Two-colour QCD at finite fundamental quark-number density and related theories. *

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We are simulating SU(2) Yang-Mills theory with four flavours of dynamical quarks in the fundamental representation of SU(2) ‘colour’ at finite chemical potential, μ for quark number, as a model for QCD at finite baryon number density. In particular we observe that for μ large enough this theory undergoes a phase transition to a state with a diquark condensate which breaks quark-number symmetry. In this phase we examine the spectrum of light scalar and pseudoscalar bosons and see evidence for the Goldstone boson associated with this spontaneous symmetry breaking. This theory is closely related to QCD at finite chemical potential for isospin, a theory which we are now studying for SU(3) colour.

1. Introduction

QCD at finite baryon number density describes nuclear matter including neutron stars and heavy nuclei. Nuclear matter at finite temperature existed in the early universe and may be observed in relativistic heavy-ion collisions at CERN and RHIC.

QCD at finite chemical potential, μ for quark number has a complex fermion determinant. Hence current simulation algorithms fail for this theory. Working explicitly at finite baryon number density trades this complex determinant for a sign problem.

We therefore turn to studying models which have some of the properties of QCD at finite μ. One such property suggested for QCD at finite μ is diquark condensation. It is this which led us to study 2-colour (SU(2)) QCD with quarks in the fundamental representation of SU(2) colour, which has a real non-negative determinant (and pfaffian), permitting simulations.

2-colour QCD with fundamental quarks is confining and has a sensible meson spectrum but an unphysical “baryon” spectrum. At m = 0 and μ = 0 the usual SU(Nf) × SU(Nf) × U(1) chiral symmetry is enlarged to SU(2Nf). The “meson” multiplets are enlarged to include 2-quark “baryons”. Spontaneous breakdown of chiral symmetry gives SU(2Nf) → Sp(2Nf) rather than the usual SUr(Nf) × Sul(Nf) × Uv(1) → Uv(Nf). For large enough chemical potential we expect a spontaneous breakdown of quark number with a diquark condensate. This condensate is a colour singlet and has associated Goldstone bosons [1]. For true QCD the condensate is, of necessity, coloured, and breaks the gauge symmetry in the Higgs manner – colour superconductivity. For a chiral perturbation theory analysis of 2-colour QCD at finite μ see [2].

The 2 flavour version of this theory can be interpreted as 2-colour QCD at finite chemical potential for isospin. However, it is easy to see that we can treat true (3-colour) QCD at finite chemical potential for isospin since it has a real, non-negative determinant. Since nuclear matter has finite (negative) isospin density as well as finite baryon number density, this theory describes a
surface in the phase diagram for nuclear matter. For a discussion of this theory including a chiral perturbation theory analysis see [3].

2. Lattice 2-colour QCD at finite quark-number chemical potential

The standard staggered fermion transcription of this theory is:

$$S_f = \sum_{\text{sites}} \left\{ \bar{\chi} \left[ \mathcal{D}/(\mu) + m \right] \chi + \frac{1}{2} \lambda \left[ \chi^T \tau_2 \chi + \bar{\chi} \tau_2 \bar{\chi}^T \right] \right\}$$

where the chemical potential $\mu$ is introduced by multiplying links in the $+t$ direction by $e^\mu$ and those in the $-t$ direction by $e^{-\mu}$. The diquark source term (Majorana mass term) is added to allow us to observe spontaneous breakdown of quark-number on a finite lattice.

Integrating out the fermion fields gives:

$$\text{pfaffian} \left[ \begin{array}{cc} \lambda \tau_2 & \mathcal{A} \\ -\mathcal{A}^T & \lambda \tau_2 \end{array} \right] = \sqrt{\det(\mathcal{A}^\dagger \mathcal{A} + \lambda^2)}$$

where

$$\mathcal{A} \equiv \mathcal{D}(\mu) + m$$

Note that the pfaffian is strictly positive, so that we can use the hybrid molecular dynamics method to simulate this theory using “noisy” fermions to take the square root, giving $N_f = 4$.

When $m = \lambda = \mu = 0$, the chiral symmetry of the above action is expanded to $U(2)$, from the $U(1) \times U(1)_V$ of true lattice QCD. This breaks spontaneously $U(2) \rightarrow U(1)_V$ – yielding 3 Goldstone bosons. For $m = \lambda = 0$, $\mu \neq 0$ we expect spontaneous breakdown of quark number and 2 Goldstone bosons – a scalar diquark and a pseudoscalar diquark. For $\lambda = 0$, $m \neq 0$, $\mu \neq 0$ we have no spontaneous symmetry breaking for small $\mu$. For $\mu$ large enough ($\mu > m_\pi/2$ ?) we expect spontaneous breakdown of quark number and one Goldstone boson – a scalar diquark. (See [1] for a full discussion of these symmetries, and for early simulations of the 8 flavour theory at $\lambda = 0$.)

We are simulating this $N_f = 4$ theory on an $8^4$ lattice, measuring the chiral and diquark condensates, the fermion number density, the Wilson/Polyakov line, etc. We are repeating these simulations on $12^3 \times 24$ lattices, where, in addition, we are measuring all local scalar and pseudoscalar meson and diquark propagators (connected and disconnected).

3. Results

We have preliminary results for a relatively large quark mass, $m = 0.1$, and $\lambda = 0.01, 0.02$ (and 0 for small $\mu$). We have recently started simulations at a small quark mass, $m = 0.025$ and $\lambda = 0.0025, 0.005$ where we should observe a more complex spectrum of Goldstone and pseudo-Goldstone bosons. Finally we will simulate at $m = 0$ where there should be 3 Goldstone bosons for $\mu = 0$ and 2 Goldstone bosons for $\mu > 0$. All our simulations are done at $\beta = 1.5 \approx \beta_c(N_t = 4)$.

