GAUSS-BONNET INTERACTION IN RANDALL-SUNDRUM COMPACTIFICATION

HYUN MIN LEE

Department of Physics and Center for Theoretical Physics, Seoul National University, Seoul 151-742, Korea
E-mail: minlee@phya.snu.ac.kr

We show that the Gauss-Bonnet term is the only consistent curvature squared interaction in the Randall-Sundrum model and various static and inflationary solutions can be found. And from metric perturbations around the RS background with a single brane embedded, we also show that for a vanishing Gauss-Bonnet coefficient, the brane bending allows us to reproduce the 4D Einstein gravity at the linearized level.

1 Introduction

The main motivation of recent brane scenarios is to solve the gauge hierarchy problem in the higher-dimensional spacetime\(^1,3\). In the supersymmetric Standard Model, the gauge couplings unify at the energy scale \(\sim 10^{16}\text{GeV}\) by the renormalization group running from the weak scale. However, we cannot make such a prediction for the gravitational coupling, i.e., the Newton constant since gravity is not renormalizable. According to the Horava-Witten’s proposal\(^2\), one finds that the 4D Planck scale becomes the low-energy artifact of a four-dimensional world. In this proposal, the strong coupling limit of the \(E_8 \times E_8\) heterotic string compactified on a Calabi-Yau(CY) manifold \(X\), is described by a 11-dimensional theory compactified on \(X \times S^1/Z_2\) (the so called ‘Heterotic M-theory’). And the gauge and ordinary matter fields sit only on the ten-dimensional boundaries defined by \(S^1/Z_2\), and gravity propagates in the bulk of the spacetime. Upon the CY compactification of the Heterotic M-theory, the eleventh dimension is larger than the CY compactification length scale (or the string scale) when the string scale is identified as the GUT scale. Therefore, in fact, the universe is five-dimensional above the compactification scale of the eleventh dimension and thus smallness of the Newton constant stems from the fact that we cannot reach for the extra dimension of the universe. However, in the Horava-Witten’s proposal, the five-dimensional(or 11-dimensional) fundamental scale cannot be below the GUT scale for the validity of running of the gauge couplings. On the other hand, the higher-dimensional fundamental scale can be pulled down to the

\*BASED ON WORK IN COLLABORATION WITH J. E. KIM AND B. KYAE.
weak scale in the large extra dimension scenario suggested by Arkani-Hamed, Dimopoulos and Dvali (ADD)\(^1\), where the higher dimensional spacetime is a factorizable geometry. In their proposal, at least two extra dimensions are required for solving the hierarchy problem. And, there is a problem related to the stabilization of the large extra dimensions, which corresponds to introduction of another hierarchy problem. By the way, the Randall-Sundrum (RS)’s proposal\(^3\) used the 5D non-factorizable geometry with one extra dimension to explain the gauge hierarchy problem. Their model setup is similar to that of the Horava-Witten’s in the sense that the SM model matter and gauge fields are assumed to live only on the 4D boundaries (or 3-branes) defined by the \(S^2/Z_2\) orbifold, but they introduced brane tensions on the boundaries and a non-zero bulk cosmological constant, which is shown to be realized from a Calabi-Yau compactification of the Heterotic M-theory\(^5\). As the physical scale varies along the bulk according to the exponential warp factor of the metric, they identify the positive (negative) tension brane as the hidden (visible) brane with the Planck (weak) scale. In this proposal, there is no large hierarchy between input parameters. In addition, the effective 4D Planck scale becomes still finite even for the infinite extra dimension, which implies an alternative compactification without small extra dimension\(^4\). If all mass scales in the RS model are given by one input scale, i.e., the 5D Planck scale, then the curvature scale is also of order of the 5D Planck scale. Therefore, the next step is to consider the higher order gravity effects in the RS model.

In this paper, we show that in the existence of the Gauss-Bonnet term, various static and inflationary solutions can be found and properties and peculiarities of the RS model can be maintained\(^12\). Then, through the perturbative analysis and the brane bending effect, we consider the second RS model with the Gauss-Bonnet term as a 4D effective gravity theory\(^13\).

