Introduction

The language of superconnections

Our formulation has a natural interpretation in the Cheeger-Simons theory written by
in Spin(7). As an example of the metric is the Calabi-Yau manifold on the sphere that
in dimensions. We present a new construction of the Ricci flat metric with holonomy
involving volume-preserving vector fields. We give special attention to the case of 5-
We discuss the higher dimensional generalization of gravitational instantons by

Abstract

Spin(7) holonomy manifold and superconnection

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where $\Omega_{\alpha\beta\gamma\delta}$ is a duality operator, identified with the components of a certain 4-form (Cayley 4-form) on an 8-dimensional manifold [6, 3]. The manifolds with this “self-duality” condition have holonomy in $\text{Spin}(7)$ and can be thought of as 8-dimensional gravitational instantons.

In 4-dimensions, there is another construction of gravitational instantons (hyperkähler manifolds) based on Ashtekar gravity. In this approach the instantons are given by the solutions to the differential equations for volume-preserving vector fields. The following proposition summarizes the results of [7, 8, 9, 10] relevant to this construction.

**Proposition 1.** Let $(M, \omega)$ be a 4-dimensional manifold with volume form $\omega$ and $V_\mu (\mu = 0, 1, 2, 3)$ be linearly independent four vector fields satisfying the conditions:

\begin{align}
(\text{a}) \quad L_{V_\mu} \omega &= 0, \\
(\text{b}) \quad \tilde{\eta}^a_{\mu \nu} [V_\mu, V_\nu] &= 0,
\end{align}

where $\tilde{\eta}^a_{\mu \nu}$ $(a = 1, 2, 3)$ are the ’t Hooft matrices defined as

\begin{align}
\tilde{\eta}^a_{i,j} &= \epsilon_{aij}, \\
\tilde{\eta}^a_{ik} &= \delta_{ai} \quad (a, i, j = 1, 2, 3), \\
\tilde{\eta}^a_{\mu \nu} &= -\tilde{\eta}^a_{\nu \mu}.
\end{align}

Then these vector fields induce a hyperkähler metric on $M$:

\begin{align}
g = \phi \ W^\mu \otimes W^\mu, \quad \phi = \omega(V_0, V_1, V_2, V_3),
\end{align}

where $W^\mu$ is the dual basis of $V_\mu$ and the 2-forms

\begin{align}
\Sigma^a = \frac{1}{2} \phi \ \tilde{\eta}^a_{\mu \nu} W^\mu \wedge W^\nu \quad (a = 1, 2, 3)
\end{align}

give the hyperkähler forms.

This proposition yields that the vector fields $V_\mu$ may be identified with the components of a spacetime-independent Yang-Mills connection on $\mathbb{R}^4$. Indeed, the condition (a) asserts that the gauge group is the diffeomorphism group on $M$ preserving the volume form $\omega$, and the condition (b) is equivalent to the reduced anti-self-dual Yang-Mills equation.

Now we can ask the question whether it is possible to generalize the proposition 1 to the case of $\text{Spin}(7)$ holonomy manifolds. The purpose of this paper is to give an answer to this question. In Section 2 we provide a new construction of $\text{Spin}(7)$ holonomy manifolds including the brief review of the geometry for these manifolds. Our formulation actually gives an 8-dimensional generalization of the Proposition 1, and the result is summarized in the Proposition 2. As an example we also present a metric with holonomy $\text{Spin}(7)$ on the 8-dimensional space $\mathbb{R}^2 \times S^3 \times S^3$. This is the same as the metric constructed by using different methods [11, 12]. In Section 3 we give an interpretation of the Proposition 2 from the viewpoint of 8-dimensional Yang-Mills theory (generalized Chern-Simons theory). We then formulate the theory using the superconnection introduced by Quillen to describe the characteristic classes of fiber bundles. The solutions to the equations of motion are directly related to the metrics of $\text{Spin}(7)$ holonomy manifolds by 0-dimensional reduction. Section 4 is devoted to discussion. We argue the application of our formulation to $G_2$ holonomy manifolds and also make a remark on higher dimensional Ashtekar gravity.
2 Spin(7) holonomy manifold

