Twisted Kac-Moody Algebras
And
The Entropy Of AdS$_3$ Black Hole

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Abstract

We show that an $SL(2,R)_L \times SL(2,R)_R$ Chern-Simons theory coupled to a source on a manifold with the topology of a disk correctly describes the entropy of the AdS$_3$ black hole. The resulting boundary WZNW theory leads to two copies of a twisted Kac-Moody algebra, for which the respective Virasoro algebras have the same central charge $c$ as the corresponding untwisted theory. But the eigenvalues of the respective $L_0$ operators are shifted. We show that the asymptotic density of states for this theory is, up to logarithmic corrections, the same as that obtained by Strominger using the asymptotic symmetry of Brown and Henneaux.

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1 Introduction

The entropy of the AdS$_3$ black hole [1, 2], has been investigated from a variety of points of view. Some of the more prominent approaches to this problem have been compared and contrasted by Carlip [3]. In this work we will address this problem in the framework of pure Gravity in 2 + 1 dimensions. Within this framework, a direct method of obtaining the entropy of the BTZ black hole was given by Strominger [4], in which use is made of the earlier work of Brown and Henneaux [5]. Using their results, he demonstrated that the asymptotic symmetry of the BTZ black hole is generated by two copies of the Virasoro algebra with central charges

\[ c_L = c_R = \frac{3l}{2G}, \]

where \( l \) is the radius of curvature of the AdS$_3$ space, and \( G \) is Newton’s constant. Then, assuming that the ground state eigenvalue \( \Delta_0 \) of the Virasoro generator \( L_0 \) vanishes, he obtained the Bekenstein-Hawking expression for the entropy. As pointed out by Strominger [4], in this derivation one must take for granted the existence of a quantum gravity theory with appropriate symmetries. In the absence of such a quantum theory, there will be no practical way of computing either \( \Delta_0 \) or the value of the classical central charge given by Eq. (1) from first principles.

Other approaches to the entropy problem make use of the Chern-Simons theory representation of gravity in 2 + 1 dimensions [6, 7]. One common feature among them is that to account for the microscopic degrees of freedom of the black hole, the free Chern-Simons theory is formulated on a manifold with boundary [8, 9, 10]. Although the significance of the boundary in these works differ, they all lead to WZNW theories [11]. More recently, these scenarios have been further refined, improved, and extended [8, 13, 14, 15]. One important feature of a typical conformal field theory obtained in this way is that its central charge varies between the rank and the dimension of the gauge group. The relevant gauge groups for the AdS$_3$ black hole are two copies of the group \( SL(2, \mathbb{R}) \), so that in the corresponding Virasoro algebras the central charges vary in the range \( 1 \leq c \leq 3 \). On the other hand, the values of the central charges given by Eq. (1) are very large and seem to be unrelated to Kac-Moody algebras arising from relevant gauge groups. Thus, it appears that in the Chern-Simons approach one reaches an impasse in providing a quantum mechanical basis for the classical results of Brown and Henneaux.

In this work, we describe a way to resolve this apparent contradiction by interpreting the classical asymptotic Virasoro algebra of Brown and Henneaux [5] as an “effective” symmetry characterized by an “effective central charge” in the sense defined by Carlip [3]. Then, rather than naively comparing central charges, we derive the consequences of this effective theory, including its “effective central charge” from yet another approach which makes use of Chern-Simons theory but which is physically very different from the ones mentioned above. To begin with, in contrast to previous works, in our approach the Chern-Simons theory is coupled to a source. Then, since the BTZ black hole is a solution of source-free Einstein’s equations [1, 2], it is clear that the manifold \( M \) on which the Chern-Simons theory is defined cannot be identified with space-time. Instead, as shown in previous work [16, 17], the classical black hole space-time can be constructed from the information encoded in the manifold \( M \). In particular, this information supplied, mass, angular momentum, and the
all important discrete identification group [1, 2] which distinguishes the black hole from
anti-de Sitter space. One important advantage of this point of view is that the manifold \( M \)
is specified by its topology (no metric). As a result, for a manifold with the topology of,
say, a disk, the "size" of \( M \) and the location of the boundary relative to the source does
not enter into the formalism, and a conformal field theory constructed on its boundary is
independent of where that boundary is. In other words, it is unnecessary to specify whether
the boundary refers to a horizon or to asymptotic infinity.

