Four–quark operators in hadrons

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We present calculations of matrix elements of 4–quark operators in the pion and in the nucleon extracted from quenched Monte Carlo simulations at $\beta=6.0$ using Wilson fermions. These operators are relevant for higher–twist effects. We are particularly careful to avoid mixing with lower–dimensional operators by choosing appropriate flavour structures.

1. INTRODUCTION

The operator product expansion (OPE) expresses (Nachtmann) moments of hadronic structure functions in terms of Wilson coefficients (usually calculated in perturbation theory) and nonperturbative hadronic matrix elements. The leading contribution in the deep–inelastic limit $Q^2 \to \infty$ is provided by operators of twist 2 with corrections suppressed by powers of $1/Q^2$ coming from operators of twist 4 and higher. Schematically one finds for $n = 2, 4, 6, \ldots$

\begin{equation}
\int_0^1 \text{d}x x^{n-2} F_2(x, Q^2) |_{\text{Nachtmann}} = c_n^{(2)}(Q^2/\mu^2, g(\mu)) A_n^{(2)}(\mu) + O\left(\frac{1}{Q^4}\right).
\end{equation}

The reduced matrix elements $A_n^{(t)}$ of twist $t$ and spin $n$ depend on the renormalisation scale $\mu$. Whereas the corresponding Wilson coefficients $c_n^{(t)}$ are dimensionless, the mass dimension of $A_n^{(t)}$ is $t - 2$. In the flavour–nonsinglet channel, the twist–2 operators are 2–quark operators,

\begin{equation}
\bar{\psi} \gamma_{\mu_1} \hat{D}_{\mu_2} \cdots \hat{D}_{\mu_n} \psi,
\end{equation}
symmetrised in all indices and with trace terms subtracted.

An important class of twist–4 operators are 4–quark operators. In particular, the twist–4, spin–2 matrix element $A_2^{(4)}$ is given by

\begin{equation}
\langle p|\hat{A}_2^{(4)}|p\rangle = 2 A_2^{(4)}(p\mu p\nu - \text{traces})
\end{equation}
in terms of the 4–quark operator

\begin{equation}
A_2^{(4)} = \bar{\psi} G \gamma_\mu \gamma_5 t^a \psi \hat{G} \gamma_\nu \gamma_5 t^a \psi.
\end{equation}
The quark field $\psi$ carries flavour, colour, and Dirac indices, the matrices $t^a$ are the usual generators of colour SU(3), and the flavour matrix $G$ contains the quark charges:

\begin{equation}
G = \text{diag}(e_u, e_d) = \text{diag}(2/3, -1/3)
\end{equation}
for two flavours. The Wilson coefficient reads [1]

\begin{equation}
c_2^{(4)} = g^2 (1 + O(g^2)).
\end{equation}
These expressions are to be compared with their twist–2 counterparts:

\begin{equation}
\langle p|O_{(\mu\nu)}|p\rangle = 2 A_2^{(2)}(p\mu p\nu - \text{traces})
\end{equation}
with the operator
\[ O_{\mu\nu} = \frac{i}{2} \bar{\psi} G^{\mu} \gamma_\mu \nabla_\nu \psi \] (7)
and the Wilson coefficient \( c_2^{(2)} = 1 + O(g^2) \).

The operators (4) and (7) transform identically under Lorentz transformations, but (4) has dimension 6, whereas (7) has only dimension 4: 4–quark operators will in general mix with 2–quark operators of lower dimension. This fact complicates the investigation of 4–quark operators, because the mixing with lower–dimensional operators cannot be calculated reliably within perturbation theory. For the time being, we do not attempt a nonperturbative calculation of the renormalisation and mixing coefficients of 4–quark operators. Instead we restrict ourselves to cases where mixing with lower–dimensional operators is prohibited by flavour symmetry.

In the following we present Monte Carlo data from quenched simulations at \( \beta = 6.0 \) with Wilson fermions on a \( 16^3 \times 32 \) lattice. From now on all operators are written down in Euclidean space.

2. PION

In the case of the pion, we consider the symmetry SU(2)_F, i.e. isospin symmetry. While 2–quark operators can have at most isospin \( I = 1 \), there are 4–quark operators with \( I = 2 \). Such operators are protected from mixing with lower–dimensional operators and it makes sense to renormalise them perturbatively. Therefore we study the \( I = 2 \) pion structure function
\[ F_2^{I=2} = F_2^+ + F_2^- - 2F_2^0, \] (8)
which is purely higher twist and receives no contributions from 2–quark operators.