We begin by reviewing some of the properties of Spin(7) holonomy manifold $M$ that we shall need in this paper [12, 13]. Such a manifold is 8-dimensional and admits a covariant constant Majorana-Weyl spinor $\zeta$ invariant under the action of Spin(7). It is interesting to note that the existence of $\zeta$ automatically implies that the manifold $M$ must be Ricci-flat. According to the standard isomorphism between the space of forms and tensor product of the Clifford module, one can immediately construct a 4-form $\Omega$ on $M$, with components given by

$$\Omega_{\alpha\beta\gamma\delta} = \zeta^T \Gamma_{\alpha\beta\gamma\delta} \zeta, \quad (2.1)$$

where $\Gamma_{\alpha\beta\gamma\delta}$ denotes the anti-symmetrized 4-fold product of the $\gamma$-matrices $\Gamma_{\alpha}$ in 8-dimensions. This 4-form enjoys a couple of remarkable properties; (a) closedness, (b) self-duality with respect to usual Hodge star operator, (c) Spin(7)-invariance. Conversely, a manifold has holonomy in Spin(7) if there exists a 4-form $\Omega$ satisfying the conditions (a), (b) and (c). Such a 4-form $\Omega$ is known as the Cayley 4-form.

2.1 Cayley 4-form

The purpose of this section is to study Spin(7) holonomy manifolds by using volume-preserving vector fields. We will see that these vector fields induce the Cayley 4-form on an 8-dimensional space.

Before stating our proposition, we give a brief exposition of the Cayley 4-form

$$\Omega = \frac{1}{4!} \Omega_{\alpha\beta\gamma\delta} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\delta \quad (2.2)$$

on the Euclidean space $\mathbb{R}^8 = \{(x^1, x^2, \ldots, x^8)\}$. A more detailed treatment may be found in [14, 13]. The components $\Omega_{\alpha\beta\gamma\delta}$ are related to the structure constants of octonionic algebra $\mathcal{O}$. A basis for $\mathcal{O}$ is provided by the eight elements $1, e_a \ (a = 1, 2, \cdots, 7)$ which satisfy the relation

$$e_a e_b = \varphi_{abc} e_c - \delta_{ab}. \quad (2.3)$$

The structure constant $\varphi_{abc}$ is totally anti-symmetric with

$$\varphi_{abc} = 1 \quad (abc) = (123), (516), (624), (435), (471), (673), (572). \quad (2.4)$$

Then $\Omega_{\alpha\beta\gamma\delta}$ is given by

$$\Omega_{abcs} = \varphi_{abc}, \quad \Omega_{abcd} = \frac{1}{3!} \epsilon_{abcdefg} \varphi_{efg}. \quad (2.5)$$

The 4-form $\Omega$ is obviously self-dual and, in addition, it is invariant under the action of Spin(7). The last property can be confirmed as follows. For the space of 2-forms $\Lambda^2(\mathbb{R}^8) \cong \mathfrak{so}(8)$, the decomposition in irreducible Spin(7)-modules is given by

$$\Lambda^2(\mathbb{R}^8) = \Lambda^2_+ \oplus \Lambda^2_-, \quad (2.6)$$
and the dimensions turn out to be $\dim \Lambda^1_+ = 7$ and $\dim \Lambda^2_- = 21$. $\Lambda^2_-$ is isomorphic to $\text{spin}(7)$ and the projection is explicitly written as

$$M_{\alpha\beta} = \frac{3}{4} (M_{\alpha\beta} + \frac{1}{6} \Omega_{\alpha\beta\gamma\delta} M_{\gamma\delta}) \in \Lambda^2_-.$$  \hfill (2.7)

If we identify $M_{\alpha\beta}$ with the standard generators of $\text{SO}(8)$, it is easy to show that the Spin($7$) generators $M_{\alpha\beta}$ leave the 4-form $\Omega$ invariant.