Just as in obtaining the classical features of the black hole space-time [16, 17], the
coupling to a source turns out to be essential in arriving at a microscopic description of the
black hole entropy. In particular, it results in a conformal field theory on the boundary with
two copies of a twisted affine Kac-Moody algebra. In the corresponding Virasoro algebra,
the value of the central charge remains the same as the theory without a source, but the
eigenvalues of the operator \( L_0 \) are shifted and are non-vanishing. Taking these features as
well as the subtleties that arise from the non-compactness of \( SL(2,R) \) into account, we find
that the asymptotic density of states for this microscopic theory agree with that given by
Strominger [4] if we identify the Brown-Henneaux values for the central charge [5] with the
effective central charge \( c_{eff} \) of our theory.

2 Chern-Simons Action and Boundary Effects

For a simple or a semi-simple Lie group, the Chern Simons action has the form

\[ I_{cs} = \frac{k}{4\pi} Tr \int_M A \wedge \left( dA + \frac{2}{3} A \wedge A \right) \]  

(2)

where \( Tr \) stands for trace and

\[ A = A_\mu dx^\mu \]  

(3)

We require the 2+1 dimensional manifold \( M \) to have the topology \( R \times \Sigma \), with \( \Sigma \) a two-
manifold and \( R \) representing the time-like coordinate \( x^0 \). Moreover, we take the topology of
\( \Sigma \) to be trivial in the absence of sources, with the possible exception of a boundary. Then,
subject to the constraints

\[ F^b[A] = \frac{1}{2} \epsilon^{ij} (\partial_i A_j^b - \partial_j A_i^b + \epsilon^{bjc} A_c^i A_d^j) = 0 \]  

(4)

the Chern-Simons action for a simple group \( G \) will take the form

\[ I_{cs} = \frac{k}{2\pi} \int_R dx^0 \int_\Sigma d^2x \left( -\epsilon^{ij} \eta_{ab} A_i^a \partial_0 A_j^b + A_0^b F_{ab} \right) \]  

(5)

where \( i, j = 1, 2 \).

We want to explore the properties of the Chern-Simons theory coupled to a source for
the group \( SL(2,R)_L \times SL(2,R)_R \) on a manifold with boundary. Since the gauge group is
semi-simple, the theory breaks up into two parts, one for each \( SL(2,R) \), where by \( SL(2,R) \)
we mean its infinite cover. So, to simplify the presentation, we will study a single \( SL(2,R) \).
Much of what we discuss in this and the next section hold for any simple Lie group, \( G \).
Also, to establish our notation, we consider first the theory in the absence of the source.
The main features of a Chern-Simons theory on a manifold with boundary has been known for sometime [11, 18]. Here, with \( M = R \times \Sigma \), we identify the two dimensional manifold \( \Sigma \) with a disk \( D \). Then, the boundary of \( M \) will have the topology \( R \times S^1 \). We parametrize \( R \) with \( \tau \) and \( S^1 \) with \( \phi \). In this parametrization, the Chern-Simons action on a manifold with boundary can be written as

\[
S_{cs} = \frac{k}{4\pi} \int_M Tr(AdA + \frac{2}{3}A^3) + \frac{k}{4\pi} \int_{\partial M} A_\phi A_\tau. \tag{6}
\]

The surface term vanishes in the gauge in which \( A_\tau = 0 \) on the boundary. In this action, let \( A = \tilde{A} + A_\tau \) and \( d = d\tau \frac{\partial}{\partial \tau} + \tilde{d} \). Then, the resulting constraint equations for the field strength take the form

\[
\tilde{F} = 0. \tag{7}
\]

They can be solved exactly by the ansatz [11, 18]

\[
\tilde{A} = -\tilde{d}UU^{-1}, \tag{8}
\]

where \( U = U(\phi, \tau) \) is an element of the gauge group \( G \). Using this solution, the Chern-Simons action given by Eq. (5) can be rewritten as

\[
S_{WZNW} = \frac{k}{12\pi} \int_M Tr(U^{-1}dU)^3 + \frac{k}{4\pi} \int_{\partial M} Tr(U^{-1}\partial_\phi U)(U^{-1}\partial_\tau U)d\phi d\tau. \tag{9}
\]

We thus arrive at a WZNW action and can take over many result already available in the literature for this model. As in any WZNW theory, the change in the integrand of this action under an infinitesimal variation \( \delta U \) of \( U \) is a derivative. We interpret this to mean that \( U = U(\phi, \tau) \), i.e., it is independent of the third (radial) coordinate of the bulk. In other words, the information encoded in the disk depends only on its topology and is invariant under any scaling of the size of the disk.

