension of our results to other cases is straightforward.

A generic configuration consists of a three-dimensional grid of $k$ NS branes, $k'$ NS' branes and $k''$ NS'' branes dividing the 246 torus into a set of $kk'k''$ boxes. We will often think about these configurations as infinite periodic arrays of boxes in $\mathbb{R}^3$, quotiented by an infinite discrete group of translations in a three-dimensional lattice $\Lambda$. This point of view is particularly useful to define models in which the unit cell has non-trivial identifications of sides [28].

(0,2) Effective Theory on the D4 Brane

The effective field theory on the only non-compact directions 01 of the D4 branes world-volume is a (0,2) gauge theory in two dimensions. These theories are described in the (0,2) superspace $(y^\alpha, \theta^+, \eta^+ )$. There are three basic kinds of multiplets which we will use.

- The (0,2) gauge multiplet $V'$, which contains gauge bosons $v_\alpha$, $\alpha = 0, 1$, and one fermion $\chi_−$.
- The (0,2) chiral multiplet $\Phi'$, contains one complex scalar $\phi$ and one chiral fermion $\psi_+$.
- The (0,2) Fermi multiplet, $\Lambda$, is described by an anticommuting superfield. Its complete $\theta$ expansion contains a chiral spinor $\lambda_-$, an auxiliary field $G$, and a holomorphic function $E$ depending on the chiral (0,2) superfields $\Phi'$. The Fermi multiplet $\Lambda$ satisfies the constraint $\overline{D}_+ \Lambda = \sqrt{2} E(\Phi')$, with $\overline{D}_+ E = 0$. Here $\overline{D}_+$ represents the supersymmetric covariant derivative. The expansion in components for the Fermi superfield is

$$\Lambda = \lambda_− - \sqrt{2}\theta^+ G - i\theta^+ \eta^+ (D_0 + D_1) \lambda_− - \sqrt{2}\theta^+ E(\Phi')$$

(4.12)
with $D_\alpha$ denoting the usual supersymmetric derivative.

Gauge theories involving these fields are described by a Lagrangian with the

$$L = L_{\text{gauge}} + L_{\text{ch}} + L_F + L_{D,\theta} + L_J.$$  \hfill (4.13)

As usual, $L_{gauge}$ is the kinetic term of the gauge multiplet given by

$$L_{\text{gauge}} = \frac{1}{8g^2} \int d^2 y d\theta^+ d\overline{\theta}^+ \text{Tr}(\overline{\Upsilon} \Upsilon)$$  \hfill (4.14)

where $\Upsilon$ is the field strength of $V'$.

The term $L_{ch}$ contains the kinetic energy and gauge couplings of the $(0,2)$ chiral superfields $\Phi'_i$. It is given by

$$L_{\text{ch}} = -\frac{i}{2} \int d^2 y d^2 \theta \sum_i \left( \overline{\Phi}'_i (D_0 + D_1) \Phi'_i \right),$$  \hfill (4.15)

where $D_0$ and $D_1$ are the $(0,2)$ gauge covariant derivatives with respect to $V'$.

The term $L_F$ describes the dynamics of the Fermi multiplets $\Lambda$, and certain interactions. It is given by

$$L_F = -\frac{1}{2} \int d^2 y d^2 \theta \sum_a (\overline{\Lambda}_a \Lambda_a).$$  \hfill (4.16)

Substitution of (4.12) into (4.16) leads to

$$L_F = \int d^2 y \sum_a \left\{ i \overline{\lambda}_{-\alpha} (D_0 + D_1) \lambda_{-\alpha} + |G_a|^2 - |E_a|^2 - \sum_j \left( \overline{\lambda}_{-\alpha} \frac{\partial E_a}{\partial \phi_j} \psi_{+,j} + \overline{\partial E_a} \frac{\partial \phi_j}{\partial \phi_j} \psi_{-,j} \right) \right\}.$$  \hfill (4.17)

The Fayet-Iliopoulos and theta angle terms are encoded in the $(0,2)$ Lagrangian $L_{D,\theta}$ which is written as

$$L_{D,\theta} = \frac{t}{4} \int d^2 y d\theta^+ \text{Tr}(\Upsilon|_{\overline{\theta}^+ = 0}) + h.c.$$  \hfill (4.18)

where $t = \frac{\theta}{2\pi} + ir$.