Now we proceed to our main result, which is a generalization of the Proposition 1 to Spin($7$) holonomy manifolds.

Proposition 2. Let $(M, \omega)$ be an 8-dimensional manifold with volume form $\omega$. Let $V_\alpha (\alpha = 1, 2, \cdots, 8)$ be linearly independent eight vector fields on $M$ that satisfy the following two conditions:

(a) volume-preserving condition

$$L_{V_\alpha} \omega = 0,$$ \hfill (2.8)

(b) 2-vector condition

$$\Omega_{\alpha\beta\gamma\delta}[V_\alpha \wedge V_\beta \wedge V_\gamma \wedge V_\delta]_{\text{SN}} = 0,$$ \hfill (2.9)

where $\Omega_{\alpha\beta\gamma\delta}$ are the coefficients defined by (2.5) and $[ \ , \ ]_{\text{SN}}$ is provided by the Schouten-Nijenhuis bracket (see Appendix). Then we have a Ricci-flat metric

$$g = \phi W^\alpha \otimes W^\alpha, \quad \phi = \sqrt{\omega(V_1, V_2, \cdots, V_8)},$$ \hfill (2.10)

where $W^\alpha$ is the dual basis of $V_\alpha$ and

$$\Omega = \frac{1}{4!} \phi^2 \Omega_{\alpha\beta\gamma\delta} W^\alpha \wedge W^\beta \wedge W^\gamma \wedge W^\delta$$ \hfill (2.11)

gives the Cayley 4-form on $M$.

Proof. If we introduce the 1-form $E^\alpha = \sqrt{\phi} W^\alpha$, then the 4-form $\Omega$ takes the form,

$$\Omega = \frac{1}{4!} \Omega_{\alpha\beta\gamma\delta} E^\alpha \wedge E^\beta \wedge E^\gamma \wedge E^\delta$$ \hfill (2.12)

and the metric is orthonormal in this frame, $g = E^\alpha \otimes E^\alpha$. Thus a completely analogous result to the $\mathbb{R}^8$ case holds if we replace the 1-form $dx^\alpha$ with $E^\alpha$, i.e. $\Omega$ is a Spin($7$) invariant self-dual 4-form. We shall now prove $\Omega$ is closed. For this we rewrite the equation (2.12) in the form

$$\Omega = \frac{1}{4!} \Omega_{\alpha\beta\gamma\delta} \iota_{V_\alpha} \iota_{V_\beta} \iota_{V_\gamma} \iota_{V_\delta} \omega,$$ \hfill (2.13)

where $\iota_{V_\alpha}$ denotes the inner derivation with respect to $V_\alpha$. By using the following formulas successively,

$$L_{V_\alpha} \iota_{V_\beta} \phi - \iota_{V_\beta} L_{V_\alpha} \phi = \iota_{[V_\alpha, V_\beta]} \phi, \quad L_{V_\alpha} = d\iota_{V_\alpha} + \iota_{V_\alpha} d$$ \hfill (2.14)
we find

$$d\Omega = \frac{1}{4!} \Omega_{\alpha \beta \gamma \delta} \epsilon^{\epsilon \gamma \delta \beta} (6 \epsilon V_\alpha \epsilon V_\beta \epsilon V_\gamma \epsilon V_\delta - 4 \epsilon V_\alpha \epsilon V_\beta \epsilon L V_\gamma \epsilon + \epsilon V_\alpha \epsilon V_\beta \epsilon V_\gamma \epsilon d \omega),$$

(2.15)

and furthermore the volume-preserving condition $L V_\alpha \omega = 0$ with $d \omega = 0$ simplifies the equation,

$$d\Omega = \frac{1}{4!} \Omega_{\alpha \beta \gamma \delta} \epsilon [V_\alpha, V_\beta] \epsilon V_\gamma \epsilon V_\delta \omega.$$  

(2.16)