The above Lagrangian is invariant under the following transformations of the \( U \) field [18]:

\[
U(\phi, \tau) \rightarrow \Omega(\phi)U\Omega(\tau), \tag{10}
\]

where \( \Omega(\phi) \) and \( \Omega(\tau) \) are any two elements of \( G \). To obtain the conserved currents, let \( U \rightarrow U + \delta U \). The corresponding variation of the action leads to \( S_{WZNW} \rightarrow S_{WZNW} + \delta S_{WZNW} \), where

\[
\delta S_{WZNW} = \frac{k}{2\pi} \int_{\partial M} (\partial_\tau(U^{-1}\partial_\phi U))\delta U. \tag{11}
\]

This implies an infinite number of conserved currents:

\[
J_\phi = -kU^{-1}\partial_\phi U = J_\phi^a T_a. \tag{12}
\]

Here, \( T_a \) are the generators of the algebra \( g \) of the group \( G \), and \( J_\phi \) is a function of \( \phi \) only because \( \partial_\tau J_\phi = 0 \).

If we expand \( J_\phi \) in a Laurent series, we obtain

\[
J_\phi = \Sigma J_n z^{-n-1}, \tag{13}
\]
where \( z = \exp(i\phi) \). As usual, \( J_n \) satisfy the Kac-Moody algebra

\[
[J_n^a, J_m^b] = f_{c}^{ab} J_{n+m}^c + k n g^{ab} \delta_{n+m,0}
\]  

(14)

The corresponding energy momentum tensor for the action \( S_{WZNW} \) can be computed using the Sugawara-Sommerfield construction. For example, for the gauge group \( SL(2,R) \),

\[
T_{\phi\phi} = \frac{1}{(k-2)} g_{ab} J_{\phi}^a(z) J_{\phi}^b(z) = \frac{1}{k-2} \Sigma : J_{n-m}^a J_{m}^a : z^{-n-2} = \Sigma L_n z^{-n-2},
\]  

(15)

where

\[
L_n = \frac{1}{k-2} \Sigma : J_{n-m}^a J_{m}^a :.
\]  

(16)

The \( L_n \) operators satisfy the following Virasoro algebra:

\[
[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0},
\]  

(17)

with \( c \) the central charge. For \( SL(2,R) \), it is given by \( c = \frac{3k}{k-2} \). We note that for large negative values of \( k \), the value of \( c \) approaches 3 which is the dimension of the group. We also note that this boundary WZNW theory has one, not the more usual two, Virasoro algebra. It will be shown below that when the Chern-Simons theory is coupled to a source on a manifold with the topology of a disk, the central charge of the Virasoro algebra of the corresponding modified WZNW theory remains the same as that in the source-free theory discussed above.

### 3 The Coupling of a source

Next, we couple a source to the Chern-Simons action on the manifold \( M \) with disk topology, which, as in the previous section, has the boundary \( R \times S^1 \). In general, we take the source to be a unitary representation of the group \( G \). To be more specific, let us consider a source action given by [11, 18]

\[
S_{source} = \int d\tau Tr[\lambda \omega(\tau)^{-1}(\partial_{\tau} + A_{\tau})\omega(\tau)].
\]  

(18)

Here \( \lambda = \lambda^i H_i \), where \( H_i \) are elements of the Cartan subalgebra \( H \) of \( G \). We will take \( \lambda^i \) to be appropriate weights. The quantity \( \omega(\tau) \) is an arbitrary element of \( G \). The above action is invariant under the transformation \( \omega(\tau) \rightarrow \omega(\tau) h(\tau) \), where \( h(\tau) \) commutes with \( \lambda \).