Finally $(0,2)$ models do admit an additional interaction term $L_J$ which depends on a set of holomorphic functions $J^a(\Phi')$ of the chiral superfields. There is one such function for each Fermi superfield. They satisfy the relation $\sum_a E_a J^a = 0$. This interaction is the $(0,2)$ analog of the superpotential, and its Lagrangian $L_J$ is given by

$$\text{...}$$
\[ L_J = -\frac{1}{\sqrt{2}} \int d^2 y d\theta^+ \sum_a \left( \Lambda_a J^a_{\theta^+ = 0} \right) - h.c. \] \hspace{1cm} (4.19)

The expansion of this term in components is

\[ L_J = -\int d^2 y \sum_a \left( G^a J^a + \sum_j \lambda_{-,a} \psi_{+,j} \frac{\partial J^a}{\partial \phi_j} \right) - h.c. \] \hspace{1cm} (4.20)

After combining the Lagrangians \( L_F \) and \( L_J \) and solving for the equations of motion for the auxiliary fields \( G \), the relevant interaction terms in the Lagrangian (we are not listing the gauge interactions and D-terms here) are

\[ \sum_a \left( |J^a(\phi)|^2 + |E^a(\phi)|^2 \right) - \sum_{a,j} \left( \lambda_{-,a} \psi_{+,j} \frac{\partial J^a}{\partial \phi_j} + \lambda_{+,a} \frac{\partial J^a}{\partial \phi_j} \psi_{+,j} + h.c. \right). \] \hspace{1cm} (4.21)

The first term contains the scalar potential, and the second the Yukawa couplings.

Notice that the choice of the functions \( E \) and \( J \) completely defines the interactions of the theory.

For (0,2) theories in two dimensions we have just one U(1) R-symmetry group acting on the superspace coordinates \((\theta^+, \bar{\theta}^+)\). This is right-moving R-symmetry and it acts as \( \theta^+ \to e^{i\beta} \theta^+ \), \( \bar{\theta}^+ \to e^{-i\beta} \bar{\theta}^+ \), leaving \( \theta^- \), \( \bar{\theta}^- \) invariant.

**Interpretation of the Linear Sigma Model**

Up to here we have introduced a large family of two-dimensional (0, 2) gauge theories. Since (2, 2) and (0, 2) theories have been traditionally used as world-sheet descriptions of string theories propagating on some target space, it is a natural question whether the (classical) Higgs branch or our models has any geometrical interpretation of the kind. In this section we are to show that it describes the dynamics of a type IIB D1 brane on a \( IC^4/\Gamma \) singularity, with \( \Gamma \) an abelian subgroup of \( SU(4) \). The main tool for reaching this conclusion will be a T-duality performed on the brane box model along the directions 246.

In this section we perform a T-duality on the brane box models along the directions 246. The main tool will be the well known T-duality relation between a set of \( n \) parallel NS fivebranes and \( n \) Kaluza-Klein monopoles. The discussion in this subsection parallels that in [28].
Let us start with the simplest case of a brane box model formed by a unit cell of \( k \times k' \times k'' \) boxes, with trivial identifications of faces. In this case the T-duality along the directions 246 is particularly easy. We start with \( k \) NS branes along 012345, \( k' \) NS' branes along 012367, and \( k'' \) NS'' branes along 014567. The T-duality along 2 transforms the NS'' branes into \( k'' \) Kaluza-Klein monopoles. These will be described by a multi-center Taub-NUT metric, with non-trivial geometry on the directions 2',3,8,9, with 2' denoting the coordinate dual to 2. Notice that, since the 3,8,9 coordinates of the initial NS'' branes coincided, so do the coordinates of the corresponding \( k'' \) centers of the Taub-NUT metric, so that it contains singularities of type \( A_{k''-1} \).