Finally, noting the identity

$$[V_\alpha \wedge V_\beta, V_\gamma \wedge V_\delta]_\mathcal{S}_N = [V_\alpha, V_\beta] \wedge V_\gamma \wedge V_\delta - [V_\alpha, V_\delta] \wedge V_\beta \wedge V_\gamma - [V_\beta, V_\gamma] \wedge V_\alpha \wedge V_\delta + [V_\alpha, V_\gamma] \wedge V_\beta \wedge V_\delta,$$

$$+ \epsilon \Omega_{\alpha \beta \gamma \delta} ([V_\gamma, V_\delta] \wedge V_\alpha \wedge V_\beta + [V_\alpha, V_\beta] \wedge V_\gamma \wedge V_\delta) - [V_\beta, V_\gamma] \wedge V_\alpha \wedge V_\delta + [V_\alpha, V_\gamma] \wedge V_\beta \wedge V_\delta,$$

(2.17)

and the 2-vector condition (2.9), we see that $\Omega$ is closed. Thus $\Omega$ is the Cayley 4-form on $M$, and $g$ a Ricci-flat metric with the holonomy in Spin(7).

**Remark.** If $V_\alpha$ is a solution to (2.8) and (2.9), then an infinitesimal deformation by the volume-preserving vector field $X$, 

$$\bar{V}_\alpha = V_\alpha + \varepsilon [X, V_\alpha]$$

(2.18)

gives a new solution. Indeed, we have $L V_\alpha \omega = 0$ and

$$\Omega_{\alpha \beta \gamma \delta} \bar{V}_\alpha \wedge \bar{V}_\beta \wedge \bar{V}_\gamma \wedge \bar{V}_\delta \mathcal{S}_N = \Omega_{\alpha \beta \gamma \delta} [V_\alpha \wedge V_\beta, V_\gamma \wedge V_\delta]_\mathcal{S}_N$$

$$+ \epsilon \Omega_{\alpha \beta \gamma \delta} ([X, V_\alpha \wedge V_\beta] \wedge V_\gamma \wedge V_\delta) + [V_\alpha \wedge V_\beta, [X, V_\gamma \wedge V_\delta]_\mathcal{S}_N]_\mathcal{S}_N,$$

(2.19)

which vanishes by (2.9) and the super Jacobi identity (A.4) up to the order $\varepsilon$. This symmetry will be properly realized in Section 3 as the gauge transformations of superconnection. It should be also noticed that (2.18) induces the metric deformation $g \rightarrow g + L_X g$, and hence this deformation can be absorbed in the coordinate transformation.

### 2.2 Example

We illustrate our formulation by concentrating on the 8-dimensional space $M = \mathbb{R}^2 \times S^3 \times S^3$. Let $(X, Y)$ be the natural coordinates on $\mathbb{R}^2$ and choose a volume form

$$\omega = dX \wedge dY \wedge \Sigma^1 \wedge \Sigma^2 \wedge \Sigma^3 \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3,$$

(2.20)

where $\Sigma^i$ and $\sigma^i$ ($i = 1, 2, 3$) are SU(2) left-invariant 1-forms on the three-spheres satisfying the relations,

$$d\Sigma^i = -\frac{1}{2} \varepsilon_{ijk} \Sigma^j \wedge \Sigma^k, \quad d\sigma^i = -\frac{1}{2} \varepsilon_{ijk} \sigma^j \wedge \sigma^k.$$

(2.21)

We take the following ansatz for vector fields $V_\alpha$ ($\alpha = 1, 2, \cdots, 8$) on $M$:

$$V_i = a(X)(\alpha_{11}(Y)\theta_i + \alpha_{12}(Y)\theta_i) \quad (i = 1, 2, 3),$$

(2.22a)

$$V_\hat{i} = b(X)(\alpha_{21}(Y)\theta_i + \alpha_{22}(Y)\theta_i) \quad (\hat{i} = 4, 5, 6),$$