### 2 A review of RS model

The large extra dimension scenario\(^1\) is the simplest case to use the higher-dimensional mechanism to solve the gauge hierarchy problem. The effective 4D Planck scale \(M_P\) is determined by the \((4+n)\)-dimensional Planck scale \(M\) and the geometry of the extra dimensions. Since the higher-dimensional spacetime is a product of a 4-dimensional spacetime with a \(n\)-dimensional compact space in the large extra dimension scenario, the effective 4D Planck scale \(M_P\) is given by the formula \(M_P^2 = \frac{M^{4+n} V_n}{n}\), where \(V_n\) is the volume of the compact space. For the \((4+n)\)-dimensional Planck scale \(M\) to be the weak scale, the compactification scale \(\mu_c \sim 1/V_n^{1/n}\) would have to be much smaller than the weak scale, which requires that the SM particles and forces are
confined to a 4-dimensional subspace while gravity is allowed to propagate in the bulk of the spacetime. However, it gives rise to a new hierarchy problem related to the compactification scale. On the other hand, the small extra dimension scenario\(^3\) considers the higher-dimensional spacetime as the case of the non-factorizable geometry with \(S^1/Z_2\):

\[
\begin{align*}
    ds^2 &= e^{-2kb_0|y|}\eta_{\mu\nu}dx^\mu dx^\nu + b_0^2dy^2 \\
    &= \bar{g}_{MN}dx^M dx^N \quad (1)
\end{align*}
\]

where \(k\) is the AdS curvature scale given by \(k = \sqrt{-\frac{\Lambda}{6M^3}}\) and \(y\) is the fifth coordinate with \(y \in [-\frac{1}{2}, \frac{1}{2}]\). Then, the effective 4D Planck scale is determined by \(M_P^2 = M^3 \int_{-\frac{1}{2}}^{\frac{1}{2}} dy e^{-2kb_0|y|} = \frac{M^3}{k}(1 - e^{-kb_0})\), which implies the weak dependence of the 4D Planck scale on the extra dimension. Even the non-compact extra dimension also allows the finite 4D Planck scale. Since there exists two 3-branes with brane tensions \(\Lambda_1\), \(\Lambda_2\) at \(y = 0\) and \(y = \frac{1}{2}\), respectively, by the consistency of the boundary conditions on the branes, the following finetuning condition is required between brane and bulk cosmological constants,

\[
\Lambda_1 = -\Lambda_2 = \sqrt{6M^3\Lambda_b}. \quad (2)
\]

And, as the warp factor exponentially decreases along the bulk, we can obtain the weak scale as the physical scale of the brane at \(y = \frac{1}{2}\) by appropriately choosing the distance between two branes (i.e., \(b_0 \sim 74/k \sim 74/M_P\)) without introducing another large hierarchy.

In a next step, cosmological considerations in the extra dimension scenarios follow essentially. The cosmological bound on the ADD scenario comes from the effects of the light Kaluza-Klein (KK) graviton excitations. However, the masses of the KK graviton modes should be larger than about a few GeV for nucleosynthesis\(^1\), which corresponds to \(b_0 < 80/k\), so it seems that the light KK graviton problem may be avoided in the RS model. On the other hand, we have to consider the cosmological expansion of 3-brane universes in the bulk and check whether the normal Hubble expansion rate can be reproduced on the brane. For the sake of this, we assume that the 3-branes are homogeneous and isotropic such that the 5D metric reads,

\[
\begin{align*}
    ds^2 &= -n^2(\tau, y)d\tau^2 + a^2(\tau, y)A_{ij}dx^i dx^j + b^2(\tau, y)dy^2. \quad (3)
\end{align*}
\]

Then, from the Einstein equations of motion with the above metric, we have the following non-trivial equation for the Hubble expansion rate on each
brane$^6$, 

\[
H^2 = \frac{\Lambda_b}{3M^3} + \frac{\rho_i + \Lambda_i}{36M^6} + \frac{K}{2M^4\epsilon_i} \\
= \text{sgn}(\Lambda_i)\left(\frac{\rho_i}{3M^2_p}\right) + \frac{\rho_i^2}{3|\Lambda_i|M^2_p} + \frac{K}{2M^4\epsilon_i}, \quad i = 1, 2
\]  

(4)

where we used the Eq. (2) and $K$ is a constant of motion determined from the initial condition and the last term is so called the dark radiation$^{10}$. Consequently, the $\rho_2^2$ term and the dark radiation term in the above Hubble expansion would drastically affect a later cosmology in our brane, e.g., the big bang nucleosynthesis and there is the wrong sign problem in the Hubble parameter from the linear term in $\rho_2$. To avoid the effects to the nucleosynthesis, the brane tension must be $|\Lambda_2| \gg (MeV)^4$ and the dark radiation density should be diluted by inflation and/or reheating processes. And the wrong sign problem can be solved by having the positive tension brane $\Lambda_2 > 0$$^{12}$ or by introducing a mechanism for stabilizing the size of the extra dimension while the branes expand$^{8,9}$.