Now the total action on \( M \) is,

\[
S_{total} = \frac{k}{4\pi} \int Tr(AdA + \frac{2}{3} A^3) + \frac{k}{4\pi} \int A_{\tau} A_{\phi} + \int d\tau Tr[\lambda \omega(\tau)^{-1}(\partial_{\tau} + A_{\tau})\omega(\tau)]
\]  

(19)

The new constraint equation takes the form,

\[
\frac{k}{2\pi} \tilde{F}(x) + \omega(\tau) \lambda \omega^{-1}(\tau) \delta^2(x - x_p) = 0,
\]  

(20)
where \( x_p \) specifies the location of the source, heretofore taken to be at \( x_p = 0 \). The solution to the above equation is given by

\[
\tilde{A} = -\tilde{d} \tilde{U} \tilde{U}^{-1},
\]

where [18]

\[
\tilde{U} = U \exp\left(\frac{1}{k} \omega(\tau) \lambda \omega^{-1}(\tau) \phi\right)
\]

The new effective action on the boundary \( \partial M \) is then

\[
S_{\text{total}} = S_{\text{WZNW}} + \frac{1}{2\pi} \int_{\partial M} Tr(\lambda U^{-1} \partial \tau U).
\]

This Lagrangian is also invariant under the following transformation:

\[
U(\phi, \tau) \to \Omega(\phi) U \Omega(\tau)
\]

where \( \Omega(\tau) \) commutes with \( \lambda \). Varying the action under the above symmetry transformation, we get

\[
\delta S_{\text{total}} = \delta S_{\text{WZNW}} + \delta S_{\text{source}},
\]

where

\[
\delta S_{\text{source}} = \frac{1}{2\pi} \int Tr \left( -U^{-1} \delta U [U^{-1} \partial \tau U, \lambda] \right).
\]

Hence, the requirement that \( \delta S_{\text{total}} = 0 \) will give rise to the conservation equation [20]

\[
\partial \tau \left( -k U^{-1} \partial \phi U \right) + [U^{-1} \partial \tau U, \lambda] = 0.
\]

The first term in this expression has the same structure as the current \( J_\phi \) of the source free theory. Hence, requiring that \( U(\phi, \tau) = U(\phi + \tau) \), we can write the new current \( \tilde{J}_\phi \) in terms of the current in the absence of the source as

\[
\tilde{J}_\phi = e^{\frac{k}{\lambda} (\phi + \tau)} J_\phi e^{-\frac{k}{\lambda} (\phi + \tau)}
\]

It is easy to check that

\[
\partial \tau \tilde{J}_\phi = 0
\]

With the new currents at our disposal, the next step is to see how this modification affects the properties of the corresponding conformal field theory. In this respect, we note from Eq. (28) that our new currents \( \tilde{J}_\phi \) are related to the currents \( J_\phi \) in the absence of the source by a conjugation with respect to the elements of the Cartan subalgebra \( H \) of the group \( G \). This kind of conjugation has been noted in the study of Kac-Moody algebras [19, 21, 22]: The algebra satisfied by the new currents fall in the category of twisted affine Kac-Moody algebras. So, to understand how the coupling to a source modifies the structure of the source-free conformal field theory, we follow the analysis of reference [19] and express the Lie algebra of the group \( G \) of rank \( r \) in the Cartan-Weyl basis. Let \( H^i \) be the elements of the Cartan subalgebra and denote the remaining generators by \( E^\alpha \). Then, with label \( a = (i, \alpha) \),

\[
[H^i, H^j] = 0; \quad [H^i, E^\alpha] = \alpha^i E^\alpha
\]
In this expression, \(1 \leq i, j \leq r\), and \(\alpha, \beta\) are roots. Now we can rewrite the affine Kac-Moody algebra \(g\) of the source free theory of the last section in this basis as follows:

\[
[H^i_m, H^j_n] = km\delta^{ij}\delta_{m,-n}; \quad [H^i_m, E^\alpha_n] = \alpha^i E^\alpha_{m+n}
\]

(32)

We also note from the last section that in the absence of the source the element \(L_0\) of the Virasoro algebra of the source-free theory and the currents \(J^a_n\) have the following commutation relations:

\[
[L_0, J^a_n] = -nJ^a_n.
\]

(34)

It can be seen from Eq. (28) that the new currents can be viewed as an inner automorphism of the algebra \(g\) in the form \(\zeta(J) = \gamma J \gamma^{-1}\). The effect of this on the component currents can be represented by

\[
e^{i\chi \cdot H}
\]