(2.22b)

$$V_7 = \beta(Y) \frac{\partial}{\partial X},$$

(2.22c)

$$V_8 = c(X) \frac{\partial}{\partial Y},$$

(2.22d)
where $\Theta_i$ and $\theta_i$ are the dual basis of $\Sigma^i$ and $\sigma^i$, respectively. Then these vector fields obviously preserve the volume form $\omega$, i.e. $L_{V_i}\omega = 0$. It is straightforward to compute the Schouten-Nijenhuis bracket $[V_\alpha \wedge V_\beta, V_\gamma \wedge V_\delta]_{SN}$ by the formula (2.17). Substituting (2.22) into (2.9), we find

$$
\alpha_{11} = 2, \quad \alpha_{12} = 1 - \tanh(Y), \quad \alpha_{21} = 0, \quad \alpha_{22} = \beta = \text{sech}(Y),
$$

(2.23)

and

$$
\frac{da}{dX} = \frac{1}{2} \left( \frac{a^2}{b} - ab \right), \quad \frac{db}{dX} = -2a^2, \quad a = c.
$$

(2.24)

Applying the Proposition 2, we obtain a metric of Spin(7) holonomy

$$
d^2s = \sqrt{a^4b^2}dX^2 + \sqrt{b^2\text{sech}^2(Y)} \left( dY^2 + \frac{1}{4}(\Sigma^i)^2 \right) + \sqrt{\frac{a^4}{b}} \left( \sigma^i + \frac{1 - \tanh(Y)}{2} \Sigma^i \right)^2
$$

(2.25)

This metric was originally obtained by Bryant-Salamon [11], Gibbons-Page-Pope [12] and further discussed in [3]. We finish this section with some remarks:

(1) **Coordinate transformation**

If we introduce the new variables

$$
a = -\frac{3}{10} \left( \frac{9}{20} \right)^\frac{2}{9} r^\frac{4}{9} f^{-1}(r), \quad b = \left( \frac{9}{20} \right)^\frac{2}{9} r^\frac{4}{9},
$$

(2.26)

with the coordinate transformation

$$
\frac{dX}{dr} = -\frac{10}{3} \left( \frac{20}{9} \right)^\frac{2}{9} r^{-\frac{2}{9}} f^3(r),
$$

(2.27)

then the equation (2.24) reduces to

$$
r \frac{df}{dr} = \frac{5}{3}(1 - f^2)f.
$$

(2.28)

The solution is explicitly given by

$$
f(r) = \frac{1}{\left(1 - \left( \frac{m}{r} \right)^\frac{4}{9} \right)^\frac{1}{2}},
$$

(2.29)

which reproduces the metric given in [12].

(2) **Comparison with the Eguchi-Hanson metric**

Using the Proposition 1, we can construct the Eguchi-Hanson metric from the vector fields $V_\mu (\mu = 0, 1, 2, 3)$ on $M = \mathbb{R} \times S^3$:

$$
V_0 = \frac{\partial}{\partial X}, \quad V_1 = a(X)\Theta_1, \quad V_2 = a(X)\Theta_2, \quad V_3 = b(X)\Theta_3
$$

(2.30)
with
\[ \frac{da}{dX} = ab, \quad \frac{db}{dX} = a^2. \] (2.31)

Indeed, these vector fields preserve the volume form \( \omega = dX \wedge \Sigma^1 \wedge \Sigma^2 \wedge \Sigma^3 \) and satisfy the condition \( \tilde{\eta}_{\mu
u}[V, V] = 0 \). Thus we may regard the ansatz (2.22) as an 8-dimensional analogue of (2.30). Spin(7) manifolds have been extensively studied by Joyce [15]. However, nothing is actually known about the explicit metric except for (2.25). On the other hand, we have many other examples in 4-dimensions, Taub-NUT metric, Gibbons-Hawking metric, Atiyah-Hitchin metric and so on [16, 17]. It is interesting to investigate the Spin(7) holonomy metric starting from more general ansatz. We leave this issue to future research.

3 Superconnection and Chern-Simons theory

As mentioned in Section 1, there is a close connection between 4-dimensional hyperkähler manifolds and self-dual Yang-Mills theories. It is natural to ask whether any relation exists between Spin(7) holonomy manifolds and 8-dimensional Yang-Mills theories. The answer to this problem is positive; we rewrite the equations (2.8), (2.9) using the language of superconnections, and arrive at the equations of motion whose solution gives a stational point of the (generalized) Chern-Simons action.