Figure 1 shows the diquark condensate $\langle \chi^T \tau_2 \chi \rangle$ as a function of chemical potential potential. For small $\mu$ there is no diquark condensate and quark-

Figure 1. Diquark condensate as a function of $\mu$ on an $8^4$ lattice. The arrow indicates $\mu \approx m_\pi/2$. 
number is a good symmetry. At $\mu = \mu_c \sim m_\pi/2$ there is a phase transition, above which there is a diquark condensate, and quark-number is spontaneously broken. This condensate increases to a maximum and then falls towards zero at large $\mu$.

Figure 2 shows the chiral condensate, $\langle \bar{\chi}\chi \rangle$ as a function of $\mu$. The chiral condensate is approximately constant for $\mu < \mu_c$. For $\mu > \mu_c$ it falls, approaching zero for large $\mu$.

Figure 3 shows the quark-number density as a function of $\mu$. The quark-number density $j_0$ is zero for $\mu < \mu_c$ and increases for $\mu > \mu_c$ approaching the saturation value $j_0 = 2$ for large $\mu$.

Finally, in figure 4 we show the masses of the pion and the scalar diquark state which is orthogonal to the diquark condensate, as functions of $\mu$. This is from the $12^3 \times 24$ lattice where we have less complete ‘data’. The mass of the scalar diquark falls roughly as the expected $m_\pi - 2\mu$ as $\mu$ is increased towards $\mu_c$. For $\mu > \mu_c$ it is small enough and its $\lambda$ dependence is such that one can believe that it would become massless as $\lambda \to 0$. Thus the scalar diquark does appear to be the Goldstone boson associated with spontaneous quark-number breaking. The pion mass appears to remain constant for $\mu < \mu_c$, dropping towards zero as $\mu$ is increased beyond $\mu_c$. This mass spectrum and the behaviour of the order parameters is in good agreement with the predictions from chiral perturbation theory for $\mu \leq 0.6$.

4. QCD at finite isospin density

For QCD at finite chemical potential for isospin ($I_3$), the staggered quark action is

$$S_f = \sum_{\text{sites}} \{\bar{\chi}[\mathcal{D}(\tau_3\mu) + m] \chi + \lambda \bar{\chi}\tau_2\epsilon \chi\}$$

(4)

where the explicit symmetry breaking term proportional to $\lambda$ is needed to observe spontaneous isospin breaking on finite lattices; $\tau_i$ are isospin matrices acting on the isodoublet $\chi$, and $\epsilon = (-1)^{x+y+z+t}$. Integrating out the fermion fields...
Figure 4. Pion ($\pi$) and scalar diquark ($q\bar{q}$) masses as functions of $\mu$. The straight line is $\text{mass} = m_\pi - 2\mu$. The arrow is at $\mu = m_\pi/2$.

yields:

$$\det(A^\dagger A + \lambda^2)$$

where

$$A \equiv D(\mu) + m$$

Note $A$ is only a $1 \times 1$ matrix in isospin space. This describes 8 fermion flavours, so we use hybrid molecular dynamics with “noisy” fermions to perform the simulations, to reduce this to 2 flavours. Note this determinant has exactly the same form as that for 2-colour QCD at finite quark number density, except now we can use $SU(3)$ colour. To get a sensible flavour interpretation for this 2-flavour theory in the continuum limit requires making the field redefinition $\chi_2 \to \xi_5 \chi_2$, where $\xi_5$ is the analogue of $\gamma_5$ in $SU(4)$ flavour space.

5. Conclusions

2-colour QCD at finite chemical potential for quark number undergoes a phase transition at $\mu = \mu_c \leq m_\pi/2$. The high $\mu$ phase is characterized by spontaneous breakdown of quark number precipitated by a diquark condensate, with a single Goldstone boson. Is there a second transition at high $\mu$ to a free-field phase? What can we learn from this model about diquark condensates in true ($SU(3)$) QCD?

We need to simulate at smaller quark masses (including zero) where the competition between chiral and quark-number symmetry breaking should lead to a more complex spectrum of Goldstone and pseudo-Goldstone bosons. A more extensive analysis of the spectrum of scalar and pseudoscalar mesons and diquarks is called for.

QCD at finite chemical potential, $\mu_I$, for isospin maps a surface in the phase diagram of nuclear matter which, in addition to having a finite baryon number density, also has a finite (negative) isospin density. A reinterpretation of 2-colour QCD at finite quark number density yields 2-colour QCD at finite isospin density, and gives us a guide as to what to expect in the 3-colour case. At finite $\mu_I$, true (3-colour) QCD has a real positive determinant which allows us to simulate it. For large enough chemical potential we should get spontaneous breaking of the remaining $U(1)$ isospin symmetry generated by $I_3$ with a charged pion condensate and one true Goldstone pion.

Recently, the work of [1] has been extended to finite temperature as well as $\mu$ by the Hiroshima group[4]. In addition the eigenvalues of the Dirac matrix in [1] have been studied by [5]. Finally we would like to mention related work on gauge theories with adjoint fermions [6].

Acknowledgements

These simulations are being performed on the Cray SV1’s ($8^4$, $m = 0.1$) and the IBM SP ($12^4 \times 24$) at NERSC, and on the IBM SP’s ($8^4$, $m = 0.025$) at NPACI.

REFERENCES

5. H. Markum, *et al.*, talk presented at this meeting.