(35)

As a result of this inner automorphism on elements of the algebra \(g\), we obtain a modified algebra \(\hat{g}\) the elements of which in the Cartan-Weyl basis are given by [19]

\[
\zeta(H^i) = H^i; \quad \zeta(E^\alpha) = e^{i\chi \cdot \alpha} E^\alpha
\]

(36)

If the map \(\zeta\) is endowed with the property that \(\zeta^N = 1\), then we must have \(N\chi \cdot \alpha = 2n\pi\), where \(n\) is an integer for all roots \(\alpha \in g\):

\[
e^{i\chi \cdot \alpha} = e^{2\sin n\pi/2}\n\]

(37)

where \(n\) is a positive integer \(\leq N - 1\). As far as the currents obtained from the Chern-Simons theory coupled to a source are concerned, it appears that all possible values of \(N\) are allowed. However, for algebras of low rank such as, the value of \(N\) can be unique. This is because the automorphism \(\zeta\) divides a suitable combination of the generators of \(\hat{g}\) into eigenspaces \(\hat{g}(m)\). For this arrangement to work for \(SU(2)\) or \(SL(2, \mathbb{R})\), the only non-trivial possibility is \(N = 2\).

Thus, the basis of \(\hat{g}\) consists of the elements \(H^i_m\) and \(E^\alpha_n\) where \(m \in \mathbb{Z}\) and \(n \in (\mathbb{Z} + \frac{\chi \cdot \alpha}{2\pi})\). These operators satisfy a Kac-Moody algebra which has formally the same structure as that of \(g\) but with rearranged (fractional) values of the suffices. Hence the algebra \(\hat{g}\) can be viewed as the “twisted” version of the algebra \(g\).

Since the automorphism which relates the two algebras is of inner variety [19], we must look for features, if any, that distinguish the algebra \(\hat{g}\) from its untwisted version \(g\). These features depend on the extent to which we can undo the twisting. To this end, we introduce a new basis for \(\hat{g}\)

\[
\hat{E}^\alpha_n = E^\alpha_{n + \frac{\chi \cdot \alpha}{2\pi}}; \quad \hat{H}^i_n = H^i_n + \frac{k}{2\pi} \chi^i \delta_{n,0},
\]

(38)
The new operators, $\hat{E}_n^\alpha$ and $\hat{H}_i^\alpha$, satisfy the same commutation relations as the elements of the untwisted affine Kac-Moody algebra $g$. The corresponding conformal field theories are not identical, however. This can be seen most easily if we express the Virasoro generators $\hat{L}_n$ of the twisted theory in terms of untwisted generators:

$$\hat{L}_m = L_m - \frac{1}{2\pi} \chi^i H_0^i + \frac{k}{4\pi^2} \chi^i \chi^j \delta_{n,0}. \quad (39)$$

In particular, we get for $\hat{L}_0$,

$$\hat{L}_0 = L_0 - \frac{1}{2\pi} \chi^i H_0^i + \frac{k}{4\pi^2} \chi^i \chi^j. \quad (40)$$

Thus the eigenvalues $\hat{\Delta}$ of the operator $\hat{L}_0$ are shifted relative to the eigenvalues $\Delta$ of $L_0$. But, as can be verified directly, the value of the central charge $c$ remains unchanged [19, 21, 22]. More specifically, we have

$$\hat{L}_0 |\hat{\Delta}, \mu> = \hat{\Delta} |\hat{\Delta}, \mu>, \quad (41)$$

where $\mu$ is a weight and

$$\hat{\Delta} = \Delta - \frac{1}{2\pi} \chi^i \mu^i + \frac{k}{4\pi^2} \chi^i \chi^j. \quad (42)$$

So, for the highest (lowest) weight states, we get

$$\hat{\Delta}_0 = \Delta_0 - \frac{1}{2\pi} \chi^i \mu^i_0 + \frac{1}{4\pi^2} k \chi^2. \quad (43)$$

With minor exceptions, most of the derivation of our twisted Kac-Moody algebras from the Chern-Simons theory applies to any gauge group. But the relation between the irreducible representations of an affine Kac-Moody algebra and its Lie subalgebra imposes restrictions on the value of the central term $k$. For example, for $SU(2)$, the value of $k$ is restricted to the non-negative values [19]. But for discrete unitary representations of $SL(2, R)$ with a lowest weight, the quantity $k$ is restricted to [23]

$$k < -1. \quad (44)$$

It follows that in this case large negative values of $k$ are allowed. We will take advantage of this feature in the application of this formalism to entropy of the AdS$_3$ black hole in the next section.