### 3 Static and inflationary solutions

When the higher curvature terms are added as correction terms in the action, the higher derivatives are generically induced in the equations of motion, which gives rise to runaway solutions and tends to make the system unstable. In particular, since the first derivative of the RS metric has to be discontinuous along the bulk to compensate the delta function sources due to the branes, we have to choose the higher curvature terms such that there don’t appear higher derivatives of the metric with respect to the $y$ coordinate than the second. The Gauss-Bonnet term, $E = R^2 - 4R_{MN}R^{MN} + R_{MNPQ}R^{MN}PQ$, one of particular choices of the curvature squared terms, is a topological term in $D = 4$ and it does not affect the graviton propagator even for the $D > 4$ flat spacetime background$^{14}$. Since there are no higher order derivatives induced from the Gauss-Bonnet term, it seems that the Gauss-Bonnet term is consistent with the RS model as the effective interaction.

When the Gauss-Bonnet term is included as the effective interaction in the RS model, we obtain two RS type static solutions with the AdS curvature scale $k$ as follows$^{12}$,

\[
k = k_{\pm} \equiv \left(\frac{M^2}{4\alpha} \left[1 \pm \sqrt{1 + \frac{4\alpha\Lambda_b}{3M^5}}\right]\right)^{1/2}
\]  

(5)
where $M$, $\alpha$ and $\Lambda_b$ are the 5D fundamental scale, the dimensionless parameter of the Gauss-Bonnet term and the bulk cosmological constant, respectively. By the boundary conditions on the branes, the finetuning conditions are to be satisfied between input parameters, $\alpha$, $\Lambda_1$, $\Lambda_2$ and $\Lambda_b$:\(^\dagger\dagger\),

$$\Lambda_1^+ = -\Lambda_2^+ = \mp 6k_\pm M^3 \sqrt{1 + \frac{4\alpha \Lambda_b}{3M^3}}. \tag{6}$$

From the above result, we find that for the $k_+$ solution, the bulk cosmological constant is allowed to be positive and it is possible to have a positive tension brane as the visible brane at $y = \frac{1}{2}$. On the other hand, the $k_-$ solution is connected with the RS solution in the limit of vanishing $\alpha$, for which the visible brane has a negative tension and the bulk cosmological constant has to be negative as in the RS case.

Unless the input parameters are finetuned like the Eq. (6), the branes and the bulk space are not static any more\(^\dagger\). Then, for inflationary solutions in the RS model with the Gauss-Bonnet term, we assume a separable metric ansatz like $n = f(y)$, $a = f(y)g(\tau)$ in the Eq. (3). Here we have the extra dimension static necessarily for the separable metric; $b = b_0 = const$ and $\dot{g}/g = H_0 = const$. Consequently, the inflationary solutions are two-fold as follows\(^\dagger\dagger\),

$$ds^2 = \left(\frac{H_0}{k_\pm}\right)^2 \sinh^2(-k_\pm b_0|y| + c_0)[-d\tau^2 + e^{2H_0 \tau} \delta_{ij}dx^idx^j] + b_0^2 dy^2 \tag{7}$$

where the constants $b_0$ and $c_0$ are determined from the boundary conditions on the branes. In the limit of $H_0 \to 0$ and $c_0 \to +\infty$ with keeping the ratio $(H_0e^{c_0})/(2k_\pm) \to 1$ fixed, the two RS type static solutions are recovered along with the consistency from the boundary conditions, Eq. (6). Therefore, one can see the possibility of the visible brane with the positive tension again. By making the 4-dimensional part of the metric be in the form $ds^2_4 = -dt^2 + e^{2H(y)\tau} \delta_{ij}dx^idx^j$, we get the Hubble parameter at the visible brane expressed as $H_{\text{vis},\pm} = \sqrt{(k_{\text{vis},\pm})^2 - k_\pm^2}$. Here $k_{\text{vis},\pm}^2 = k_\pm^2$ for the static solutions and the two parameters corresponding to the $k_+$ and $k_-$ solutions at the visible brane, $k_{\text{vis},\pm}$, are given by

$$k_{\text{vis},\pm} = \frac{(\Lambda_2^\pm + \rho_{\text{vis}})}{6M^3 \sqrt{1 + (4\alpha \Lambda_b/3M^3)}} \tag{8}$$

where $\Lambda_2^\pm \gg \rho_{\text{vis}}$. Thus the Hubble parameter at the visible brane is given by

$$H_{\text{vis},\pm}^2 = \frac{\rho_{\text{vis}} (\rho_{\text{vis}} + 2\Lambda_2^\pm)}{36M^6 (1 + 4\alpha \Lambda_b/3M^3)} = \frac{\rho_{\text{vis}}}{3M_p^2 \sqrt{1 + 4\alpha \Lambda_b/3M^3}} \left[1 + \frac{\rho_{\text{vis}}}{2\Lambda_2^\pm}\right]. \tag{9}$$
Therefore, with the $k_+$ solution we can obtain a plausible FRW universe at low temperatures. As a result, our additional solution could be proposed to solve the negative tension problem in the RS model. However, as we will see in subsequent sections, we will show that the $k_+$ solution may be unstable under perturbations.