Similarly, the T-duality along 4 transforms the \( k' \) NS' branes into \( k' \) Kaluza-Klein monopoles with world-volume along 012367, and represented by a nontrivial geometry on 4',5,8,9. Again, since the centers of the Kaluza-Klein monopoles coincide, such geometry will contain singularities of type \( A_{k'-1} \). Finally, the T-duality along 6 turns the \( k \) NS branes into \( k \) Kaluza-Klein monopoles. Their world-volume spans 012345, and they are represented by a non-trivial geometry along 6',7,8,9. Since again all the centers coincide, there will be \( A_{k-1} \) singularities.

Thus, the final T-dual of the grid of NS, NS' and NS'' branes is type IIB string theory with a complicated geometry in the directions 2',3,4',5,6',7,8,9. One can think of it roughly as some ‘superposition’ of the Kaluza-Klein monopoles we have described. Even without a quantitative knowledge of such metric, we can describe the relevant features for our purposes. One such feature is that the number of unbroken supersymmetries constrains the manifold to be a Calabi-Yau four-fold. Also, from our remarks above we know the existence of certain (complex) surfaces of singularities of type \( A_{k-1} \), \( A_{k'-1} \) and \( A_{k''-1} \) singularities. If we introduce complex coordinates \( w_1 = \exp(x^7 + ix^6') \), \( w_2 = \exp(x^5 + ix^4') \), \( w_3 = \exp(x^3 + ix^2') \), and \( w_4 = x^9 + ix^8 \), the surface of \( A_{k-1} \) singularities is defined roughly by \( w_1 = w_4 = 0 \), the surface of \( A_{k'-1} \) singularities is defined by \( w_2 = w_4 = 0 \), and the surface of \( A_{k''-1} \) singularities is given by \( w_3 = w_4 = 0 \). At the origin all surfaces meet and the singularity is worse. It can be described as a quotient singularity of type \( \mathbb{C}^4/\Gamma \), with \( \Gamma = \mathbb{Z}_k \times \mathbb{Z}_{k'} \times \mathbb{Z}_{k''} \). This discrete group is generated by elements \( \theta \), \( \omega \), \( \eta \), whose action on \((z_1,z_2,z_3,z_4) \in \mathbb{C}^4\) is as follows:

\[
\begin{align*}
\theta : \quad (z_1,z_2,z_3,z_4) & \rightarrow (e^{2\pi i/k}z_1,z_2,z_3,e^{-2\pi i/k}z_4) \\
\omega : \quad (z_1,z_2,z_3,z_4) & \rightarrow (z_1,e^{2\pi i/k'}z_2,z_3,e^{-2\pi i/k'}z_4) \\
\eta : \quad (z_1,z_2,z_3,z_4) & \rightarrow (z_1,z_2,e^{2\pi i/k''}z_3,e^{-2\pi i/k''}z_4).
\end{align*}
\]
In this description it becomes clear that there may be further surfaces of singularities when the greatest common divisor of any two of \( k, k', k'' \) is not 1, in analogy with the discussion in [28]. This will not be relevant for our purposes and we do not develop the issue further.

After the T-duality, the initial D4 branes become D1 branes located at a point in the four-fold. When the initial D4 branes are bounded by the grid of NS, NS', and NS'' branes, the T-dual D1 branes will be located precisely at the \( \mathbb{C}^4/\Gamma \) singular point. The field theories introduced previously correspond to the field theories appearing in the world-volume of such D1 brane probes. In addition, the structure of the singularity controls the spectrum and dynamics of the field theory (for a recent review see [29].

Brane boxes models generating gauge theories in two dimensions with enhanced chiral (0,4), (0,6) and (0,8) supersymmetry can be constructed and are also described at [25].