4 The Entropy of the AdS$_3$ Black Hole

As pointed out in the introduction, in the derivation of the entropy of the AdS$_3$ black hole by Strominger [4], use was made of the expression for the central charges of the two asymptotic Virasoro algebras obtained by Brown and Henneaux [5] using classical (non-quantum) arguments. They are given by

$$c_L = c_R = \frac{3l}{2G}. \quad (45)$$
where \( l \) is the radius of curvature of the AdS\(_3\) space, and \( G \) is Newton’s constant. The presence of such a symmetry indicates that there is a conformal field theory at the asymptotic boundary. It was shown by Strominger that the BTZ solution satisfies the Brown-Henneaux boundary conditions so that it possessed an asymptotic symmetry of this type. So, he identified the degrees of freedom of the black hole in the bulk with those of the conformal field theory at the infinite boundary. Then, using Cardy’s formula [25] for the asymptotic density of states, he showed that for \( l \gg G \) the entropy of this conformal field theory is given by

\[
S = \frac{2\pi r_+}{4G},
\]

(46)
in agreement with Bekenstein-Hawking formula. Here, the quantity \( r_+ \) is the outer horizon radius of the black hole. An important assumption in this derivation was that the ground state eigenvalue \( \Delta_0 \) of the operator \( L_0 \) vanishes.

The formula by Cardy [25] for the asymptotic density of states, leading to the above expression for entropy is given by

\[
\rho(\Delta) \approx e^{\pi \sqrt{\frac{c\Delta}{6}}},
\]

(47)

where \( \rho(\Delta) \) is the number of states for which the eigenvalue of \( L_0 \) is \( \Delta \). The result holds when \( \Delta \) is large and the lowest eigenvalue \( \Delta_0 \) vanishes. From the analysis of the previous section, it is clear that in the conformal field theory arising from a Chern-Simons theory coupled to a source the eigenvalue \( \Delta_0 \) does not vanish, so that the above Cardy formula must be appropriately modified. In such a case, the asymptotic density of states for large \( \Delta \) is given by [3]:

\[
\rho(\Delta) \approx e^{\pi \sqrt{\frac{(c - 24\Delta_0)\Delta}{6}}} \rho(\Delta_0) = e^{\pi \sqrt{\frac{c_{\text{eff}}\Delta}{6}}} \rho(\Delta_0).
\]

(48)

Thus, it is the latter formula which must be used in the application of our formalism to the black hole entropy. It is important to note that the expression for the asymptotic density of states given by Eq. (48) rests on the existence of a consistent conformal field theory with a well defined partition function. For Kac-Moody algebras based on compact Lie groups this can be established rigorously. But for Kac-Moody algebras based on non-compact groups such as \( SL(2, R) \), no general proof exists. So, all the conformal field theories based on \( SL(2, R) \), which have been made use of in connection with the AdS\(_3\) black hole, including the present work, share this common weakness.

We want to show that the results obtained from such a microscopic analysis are in agreement with those given by Strominger [4]. In so doing, we will rely heavily on our previous results which dealt with the understanding of the macroscopic features of the BTZ black hole [16, 17]. We recall from these references that the unitary representations of \( SL(2, R) \) which are relevant to the description of the macroscopic features of the black hole are the infinite dimensional discrete series which are bounded from below. These irreducible representations are characterized by a label \( F \) which can be identified with the lowest eigenvalue of the \( SL(2, R) \) generator which is being diagonalized. In the literature of the \( SL(2, R) \) Kac-Moody algebra [23], this (lowest weight) label, which is convenient for
the description of this series, is often referred to as $-j$. Thus, the Casimir eigenvalues in the two notations are related according to

$$j(j + 1) = F(F - 1), \quad (49)$$

where $F \geq 1$. On the other hand, in the description of the black holes in terms of a Chern Simons theory with gauge group $SL(2, R)_L \times SL(2, R)_R$, the mass and the angular momentum of the black hole are related to the Casimir invariants of $j^2_\pm$ of this gauge group as follows [16, 17]:

$$j^2_\pm = \frac{1}{4}(lM \pm J). \quad (50)$$

So, from Eqs. (49) and (50) we get for positive mass black holes

$$F_\pm = \frac{1}{2} \left[ 1 + \sqrt{1 + (lM \pm J)} \right]. \quad (51)$$

In particular [16, 17], for the lowest massive state, we must have $F^{(0)}_\pm \approx 1$. This means that $(lM_0 + J_0) \ll 1$. We will assume that this dimensionless quantity is proportional to the ratio of the two scales of the theory:

$$(lM_0 + J_0) \approx \frac{2\beta^2 G}{l}. \quad (52)$$

Here $\beta$ is a real positive number, and the factor 2 is put there for later convenience. The value of $\beta$ will be fixed by requiring consistency between the classical and the quantum black hole descriptions.

Next, consider the determination of the Chern-Simons couplings $k_\pm$. These were referred to as $a_\pm^{-1}$ in references [16, 17]. Since the gauge group $SL(2, R)_L \times SL(2, R)_R$ is semi-simple, the couplings $k_+$ and $k_-$ are independent. Also, since the two $SL(2, R)$ groups play a parallel role in our approach, we will focus on determining one of them, say, $k_+$. The other one can be obtained in a similar way. To this end, we recall from references [16, 17] that in our approach, the manifold $M$ on which the Chern-Simons theory is defined is not space-time. This means that from the data encoded in $M$ we must be able to obtain all the features of the classical black hole space-time. One of these is the discrete identification group of the black hole [1, 2]. The elements of this group were obtained within our framework by considering the holonomies around the source in $M$. In this respect, we note that the Cartan sub-algebra of $SL(2, R)$ is one dimensional, and the corresponding weight is $F$ as discussed above. Then, directly or using non-abelian Stokes theorem [26] and equation (20), the holonomies can be evaluated. To get the correct discrete identification group, this implies that [16, 17]

$$\frac{2F^{(0)}_\pm}{k_\pm} = \pm \frac{1}{l}(r_+ + r_-) = \pm 2\sqrt{\frac{G}{l}(lM_0 + J_0)}, \quad (53)$$

where $r_+$ and $r_-$ are, respectively, the outer and the inner horizon radii of the black hole. A similar analysis can be carried out for $k_-$. Using the values of $F^{(0)}$ and $(lM_0 + J_0)$ for the ground state given above, we find for both couplings

$$k_+ = k_- = \pm \frac{l}{2\beta G}. \quad (54)$$
The sign of \( k_\pm \) is not fixed by holonomy considerations alone. But from the discussion of the last section it is clear that we must choose the negative sign for both since \( SL(2, R) \) is non-compact.

Next, consider the ground state eigenvalue \( \hat{\Delta}_0 \) for one of the two \( SL(2, R) \) Virasoro algebras. Since the Cartan subalgebra is one dimensional, the sums in Eq. (43) consist of one term each, and, from Eq. (37), the quantity \( \chi \) is given by

\[
\chi = \frac{2\pi n}{N\alpha}.
\] (55)

The quantity \( \mu_0 \) in Eq. (43) is clearly the weight of the ground state, i.e., the weight \( F \approx 1 \) described above. We also note that the root \( \alpha \) is the weight of the adjoint representation of \( SL(2, R) \), so that \( \alpha = 1 \). Moreover, as discussed in the last section, for \( SL(2, R) \) we have \( \frac{k}{N} = \frac{1}{2} \). Then, Eq. (44) specialized to the case at hand will take the form

\[
\hat{\Delta}_0 = \Delta_0 - \frac{1}{2} + \frac{k}{4}.
\] (56)

Also substituting for \( k \) from Eq. (54), we get

\[
\hat{\Delta}_0 = \Delta_0 - \frac{1}{2} - \frac{l}{8\beta G}.
\] (57)

In this expression, the quantity \( \Delta_0 \) is the ground state eigenvalue of \( L_0 \), and its value is not known but is often taken to be zero without \( \text{\textit{a priori}} \) justification. For our purposes, it is only necessary that it be small compared to the last term.