Omitting Dirac and colour matrices, the flavour structure of the relevant operator in the OPE is given by
\[
\begin{align*}
(e_u \bar{u} u + e_d \bar{d} d)(e_u \bar{u} u + e_d \bar{d} d) \\
= \frac{1}{6} & (\bar{u} u)(\bar{u} u) + (\bar{d} d)(\bar{d} d) \\
- (\bar{u} u)(\bar{d} d) - (\bar{d} d)(\bar{u} u) - (\bar{u} d)(\bar{u} u) - (\bar{d} u)(\bar{u} u)
\end{align*}
\]
up to contributions with \( I = 0, 1 \). Although this flavour structure excludes mixing with 2–quark operators, mixing with other 4–quark operators is still possible. For spin 2, a basis of 4–quark operators whose \( I = 2 \) components are closed under renormalisation is given by the spin–2 projections of
\[
V^c_{\mu\nu} = \bar{\psi} G_{\mu\nu} t^a \psi G_{\mu\nu} t^a \psi
\]
\[ A^c_{\mu\nu} = \bar{\psi} G_{\mu\nu} \gamma_5 t^a \psi G_{\mu\nu} \gamma_5 t^a \psi \]
\[ T^c_{\mu\nu} = \bar{\psi} G_{\mu\nu} t^a \psi G_{\mu\nu} t^a \psi \]

together with the operators \( V_{\mu\nu} \), \( A_{\mu\nu} \), and \( T_{\mu\nu} \) which differ from the above by the omission of the colour matrices \( t^a \). One–loop results for the renormalisation coefficients are given in [2]. These are to be combined with the bare matrix elements extrapolated to the chiral limit. Expressing the matrix elements in terms of the pion decay constant \( f_\pi \) we get [2]
\[ A_2^{(4)I=2} = 0.133(51) f_\pi^2, \] (10)
where \( f_\pi = 131\text{MeV} \) in the real world. The order of magnitude of this result is reproduced by the vacuum insertion approximation, which yields an answer proportional to \( f_\pi^2 \). For the first moment of \( F_2 \) we find
\[
\int_0^1 dx F_2(x,Q^2)|_{\text{Nachtmann}}^{I=2} = 1.67(64) \frac{f_\pi^2 \alpha_s(Q^2)}{Q^2} + O(\alpha_s^2). \] (11)

Except for very small values of \( Q^2 \) this is considerably smaller than the leading twist–2 contribution to the \( \pi^+ \) structure function computed in a quenched Monte Carlo simulation [3]:
\[
\int_0^1 dx F_2(x,Q^2)|_{\text{Nachtmann}}^{\pi^+} = 0.152(7). \] (12)
Of course, the fact that the particular twist–4 contribution (11) is rather small does not tell us that all other higher–twist contributions will be small as well.
3. PROTON

In the case of the proton it is more difficult to find 4–quark operators that are safe from mixing with lower–dimensional operators: the expectation value of any \( I = 2 \) operator in the proton vanishes. So we have to enlarge the flavour symmetry which we consider from SU(2) to SU(3), i.e. we assume three quarks of the same mass. Correspondingly, the flavour structure of the operator in the OPE is now

\[
\mathcal{O} = (e_u \bar{u}u + e_d \bar{d}d + e_s \bar{s}s)(e_u \bar{u}u + e_d \bar{d}d + e_s \bar{s}s).
\]

Whereas 2–quark operators transform under SU(3) according to \( \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \), we have for 4–quark operators: \( \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = 2 \cdot \mathbf{1} \oplus 4 \cdot \mathbf{8} \oplus \mathbf{10} \oplus \mathbf{16} \oplus \mathbf{27} \). 4–quark operators with \( I = 0, 1 \); \( I_3 = 0 \), and hypercharge \( \mathcal{Y} = 0 \) from the \( \mathbf{10}, \mathbf{16}, \mathbf{27} \) multiplets do not mix with 2–quark operators and can be used in a proton expectation value, e.g. the \( I = 1 \) operator

\[
\mathcal{O}^{I=1}_{27} = \frac{1}{10}[(\bar{u}u)(\bar{u}u) - (\bar{d}d)(\bar{d}d)] - (\bar{s}s)(\bar{s}s) + (\bar{d}s)(\bar{d}s) + (\bar{s}d)(\bar{s}d) - (\bar{s}s)(\bar{s}s) + (\bar{u}u)(\bar{u}u) + (\bar{d}d)(\bar{d}d) + (\bar{s}s)(\bar{s}s)].
\]

belongs to the \( \mathbf{27} \) multiplet. Analogously to the case of the pion, one can find linear combinations of the structure functions of the octet baryons \( (p, n, \Lambda, \Sigma, \Xi) \) which project out the desired flavour component, e.g.