5. Non-perturbative String Theory

5.1. Strong-Weak Coupling String Duality

We have described the massless spectrum of the five consistent superstring theories in ten dimensions. Additional theories can be constructed in lower dimensions by compactification of some of the ten dimensions. Thus the ten-dimensional spacetime \( X \) looks like the product \( X = \mathcal{K}^d \times \mathbb{R}^{1,9-d} \), with \( \mathcal{K} \) a suitable compact manifold or orbifold. Depending on which compact space is taken, it will be the quantity of preserved supersymmetry.

All five theories and their compactifications are parametrized by: the string coupling constant \( g_s \), the geometry of the compact manifold \( \mathcal{K} \), the topology of \( \mathcal{K} \) and the spectrum of bosonic fields in the NS-NS and the R-R sectors. Thus one can define the string moduli space \( \mathcal{M} \) of each one of the theories as the space of all associated parameters. Moreover, it can be defined a map between two of these moduli spaces. The dual map is defined as the map \( S : \mathcal{M} \rightarrow \mathcal{M}' \) between the moduli spaces \( \mathcal{M} \) and \( \mathcal{M}' \) such that the strong/weak region of \( \mathcal{M} \) is interchanged with the weak/strong region of \( \mathcal{M}' \). One can define another map \( T : \mathcal{M} \rightarrow \mathcal{M}' \) which interchanges the volume \( V \) of \( \mathcal{K} \) for \( \frac{1}{V} \). One example of the map \( T \) is the equivalence, by T-duality, between the theories Type IIA compactified on \( S^1 \) at radius \( R \) and the Type IIB theory on \( S^1 \) at radius \( \frac{1}{R} \). The theories HE and HO constitutes another example of the \( T \) map. In this section we will follows the Sen’s review
Another useful references are [31,32,33,34,35]. Type IIB theory is self-dual with respect the $S$ map.

**Type IIB-IIB Duality**

The Type IIB theory is self-dual. In order to see that write the bosonic part of the action of Type IIB superstring theory

$$S_{\text{IIB}} = \frac{1}{2\kappa^2} \int_X d^{10}x \sqrt{-G} I I B e^{-2\Phi} \left( R + 4\partial_I \Phi \partial^I \Phi - \frac{1}{2} H_{IJK} H^{IJK} \right)$$

$$- \frac{1}{4\kappa^2} \int_X d^{10}x \sqrt{-G} I I B \left( |F(1)|^2 + |\tilde{F}(3)|^2 + \frac{1}{2} |\tilde{F}(5)|^2 \right) - \frac{1}{4\kappa^2} \int_X A_{(4)} \wedge H_{(3)} \wedge F_{(3)}, \quad (5.1)$$

where $\tilde{F}(3) = dA_{(2)} - a \wedge H_{(3)}$ and $\tilde{F}(5) = dA_{(4)} - \frac{1}{2} A_{(2)} \wedge H_{(3)} + \frac{1}{2} B_{(2)} \wedge F_{(3)}$.

This action is clearly invariant under

$$\Phi' = -\Phi, \quad G'_{IJ} = e^{-\Phi} G_{IJ},$$

$$B_{(2)} = A_{(2)} \quad A'_{(2)} = -B_{(2)}, \quad A'_{(4)} = A_{(4)}. \quad (5.2)$$

This symmetry leads to an identification of a fundamental string $F_1$ with a D1-brane ($B_{(2)} = A_{(2)}$) and the interchanging of a pair of D3-branes.

**Type I-SO(32)-Heterotic Duality**

In order to analyze the duality between Type I and SO(32) heterotic string theories we recall the spectrum of both theories. These fields are the dynamical fields of a supergravity Lagrangian in ten dimensions. Type I string theory has in the NS-NS sector the following fields: the metric $G_{IJ}$, the dilaton $\Phi$ and in the R-R sector: the antisymmetric tensor $A_{IJ}$. Also there are 496 gauge bosons $A_{I}^{\text{adj}}$ in the adjoint representation of the gauge group SO(32). For the SO(32) heterotic string theory the spectrum consist of: the spacetime metric $G_{IJ}^{H}$, the dilaton field $\Phi^{H}$, the antisymmetric tensor $B_{IJ}^{H}$ and 496 gauge fields $A_{I}^{aH}$ in the adjoint representation of SO(32). Both theories have $\mathcal{N} = 1$ spacetime supersymmetry.