Let us now compute the effective central charge \( c_{\text{eff}} \), defined via Eq. (48), for our theory. It is given by

\[
c_{\text{eff}} = c - 24\hat{\Delta}_0 = c - 24\Delta_0 - 12 + \frac{3l}{\beta G}.
\] (58)

In this expression, \( \frac{l}{G} \gg 1 \) whereas \( 1 \leq c \leq 3 \). Assuming that \( \Delta_0 \) is also relatively small, we get

\[
c_{\text{eff}} \approx \frac{3l}{\beta G}.
\] (59)

To determine the quantity \( \beta \), we note that our starting point, i.e., the Chern-Simons theory coupled to a source leads, on the one hand, to the twisted Kac-Moody algebras and conformal field theories described above and, on the other hand [16, 17], to the classical BTZ solution [1, 2] for which the central charge for the asymptotic Virasoro algebra was given by Brown and Henneaux [5]. It is then necessary that the classical and quantum results which follow from the same Chern-Simons theory be consistent with each other. So, it is reasonable to require that the asymptotic density of states of the above quantum theory, as computed from Cardy-Carlip [3], agree, up to logarithmic terms, with the asymptotic density of states obtained by Strominger [4] using the traditional Cardy formula and the classical Brown-Henneaux value of central charge (45). The simplest way to satisfy this requirement is to set our effective central charge \( c_{\text{eff}} \) equal to the Brown-Henneaux central charge. This fixes \( \beta = 2 \). That this requirement makes sense can also be seen by noting that for the
Virasoro algebra obtained by Brown and Henneaux, the underlying Kac-Moody algebra is not known, so that there is no direct way of calculating its central charge $c$ or its ground state eigenvalue from a fundamental Kac-Moody algebra. Then, taking the classical theory to be an “effective theory”, we see that we can compute its “effective central charge” and its “effective ground state eigenvalue” from the above quantum theory up to a proportionality constant. Fixing the value of $\beta$ in our approach using consistency is to be compared with the fixing of the ground state eigenvalue $\Delta_0$ by Strominger [4]. In that work, the conformal field theory alone does not limit the continuous infinity of the possible values of $\Delta_0$, and the requirement that it vanish has only $á$ posteriori justification.

With $\beta = 2$, Eq. (54) implies that $k_\pm = -\frac{\Delta}{2h}$. Coincidentally, the magnitudes of these quantities are the same as those obtained by another approach [14] in which the manifold $M$ was taken to be space-time and the free Chern-Simons action led to the classical black hole solution in $M$. In that case, the signs of $k_+$ and $k_-$ are opposite each other, so that one of them would have to be positive. This appears to be inconsistent with what we know about $SL(2, R)$ Kac-Moody algebras.

The main result of this section is that for this conformal field theory, the expression for the asymptotic density of states given by Eq. (48) reduces to

$$\rho(\hat{\Delta}) \approx \exp\left\{ 2\pi \sqrt{\frac{c_{\text{eff}} \hat{\Delta}}{6}} \right\} \rho(\hat{\Delta}_0) = \exp\left\{ 2\pi \sqrt{\frac{l \hat{\Delta}}{4G}} \right\} \rho(\hat{\Delta}_0).$$

(60)

Modulo a logarithmic correction, this expression is identical with that used by Strominger [4]. This resolves the longstanding controversy in the traditional method of comparing the central charge of a conformal field theory obtained from the Chern-Simons approach with the (effective) classical results of Brown and Henneaux. One of the crucial features of our work which led to this resolution was the recognition that for $SL(2, R)$ Kac-Moody algebras, large negative values of $k$ are allowed.

So far, we have dealt with the density of states for one $SL(2, R)$, say, $SL(2, R)_L$. This will contribute an amount $S_L$ to the black hole entropy. Clearly, we can repeat this computation for the density of states of $SL(2, R)_R$. Then, to the extent that the logarithmic corrections can be neglected, the black hole entropy $S = S_L + S_R$ is in agreement with that given by Strominger [4]. More recently, the logarithmic contributions to the black hole entropy have been discussed in the literature [27, 28]. One would then have to assess the relative size of our logarithmic term compared to those given in these works.

This work was supported, in part by the Department of Energy under the contract number DOE-FG02-84ER40153. We would like to thank Philip Argyres and Alex Lewis for reading the manuscript and suggesting improvements.

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