Let us recall Quillen's concept of a superconnection [18, 19]. If \( M \) is a manifold and \( V = V^+ \oplus V^- \) is a \( \mathbb{Z}_2 \)-graded vector space, let \( \mathcal{A}(M, \text{End}(V)) \) be the space of \( \text{End}(V) \)-valued differential forms on \( M \). On the superalgebra \( \text{End}(V) \), there is a canonical commutator satisfying the axioms of the Lie superalgebra (see Appendix):

\[ [a, b] = ab - (-1)^{|a||b|}ba, \quad a, b \in \text{End}(V). \] (3.1)

The space \( \mathcal{A}(M, \text{End}(V)) \) has a \( \mathbb{Z} \)-grading \( \mathcal{A}^n(M, \text{End}(V)) \) given by the degree of differential forms. In addition, we have the total \( \mathbb{Z}_2 \)-grading, which we will denote by

\[ \mathcal{A}(M, \text{End}(V)) = \mathcal{A}^+(M, \text{End}(V)) \oplus \mathcal{A}^-(M, \text{End}(V)), \] (3.2)

where

\[ \mathcal{A}^\pm(M, \text{End}(V)) = \sum_k \mathcal{A}^{2k}(M, \text{End}^\pm(V)) \oplus \sum_k \mathcal{A}^{2k+1}(M, \text{End}^\mp(V)). \] (3.3)

**Definition.** A superconnection (super covariant derivative) on \( \mathcal{A}(M, \text{End}(V)) \) is an odd-parity first-order differential operator

\[ D_A : \mathcal{A}^\pm(M, \text{End}(V)) \longrightarrow \mathcal{A}^\mp(M, \text{End}(V)) \] (3.4)

\footnote{The algebra of endmorphisms \( \text{End}(V) \) is a superalgebra, when graded in the usual way: \( \text{End}^+(V) = \text{Hom}(V^+, V^+) \oplus \text{Hom}(V^-, V^-) \), \( \text{End}^-(V) = \text{Hom}(V^+, V^-) \oplus \text{Hom}(V^-, V^+) \).}
which satisfies Leibniz’s rule in $\mathbb{Z}_2$-graded sense: if $\alpha$ denotes an ordinary differential form and $\theta \in \mathcal{A}(M, \text{End}(V))$ then
\[ D_A(\alpha \wedge \theta) = d\alpha \wedge \theta + (-1)^{|\alpha|} \alpha \wedge D_A \theta. \tag{3.5} \]

In order to better understand the content of the superconnection, we may write
\[ D_A = d + A \]
and
\[ A = A^{[k]} + A^{[1]} + A^{[2]} + \cdots \]

Here, $A \in \mathcal{A}^{-}(M, \text{End}(V))$ and $A^{[k]}$ lies in $\mathcal{A}^{k}(M, \text{End}^{-}(V))$ if $k$ is even, and in $\mathcal{A}^{k}(M, \text{End}^{+}(V))$ if $k$ is odd.

The supercurvature defined by
\[ F = dA + A \wedge A \in \mathcal{A}(M, \text{End}(V)) \tag{3.6} \]
has total degree even, and satisfies the Bianchi identity
\[ dF + A \wedge F - F \wedge A = 0. \tag{3.7} \]

For a Spin(7) manifold $M$ with Cayley 4-form $\Omega$, we consider the Chern-Simons action,
\[ S = \int_M \Omega \wedge \text{str}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \tag{3.8} \]

The difference with the Chern-Simons 7-form studied in [20] is that $A$ is not an ordinary 1-form connection, but a superconnection including high degree of differential forms, so that the integrand has 8-form components. Under variations of the superconnection, we have
\[ \delta S = \int_M \Omega \wedge \text{str}\{(dA + \frac{1}{2} A \wedge A) \wedge \delta A\}. \tag{3.9} \]

Thus the equation
\[ \Omega \wedge F = 0 \tag{3.10} \]
gives a stational point of the Chern-Simons action.