The effective action for the massless fields of the Type I supergravity effective action $S_I$ is defined as

$$S_I = \frac{1}{2\kappa^2} \int_X d^{10}x \sqrt{-G} I I e^{-2\Phi} \left( R + 4(\nabla \Phi)^2 - \frac{1}{12} |\tilde{F}(3)|^2 \right)$$

$$41$$
\[ -\frac{1}{g^2} \int_X d^{10}x \sqrt{-G} e^{-\Phi} Tr(F_{IJ}^I F^{IJ}) \]  

(5.3)

where \( \tilde{F}^{(3)}_I = F^{(3)}_I - \frac{\alpha'}{4}[\omega_3 (A) - \omega_3 (\omega)] \).

While the heterotic action \( S_H \) is defined as

\[
S_H = \frac{1}{2\kappa^2} \int_X d^{10}x \sqrt{-G} e^{-2\Phi} \left[ R + 4(\nabla \Phi)^2 - \frac{1}{12} |\tilde{H}^{(3)}|^2 - \frac{\alpha'}{8} Tr(F_{IJ}^H F^{HIJ}) \right] 
\]

(5.4)

where \( \tilde{H}^{(3)} = dB^{(2)} - \frac{\alpha'}{4}[\omega_3 (A) - \omega_3 (\omega)] \).

The comparison of these two actions leads to the following identification of the fields

\[
G_{I,J}^I = e^{-\Phi^H} G_{I,J}^H, \quad \Phi^I = -\Phi^H, \\
A_{a}^I = A_{a}^H, \quad \tilde{F}^{I}_I = \tilde{H}^{H} \]  

(5.5)

This gives us many information. The first relation tells us that the metrics of both theories are the same. The second relation interchanges the \( B_{(2)} \) field in the \text{NS-NS} sector and the \( A_{(2)} \) field in the \text{R-R} sector. That implies the interchanging of heterotic strings and Type I D1-branes. The third relation identifies the gauge fields coming from the Chan-Paton factors from the Type I side with the gauge fields coming from the 16 compactified internal dimensions of the heterotic string. Finally, the opposite sign for the dilaton relation means that the string coupling constant \( g_i^I \) is inverted \( g_h^H = 1/g_i^I \) within this identification, and interchanges the strong and weak couplings of both theories leading to the explicit realization of the \( S \) map.

5.2. M-Theory

We have described how to construct dual pairs of string theories. By the uses of the \( S \) and the \( T \) maps a network of theories can be constructed in various dimensions all of them related by dualities. However new theories can emerge from this picture, this is the case of M-theory. M-theory (the name come from ‘mystery’, ‘magic’, ‘matrix’, ‘membrane’, etc.) was originally defined as the strong coupling limit for Type IIA string theory [31].

It is known that Type IIA theory can be obtained from the dimensional reduction of the eleven dimensional supergravity theory (a theory known from the 70’s years) and given by the Cremmer-Julia-Scherk Lagrangian.
\[ I_{11} = \frac{1}{2\kappa_{11}^2} \int_Y d^{11}x \sqrt{-G_{11}} \left( R - |dA_3|^2 \right) - \frac{1}{6} \int_Y A_{(3)} \wedge F(4) \wedge F(4), \]  