4 RS model with the Gauss-Bonnet term as a 4D gravity theory

In the second RS model with a single brane of positive tension\(^4\), it has been shown that gravity can be localized on the brane even if the extra dimension is non-compact. As a result, the 4D Newtonian gravity can be reproduced on the brane without the need of compactifying the extra dimension. The 4D graviton is identified as a normalizable bound state of massless graviton due to the delta function source of the brane and continuous Kaluza-Klein modes give rise to small corrections to the 4D Newtonian gravity since they are weakly coupled to the brane matters\(^4\). However, it seems that it is not plausible to detect the extra dimension in the second RS model because the effects from the extra dimension appear around the AdS curvature scale $k$, which may be about the Planck scale for giving no hierarchy. (There also exists a stringy picture of lowering the AdS curvature scale.\(^18\)) On the other hand, the localization of gravity has been also shown by decomposing the full graviton propagator\(^15,16\). As a result, it turns out that a localized source induces a localized field, which diminishes as one goes toward the AdS horizon\(^15,16\). And, the brane bending effect in the existence of matter on the brane is shown to be crucial for consistency of the linearized approximation\(^16\) and is necessary to reproduce the 4D Einstein gravity on the brane\(^15\).

For the second RS model, the extra dimension is non-compact with $y \in (-\infty, \infty)$, of which just the half $[0, \infty)$ is sufficient for discussion. Having the perturbed metric as $g_{MN} = \bar{g}_{MN} + h_{MN}$, Randall and Sundrum\(^4\) took the gauge of $h_{55} = h_{5\mu} = 0$ (Gaussian normal condition) and $\partial^{\mu}h_{\mu\nu} = h_{\mu\nu}^\mu = 0$ (transverse traceless condition) in the absence of matter on the brane, of which the advantage is that all components of the metric are decoupled. In general, however, the metric does not satisfy the RS gauge condition on the brane with matter and thus we have to maintain some degrees of freedom of the metric to satisfy the brane junction condition. As a result, there exists an additional unphysical scalar degree of freedom, which is harmful because it might couple to the trace of the energy-momentum tensor. However, it has been shown that the scalar degree can be cancelled out by a fifth coordinate transformation (or brane bending)\(^15\).
For the case in the second RS model with the Gauss-Bonnet term\textsuperscript{12,13}, we choose just the Gaussian normal condition for the metric perturbation for the case with matter on the brane. Here we put $b_0 = 1$ in the Eq. (1) and assume that the matter is localized on the brane, i.e., $T_{55} = T_{5\mu} = 0$ and $T_{\mu\nu} = S_{\mu\nu}(x)\delta(y)$. (Note that $T_{\mu\nu}$ are not including the contribution from the brane tension.) Then, the equation of motion for the trace $h$ follows\textsuperscript{13},

$$
\partial_y \left[ e^{-2ky} \partial_y (e^{2ky} h) \right] = \frac{2}{3} M^{-3} \left( 1 - \frac{4ak^2}{M^2} \right)^{-1} T_{\mu\nu}.
$$

(10)

Therefore, if $T_{\mu\nu} \neq 0$, the trace $h$ has the exponentially growing component. So, to cancel the growing component for validity of the linearized approximation, we have to take the $y$ position of the brane shifted by $-\xi^5$,

$$
\partial^\mu \partial_\mu \xi^5(x) = \frac{1}{6} M^{-3} \left( 1 - \frac{4ak^2}{M^2} \right) S_{\mu\nu} - 1 S_{\mu\nu}.
$$

(11)

In fact, $\xi^5$ is the gauge choice of the 5D coordinate transformation maintaining the metric as a Gaussian normal form. Then, we can always choose the transverse traceless condition (i.e., the RS gauge) for the metric by rewriting the Eq. (10) and the relation $\partial_y (e^{2ky} \partial^\lambda h_{\mu\lambda}) = \partial (e^{2ky} \partial_\mu h)$ in the coordinate where the brane is shifted by $-\xi^5$ along the bulk. As a result, the brane bending $\xi^5$ will play the role of the source for the metric perturbation in the RS gauge. Consequently, in the initial coordinate where the brane is perpendicular to the AdS horizon, the metric perturbation on the brane is made of two components as follows\textsuperscript{13},

$$
h_{\mu\nu}(x) = h^{(m)}_{\mu\nu}(x) + h^{(b)}_{\mu\nu}(x)
$$

(12)

where

$$
h^{(m)}_{\mu\nu}(x) = -M^{-3} \int d^4x' G_5(x, 0; x', 0) \left( S_{\mu\nu}(x') - \frac{1}{3} \eta_{\mu\nu} S^k_k(x') \right)
$$