We now show that the solutions of (3.10) describe Spin(7) holonomy manifolds under the following situation:

(a) Let us choose the Euclidean space $\mathbb{R}^8$ for the manifold $M$. Then we have the Cayley 4-form $\Omega$ given by (2.2).

(b) Let $\mathfrak{sdiff}(M^8)$ be an infinite-dimensional Lie algebra of all volume-preserving vector fields on an 8-dimensional space $M^8$ with volume form $\omega$. Then the vector space
\[
\mathfrak{g} = \bigoplus_{p=1}^{8} \Lambda^p \mathfrak{sdiff}(M^8) \quad (\Lambda^8 \mathfrak{sdiff}(M^8)) \text{ denotes the space of } p\text{-vectors } X = X_1 \wedge X_2 \wedge \cdots \wedge X_p, \quad X_i \in \mathfrak{sdiff}(M^8)) \]
becomes a Lie superalgebra $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ by the Schouten-Nijenhuis bracket $[.,.]_{\text{SN}}$. We take this Lie superalgebra, i.e. $A \in \mathcal{A}^{-}(\mathbb{R}^8, \mathfrak{g})$. 

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Table 1: Comparison of gravitational instantons

<table>
<thead>
<tr>
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<th>4-dim.</th>
<th>8-dim.</th>
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<tbody>
<tr>
<td>holonomy group</td>
<td>Sp(1)</td>
<td>Spin(7)</td>
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<tr>
<td>equations of motion</td>
<td>$\pi^a_{\mu \nu}[V_\mu, V_\nu] = 0$</td>
<td>$\Omega_{\alpha \beta \gamma \delta}[V_\alpha \wedge V_\beta, V_\gamma \wedge V_\delta]_{SN} = 0$</td>
</tr>
<tr>
<td></td>
<td>$L_{V_\mu} \omega = 0$</td>
<td>$L_{V_\mu} \omega = 0$</td>
</tr>
<tr>
<td>Yang-Mills correspondence</td>
<td>$\Sigma^a \wedge F = 0$ ($\iff F + *F = 0$)</td>
<td>$\Omega \wedge F = 0$</td>
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<td></td>
<td>$g = \mathfrak{s} \mathfrak{d} \mathfrak{i} \mathfrak{f}(M^4)$</td>
<td>$g = \bigoplus_{i=1}^{8} \wedge^8 \mathfrak{s} \mathfrak{d} \mathfrak{i} \mathfrak{f}(M^8)$</td>
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For the equation (3.10) we assume that the superconnection $A$ is independent of the coordinates $x^a (a = 1, 2, \cdots, 8)$ on $\mathbb{R}^8$, and further is of the form,

$$A = \frac{1}{2} dx^a \wedge dx^b \otimes V_{ab} \in \mathcal{A}^2(\mathbb{R}^8, \mathfrak{g}^-)$$  \hspace{1cm} (3.11)

where $V_{ab} = V_a \wedge V_b \in \mathfrak{g}^-$ denote the 2-vectors on $M^8$. Then the equation (3.10) is actually identical to the 2-vector condition (2.9). Note that the volume-preserving condition (2.8) is automatically satisfied by the choice of $\mathfrak{s} \mathfrak{d} \mathfrak{i} \mathfrak{f}(M^8)$. Thus the solution induces the Cayley 4-form on the internal space $M^8$ with the help of the Proposition 2.

4 Discussion

We have provided a new approach to Spin(7) manifolds using volume-preserving vector fields. In Table 1, we present a summary of our results and compare them with the analogous properties of 4-dimensional gravitational instantons (hyperkähler manifolds).