\[
\langle \Sigma^+ | \mathcal{O} | \Sigma^+ \rangle - 2 \langle \Sigma^0 | \mathcal{O} | \Sigma^0 \rangle + \langle \Sigma^- | \mathcal{O} | \Sigma^- \rangle = \langle p | \mathcal{O}^{I=1}_{27} | p \rangle,
\]

\[
\langle \Sigma^+ | \mathcal{O} | \Sigma^+ \rangle + \langle \Sigma^- | \mathcal{O} | \Sigma^- \rangle + 6 \langle \Lambda | \mathcal{O} | \Lambda \rangle - 2 \langle \Xi^0 | \mathcal{O} | \Xi^0 \rangle - 2 \langle \Xi^- | \mathcal{O} | \Xi^- \rangle - 2 \langle p | \mathcal{O} | p \rangle - 2 \langle n | \mathcal{O} | n \rangle = -\langle p | \mathcal{O}^{I=1}_{27} | p \rangle.
\]

The proton matrix elements are computed in the standard fashion from ratios of 3–point functions \( \langle B(t) \mathcal{O}(\tau) \bar{B}(0) \rangle \) over 2–point functions \( \langle B(t) \bar{B}(0) \rangle \) \( (0 \ll \tau \ll t) \) with suitable interpolating fields \( B \) and \( \bar{B} \). For a general 4–quark operator the 3–point function \( \langle B(t) \mathcal{O}(\tau) \bar{B}(0) \rangle \) consists of three types of contributions, which can be represented pictorially by the following diagrams

![Diagrams](image)

It is precisely through contributions of the form of the first two diagrams that the mixing with lower–dimensional operators occurs. Therefore these contributions cancel in the operators which we consider, and we are left with the contributions of the last type only.

As an example of our results we show in fig.1 the chiral extrapolation of the bare proton matrix element of \( V_{ij}^{\tau} \) – traces \( (I = 1 \) component in the \( \mathbf{27} \) representation of SU(3)) divided by the fourth power of the proton mass \( m_p \). It indicates the order of magnitude of all 4–quark matrix elements that we have studied. After renormalisation this expression contributes to the reduced matrix element \( A_{2}^{(4)} \) according to

\[
\frac{A_{2}^{(4)}}{m_p^4} = \frac{2 \langle N | \cdots | N \rangle}{3 \ m_p^4} \approx \frac{2 \langle N | \cdots | N \rangle}{3 \ m_p^4},
\]

and we obtain for the lowest moment of \( F_2 \) in our special flavour channel

\[
\int_0^1 dx F_2(x, Q^2) \bigg|_{\text{Nachtmann}}^{27,I=1} = -0.0006(5) \frac{m_p^2 \alpha_s(Q^2)}{Q^2} + O(\alpha_s^2).
\]

In the proton the corresponding twist–2 contribution is \( \approx 0.14 \). Once again, the twist–4 correction
4. COMPARISON OF RESULTS

Let us finally compare in fig. 2 the renormalised pion matrix elements $\langle \pi^+ | 6 \cdot (\cdots)^{I_{1}^{2}} | \pi^+ \rangle / m_{\pi}^2$ with the corresponding renormalised proton matrix elements $\langle p | 10 \cdot (\cdots)^{I_{27}^{2}} | p \rangle / m_{p}^2$ (in lattice units). In the operators we have set $\mu = \nu = 4$ (with traces subtracted). The normalisation of the pion and proton matrix elements is chosen such that the flavour structure $(\bar{u}u)(\bar{u}u)$ appears with the factor 1 in both cases. In view of the fact that pion and proton are very different particles it is hardly surprising that the numbers do not show many similarities.

5. SUMMARY

Contributions of higher twist to hadronic structure functions promise to challenge lattice QCD for a few more years. Still, 4–quark operators can give reasonable signals in present quenched Monte Carlo simulations. However, when the flavour structure is such that it prohibits mixing with lower–dimensional (2–quark) operators, the matrix elements turn out to be rather small.

Unfortunately, these flavour structures are somewhat exotic and not easily accessible to experiment. The investigation of physically more interesting flavour channels is of course desirable, but impossible without progress in nonperturbative renormalisation. It remains to be seen what will happen first: a measurement of structure functions in the above flavour channels or a reliable lattice calculation of twist–4 matrix elements relevant for the proton or the pion.

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