(5.6)

where \( Y \) is the eleven dimensional manifold. If we assume that the eleven-dimensional spacetime factorizes as \( Y = X \times S^1_R \), where the compact dimension has radius \( R \). Usual Kaluza-Klein dimensional reduction leads to get the ten-dimensional metric, an scalar field and a vector field. \( A_{(3)} \) from the eleven dimensional theory leads to \( A_{(3)} \) and \( A_{(2)} \) in the ten-dimensional theory. The scalar field turn out too be proportional to the dilaton field of the \textbf{NS-NS} sector of the Type IIA theory. The vector field from the KK compactification can be identified with the \( A^{\text{IIA}} \) field of the \textbf{R-R} sector. From the three-form in eleven dimensions are obtained the RR field \( A_{(3)} \) of the Type II A theory. Fin ally, the \( A_{(2)} \) field is identified with the NS-NS B-field of field strength \( H_{(3)} = dB_{(2)} \). Thus the eleven-dimensional Lagrangian leads to the Type II A supergravity in the weak coupling limit (\( \Phi \rightarrow 0 \) or \( R \rightarrow 0 \)). The ten-dimensional IIA supergravity describing the bosonic part of the low energy limit of the Type IIA superstring theory is

\[ S_{\text{IIA}} = \frac{1}{2\kappa^2} \int_X d^{10}x \sqrt{-G^{\text{IIA}}} e^{-2\Phi^{\text{IIA}}} \left( R + 4(\nabla \Phi^{\text{IIA}})^2 - \frac{1}{12} |H_{(3)}|^2 \right) \]

\[- \frac{1}{4\kappa^2} \int_X d^{10}x \sqrt{-G^{\text{IIA}}} \left( |F_{(2)}|^2 + |\tilde{F}_{(4)}|^2 \right) - \frac{1}{4\kappa^2} \int_X B_{(2)} \wedge dA_{(3)} \wedge dA_{(3)} \]  

(5.7)

where \( H_{(3)} = dB_{(2)}, \) \( F_{(2)} = dA_{(1)} \) and \( \tilde{F}_{(4)} = dA_{(3)} - A_{(1)} \wedge H_{(3)} \).

It is conjectured that there exist an eleven dimensional fundamental theory whose low energy limit is the 11 dimensional supergravity theory and that it is the strong coupling limit of the Type II A superstring theory. At the present time the degrees of freedom (dof’s) are still unknown, though at the macroscopic level they should be membranes and fivebranes (also called M2-branes and M5-branes).

\section{Horava-Witten Theory}

Just as the M-theory compactification on \( S^1_R \) leads to the Type IIA theory, Horava and Witten realized that orbifold compactifications leads to the \( E_8 \times E_8 \) heterotic theory in ten dimensions \( HE \) (see for instance \([33]\)). More precisely

\[ M/(S^1/\mathbb{Z}_2) \leftrightarrow HE \]  

(5.8)
where $S^1/\mathbb{Z}_2$ is homeomorphic to the finite interval $I$ and the $M$-theory is thus defined on $Y = X \times I$. From the ten-dimensional point of view, this configuration is seen as two parallel planes placed at the two boundaries $\partial I$ of $I$. Dimensional reduction and anomalies cancellation conditions imply that the gauge degrees of freedom should be trapped on the ten-dimensional planes $X$ with the gauge group being $E_8$ in each plane. While that the gravity is propagating in the bulk and thus both copies of $X$’s are only connected gravitationally.

5.4. F-Theory

$F$-Theory was formulated by C. Vafa, looking for an analog theory to M-Theory for describing non-perturbative compactifications of Type IIB theory (for a review see [34,30]). Usually in perturbative compactifications the parameter $\lambda = a + iexp(-\Phi/2)$ is taken to be constant. $F$-theory generalizes this fact by considering variable $\lambda$. Thus $F$-theory is defined as a twelve-dimensional theory whose compactification on the elliptic fibration $T^2 - \mathcal{K} \to B$, gives the Type IIB theory compactified on $B$ (for a suitable space $B$) with the identification of $\lambda(z)$ with the modulus $\tau(z)$ of the torus $T^2$. These compactifications can be related to the $M$-theory compactifications through the known $S$ mapping $S : IIA \to M/S^1$ and the $T$ map between Type IIA and IIB theories. This gives

$$F/\mathcal{K} \times S^1 \leftrightarrow M/\mathcal{K}.$$  \hspace{1cm} (5.9)