The present framework may be applied to the case of $G_2$ holonomy manifolds or 7-dimensional gravitational instantons. For the $G_2$ case, the Cayley 4-form $\Omega$ is replaced by a 3-form $\Phi$ and its Hodge dual 4-forms $*\Phi$. Then the condition of $G_2$ holonomy is given by $d\Phi = 0$ and $d * \Phi = 0$ [11]. Our point of view is that one must introduce a system of differential equations, eventually interpreted as a certain Yang-Mills theory. As a candidate for such equations, we propose a pair of equations:

(a) volume-preserving condition

$$L_{V_1} \omega_1 = 0, \quad L_{U_1} \omega_2 = 0$$  \hspace{1cm} (4.1)
(b) 2-vector condition

\[ \psi_{abc}[V_a \wedge V_b, V_c]_{SN} = 0, \quad \Omega_{abcd}[U_{a1} \wedge U_{b2}, U_{c3} \wedge U_{d4}]_{SN} = 0 \]  \hspace{1cm} (4.2)

where the coefficients \( \psi_{abc} \) and \( \Omega_{abcd} \) are defined by (2.4) and (2.5). Indeed, these equations reproduce the \( G_2 \) holonomy manifold given in [12] by a suitable choice of the volume forms \( \omega_1, \omega_2 \) and the vector fields \( V_a, U_q \). The details for \( G_2 \) holonomy manifolds will be presented elsewhere [21].

Finally, we would like to mention about higher dimensional Ashtekar gravity. The action of 4-dimensional Ashtekar gravity is the chiral form of the usual Einstein-Hilbert action. The variables of the chiral action consist of self-dual connection \( A_a^+ \) and self-dual 2-forms \( \Sigma^a (a = 1, 2, 3) \). Using these variables, we can easily read the \( \text{Sp}(1) \) holonomy condition, \( A_a^+ = 0 \). If we bring the self-dual space into focus, the equations of motion are reduced to \( d\Sigma^a = 0 \). On the other hand, in 8-dimensions, we know the \( \text{Spin}(7) \) holonomy condition, which is given by \( d\Omega = 0 \), i.e. \( \Omega \) is the Cayley 4-form. So we may suppose the \( \text{Spin}(7) \)-invariant self-dual 4-form \( \Omega \) as an 8-dimensional Ashtekar variable, although the precise details of the 8-dimensional Ashtekar gravity have not been given.

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Appendix Schouten-Nijenhuis bracket

In this Appendix, we assume \( M \) is an arbitrary \( N \)-dimensional manifold. Let \( \wedge^p TM \) denotes the space of \( p \)-vectors, i.e. skew symmetric contravariant tensor fields of type \((p, 0)\), and \( \wedge TM = \bigoplus_{p=0}^N \wedge^p TM \). We fix a \( \mathbb{Z}_2 \)-grading in \( \wedge TM \) as follows: the parity \( |X| \) of \( X = X_1 \wedge X_2 \wedge \cdots \wedge X_k \in TM \) equals 0 or 1 according to whether \( k \) is odd or even, respectively. A Lie superalgebra on \( \wedge TM \)

\[ [\cdot, \cdot]_{SN} : \wedge^p TM \times \wedge^q TM \longrightarrow \wedge^{p+q-1} TM \]  \hspace{1cm} (A.1)

is defined by [22]

\[ [X, Y]_{SN} = \sum_{k, l} (-1)^{k+l}[X_k, Y_l] \wedge X_1 \wedge \cdots \hat{X}_k \cdots \wedge X_p \wedge Y_1 \wedge \cdots \hat{Y}_l \cdots \wedge Y_q \]  \hspace{1cm} (A.2)

for \( X = X_1 \wedge X_2 \wedge \cdots \wedge X_p \in \wedge^p TM \), and \( Y = Y_1 \wedge Y_2 \wedge \cdots \wedge Y_q \in \wedge^q TM \). This operation is known as the Schouten-Nijenhuis bracket and satisfies the axioms of a Lie superalgebra:

\[ [X, Y]_{SN} + (-1)^{|X||Y|}[Y, X]_{SN} = 0, \]  \hspace{1cm} (A.3)

\[ [X, [Y, Z]]_{SN} = [[X, Y]_{SN}, Z]_{SN} + (-1)^{|X||Y|}[Y, [X, Z]]_{SN}. \]  \hspace{1cm} (A.4)
References