(13)

$$
h^{(b)}_{\mu\nu}(x) = 2k \eta_{\mu\nu} \xi^5(x)
$$

(14)

where $G_5$ is the 5D graviton propagator in the presence of the Gauss-Bonnet term\textsuperscript{13}. For instance, in case of a static point source with mass $m$ on the brane, i.e., for the energy-momentum tensor $S_{\mu\nu} = m \delta_{\mu\nu} \delta \delta^{(3)}(x)$, we obtain the approximate metric perturbation for a static point source on the brane as the following\textsuperscript{13},

$$
h_{00}(x) = \frac{2G_N m}{r} \left[ 1 + \frac{2}{3} \left( 1 - \frac{2\beta}{3} \right)^{-1} \left( \frac{1}{1 + 2\beta} \right)^2 \frac{1}{(kr)^2} \right],
$$

(15)

$$
h_{ij}(x) = \frac{2G_N m}{r} \left[ \left( \frac{1 + \frac{2\beta}{3}}{1 - \frac{2\beta}{3}} \right) + \frac{2}{3} \left( 1 - \frac{2\beta}{3} \right)^{-1} \left( \frac{1}{1 + 2\beta} \right)^2 \frac{1}{(kr)^2} \right] \delta_{ij},
$$

(16)
where by the Newton potential \( \Phi_N = -\frac{1}{2} h_{00} \), the Newton constant is given by

\[
G_N \equiv \frac{k}{8\pi M^2} \left( 1 - \frac{4\alpha k^2}{M^2} \right)^{-1} \left( 1 - \frac{2\beta}{1 + 2\beta} \right),
\]

(17)

\[
\beta \equiv \frac{4\alpha k^2/M^2}{1 - 12\alpha k^2/M^2}.
\]

(18)

For the \( k_+ \) solution, the Newton constant \( G_N \) would be negative and \( 1 - \frac{2\beta}{1 + 2\beta} > 0 \) always, which might give rise to massless and massive ghosts. That is, it means that the \( k_+ \) solution is unstable under perturbations and therefore we have to exclude it at the perturbative level. On the other hand, for the \( k_- \) solution, we can get the normal gravity without ghosts for \(-0.47 < \frac{4\alpha\Lambda}{3M^2} \leq 0 \) for \( \alpha > 0 \) or always for \( \alpha < 0 \). Therefore, even the \( k_- \) solution could excite ghost particles in some bulk parameter space with \( \alpha > 0 \).

As leading terms of \( h_{00} \) and \( h_{ij} \) components are not equal in the above result, elimination of the unphysical scalar degree due to the brane bending effect is incomplete in the presence of the Gauss-Bonnet term, unlike in the original RS case. Therefore, we can also show that the bending of light passing by the Sun could be modified with the Gauss-Bonnet term. For a source in \( xz \) plane on the brane, for instance, the bending of light travelling in \( z \) direction is described by a Newton-like force law, \( \ddot{x} = \frac{1}{2} (h_{00} + h_{zz})_{,x} \). If the metric perturbations due to the Sun are approximated by those from a point source, the bending of light is \( (1 - \frac{2\beta}{1 + 2\beta})^{-1} \) of that predicted from the 4D Einstein gravity. Therefore, from the experimental measurements of the bending of light\(^{17} \), we can get another bound on the Gauss-Bonnet coefficient as \(-0.20 < \frac{4\alpha\Lambda}{3M^2} < 1.2 \) for the \( k_- \) solution connected to the RS solution.

5 Conclusions

We studied static and inflationary solutions in the Randall-Sundrum framework with the Gauss-Bonnet term added to the standard Einstein term. It has been argued that the Gauss-Bonnet term is the only consistent curvature squared interaction in the Randall-Sundrum model. In particular, our additional RS type solution might solve the negative tension problem but it may be unstable under perturbations. And we showed that for a vanishing Gauss-Bonnet coefficient, the brane bending allows us to reproduce the 4D Einstein gravity at the linearized level.
Acknowledgments

The author thanks S.-H. Moon and J. D. Park for useful discussions during the works.

References