Thus the spectrum of massless states of $F$-theory compactifications can be described in terms of $M$-theory. Other interesting $F$-theory compactifications are the Calabi-Yau compactifications

$$F/CY \leftrightarrow H/K3.$$  \hspace{1cm} (5.10)
6. Non-perturbative Calabi-Yau Compactifications

M-theory Vacua

In this section we review some Calabi-Yau compactifications of M and F-Theories. In the first part of these lecture we described the perturbative CY compactifications, the purpose of the present section is see how these compactifications behaves in the light of duality and D-brane theory (for excellent reviews see [36,37]). The presence of D-branes or M-branes, in the case of M theory, modifies the perturbative CY compactifications, here we briefly describe these modifications.

Assume that the eleven-dimensional spacetime is $Y = M \times S^1/\mathbb{Z}_2 \times \mathcal{K}$, with $\mathcal{K}$ being a Calabi-Yau three-fold. Here we consider that $\mathcal{K}$ is a elliptic fibration, since they are favored by CY compactifications of M and F theories. These spacetime corresponds of having two copies (planes) of $X = M \times \mathcal{K}$ at the two boundaries of the orbifold. According to the Horava-Witten theory, anomalies cancellation involves that one $\mathcal{N} = 1$ vector supermultiplet of the $E_8$ super Yang-Mills theory has to be captured in each orbifold fixed plane $X_i$, $i = 1, 2$.

According to the perturbative description it is necessary to specify now two stable or semi-stable holomorphic vector bundles $V_i$ on $\mathcal{K}$ with arbitrary group structure. For the heterotic-M theory compactifications the structure group has to be a subgroup of $E_8$. For simplicity we restrict ourselves to $SU(n_i)$ vector bundles $V_i$ over $\mathcal{K}$. The presence of fivebranes is of extreme importance here, since it allows more flexibility to construct such vector bundles $V_i$ which leads to more realistic particle physics models. From the modified Bianchi identity and the anomaly cancellation condition of the orbifold system and the fivebranes wrapped on holomorphic two-cycles of $\mathcal{K}$, leads that these bundles are subject to the cohomological constraint of the second Chern classes $c_2(V_1) + c_2(V_2) + [W] = c_2(T\mathcal{K})$, where $[W]$ is the topological class associated to the fivebranes.

The description of the low-energy physics requires of the computation of the first three Chern classes of the holomorphic bundles $V_i$ over $\mathcal{K}$ and thus determine completely a non-perturbative vacuum. M and F theories compactifications require that $\mathcal{K}$ must be a holomorphic elliptic fibration. Thus the construction of these bundles are nontrivial.

Construction of the Gauge Bundles over Elliptic Fibrations

An holomorphic elliptically fibered Calabi-Yau three-fold is a fibration
$\mathcal{K} \xrightarrow{\pi} B$

where $B$ is an auxiliary complex two-dimensional manifold, $\pi$ is an holomorphic mapping, and for each $b \in B$, $\pi^{-1}\{b\}$ is isomorphic to an elliptic curve $E_b$. In addition we require the existence of a global section $\sigma : B \to \mathcal{K}$ of this fibration.

The elliptic fibration can be characterized by a single line bundle $\mathcal{L}$ over $B$, $\mathcal{L} \to B$, whose fiber is the cotangent space to the elliptic curve, $T^*E_b$. This bundle satisfies the condition: $\mathcal{L} = K_B^{-1}$ with $K_B$ being the canonical bundle of $B$, under the usual condition that the canonical bundle $K_\mathcal{K}$ has vanishing first Chern class $c_1(K_\mathcal{K}) = 0$. While the global section is specified giving the bundles $K_B^{-\otimes 4}$ and $K_B^{-\otimes 6}$.

These conditions are known to be satisfied by base spaces $B$ corresponding to del Pezzo, Hirzebruch and Enriques surfaces.

For elliptic fibrations, Friedman, Morgan and Witten [38] found that the second Chern class of the holomorphic tangent bundle $T\mathcal{K}$ can be written in terms of the Chern classes of $B$ as follows

$$c_2(T\mathcal{K}) = c_2(B) + 11c_1^2(B) + 12\sigma c_1(B), \quad (6.1)$$

where $c_1(B)$ and $c_2(B)$ are the first and the second class of $B$ and $\sigma$ is a two-form and represents the Poincaré dual of mentioned global section of the elliptic fibration.

One can construct the semi-stable $SU(n_i)$ holomorphic bundles $V_i$ on $\mathcal{K}$ through the specification of two line bundles $\hat{\mathcal{L}}$ with first Chern class $\eta \equiv c_1(\hat{\mathcal{L}})$ and $\hat{\mathcal{W}}$ with corresponding first Chern class $c_1(\hat{\mathcal{W}})$ depending on some parameters $n, \sigma, c_1(B), \eta$ and $\lambda$. Thus the bundle $\hat{\mathcal{W}}$ is completely specified by the elliptic fibration and the line bundle $\hat{\mathcal{L}}$.

The condition that $c_1(\hat{\mathcal{W}}) \in \mathbb{Z}$ leads to the relation $\lambda = m + \frac{1}{2}$ for $n$ odd and $\lambda = m$ and $\eta = c_1(B) \mod 2$, for $n$ even, $m \in \mathbb{Z}$. Thus the Chern classes of the $SU(n)$ gauge bundle $V$ are

- $c_1(V) = 0$
- $c_2(V) = \eta \sigma - \frac{1}{24}c_1^2(B)(n^3 - n) + \frac{1}{2}(\lambda^2 - \frac{1}{4})n\eta(\eta - nc_1(B))$
- $c_3(V) = 2\lambda \sigma \eta(\eta - nc_1(B))$.

In order to construct realistic particle physics models we take a given base space $B$ and compute its corresponding Chern classes $c_1(B)$ and $c_2(B)$. Compute the relevant Chern classes of the $SU(n)$ gauge bundles $V$. The constraints above reduce the number of
consistent physical non-perturbative vacua. Given appropriate $\eta$ and $\lambda$ one can determine completely the physical Chern classes.

Counting the Number of Families

The number of families of leptons and quarks of the four-dimensional theory is related to the number of zero modes of the Dirac operator coupled to gauge field, which is a connection on the $SU(n)$ bundle $V$ on the elliptic fibered Calabi-Yau three-fold $\mathcal{K}$. In order to count the number of the chiral fermionic zero modes, one can consider the following cases:

- $SU(3) \times E_6 \subset E_8 : 248 = (8, 1) \oplus (1, 78) \oplus (3, 27) \oplus (3^*, 27^*)$.
- $SU(4) \times SO(10) \subset E_8 : 248 = (15, 1) \oplus (1, 45) \oplus (4, 16) \oplus (4^*, 16^*)$.
- $SU(5) \times SU(5) \subset E_8 : 248 = (24, 1) \oplus (1, 24) \oplus (10, 5) \oplus (10^*, 5^*) \oplus (5, 10^*) \oplus (5^*, 10)$.

The matter representations appear in the fundamental representation of the gauge group $SU(n)$. The index of the Dirac operator gives

$$\delta = \text{index}(\mathcal{D}_\mathcal{K}) = \int_{\mathcal{K}} ch(V) td(\mathcal{K}) = \frac{1}{2} \int_{\mathcal{K}} c_3(V) \tag{6.2}$$

where $td(\mathcal{K})$ is the Todd class of $\mathcal{K}$. From explicit formula for $c_3(V)$ we get that the number of generations is given by $\delta = \lambda \eta (\eta - nc_1(B))$.

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