The relativistic 2-body problem, much like the non-relativistic one, is reduced to describing the motion of an effective particle in an external field. The concept of a relativistic reduced mass and effective particle energy, introduced some 30 years ago to compute relativistic corrections to the Balmer formula in quantum electrodynamics, is shown to work equally well for classical electromagnetic and gravitational interaction. The results for the gravitational 2-body problem have more than academic interest since they apply to the study of binary pulsars that provide precision tests for general relativity. They are compared with recent results derived by other methods.

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1. INTRODUCTION

The notion of an effective relativistic particle describing the relative motion of a 2-body system was first introduced in the context of the quasipotential approach to the quantum field theoretical eikonal approximation [1] and bound state problem in quantum electrodynamics [2], [3]. For a survey of subsequent developments - see [4]. Only later a general classical mechanical formulation of the relativistic 2-body problem was given [5] within Dirac’s constraint Hamiltonian approach [6] (for a review - see [7]).

The central concept of a relativistic reduced mass is derived in this early work by observing that the total mass $M$ of a 2-particle system should be substituted by its total centre-of-mass (CM) energy:

$$\frac{w}{c^2} = M + \frac{1}{c^2}E \quad \left( \sim M \text{ for } \frac{|E|}{Mc^2} \ll 1 \right),$$

(1)

$$M = m_1 + m_2.$$

This suggests using an energy dependent expression

$$m_w = \frac{m_1 m_2 c^2}{w} \quad \left( \rightarrow \mu = \frac{m_1 m_2}{M} \text{ for } \frac{w}{c^2} \rightarrow M \right),$$

(2)

for the relativistic generalization of the reduced mass $\mu$.

Furthermore, if we determine the off-shell momentum square $b^2(w^2)$ for a pair of free particles as the solution for $p^2$ of the equation

$$\frac{w}{c} = \sqrt{m_1^2c^2 + p^2} + \sqrt{m_2^2c^2 + p^2} \quad \Rightarrow$$

$$p^2 = b^2(w^2) = \frac{w^4 - 2(m_1^2 + m_2^2)c^4w^2 + (m_1^2 - m_2^2)^2c^8}{4w^2c^2}$$

(3)

we find for the effective particle CM energy

$$E_w = c\sqrt{m_w^2c^2 + b^2(w^2)} = \frac{w^2 - m_1^2c^4 - m_2^2c^4}{2w}.$$

(4)

The interest in this notion of an effective particle was revived recently in [8] where a modified version of it was applied to the general relativistic 2-body dynamics and the relevance (and relative simplicity) of the dimensionless counterpart of the effective particle energy

$$\epsilon = \frac{E_w}{m_w c^2} = \frac{w^2 - m_1^2c^4 - m_2^2c^4}{2m_1 m_2 c^4}$$

(5)

(which only makes sense for positive mass particles) was pointed out. On the other hand, the authors of [8] preferred to work with the non-relativistic reduced mass $\mu$ rather than with the energy dependent quantity $m_w$.

The present paper was motivated by our wish to demonstrate the advantage of the original notion of relativistic effective particle, cited in the beginning, in both the (classical) electromagnetic and gravitational 2-body problem.

We begin in Sec.2, by recalling the constraint Hamiltonian approach to a relativistic particle system. A new justification is provided on the way for formula (2) for the energy dependent reduced mass identified as the coefficient of the relative velocity in the expression for the effective particle 3-momentum in the CM frame.

Sec.3 is devoted to the electromagnetic interaction of two oppositely charged particles.

The general relativistic gravitational two-body problem (which continues to attract attention - see, e.g. [9], [10], [11], [12], [13]) is addressed in Sec.4. We compute the perihelion shift as well as the parameters of the last stable orbit in the gravitational case and compare with earlier results.

In both cases we neglect the retardation effect: it is known not to contribute to the first post-Newtonian approximation (see [14]) for which our results agree with previous calculations. The effects of the relativistic kinematics, on the other hand, are computed exactly.
The possibility to take into account the finite velocity of propagation of interactions starting with the second post-Newtonian approximation within the effective particle approach of this paper is discussed in the concluding Sec.5.

II. VELOCITY SPACE FORMULATION OF THE CONSTRAINT HAMILTONIAN APPROACH TO THE RELATIVISTIC 2-BODY PROBLEM

The mass-shell constraint for a free relativistic particle can be interpreted as a "Lorentz invariant Hamiltonian":

\[ H = \frac{1}{2\lambda} (m^2c^2 + p^2) \approx 0, \quad p^2 := p^2 - p_0^2. \]  

Indeed, the equations of motion are obtained by taking Poisson brackets with \( H \):

\[ i_\mu = \{x_\mu, H\} = \frac{1}{\lambda} p_\mu, \quad \dot{p}_\mu = \{p_\mu, H\} = 0 \]  

for \( \{x_\mu, p_\nu\} = \delta_\mu^\nu \).

Here \( \lambda \) is a Lagrange multiplier (assumed independent of \( x \)); it is linked to the choice of a time scale. The Hamiltonian constraint gives rise to a singular Lagrangian through the Legendre transform, \( \mathcal{L} = px - H \) for \( p \) determined (as a function of \( x \)) from \( \dot{x} = \frac{\partial H}{\partial p} \):

\[ \mathcal{L}(x, \dot{x}; \lambda) = (px - H) = \frac{\lambda}{2} \dot{x}^2 - \frac{m^2c^2}{2\lambda}. \]  

**Remark 1** It is important to remember – especially for the subsequent extension to a pseudo-Riemannian space-time – that the component \( p_0 \) of the covariant 4-momentum \( p_\mu \), canonically conjugate to the space-time coordinate \( x^\mu \), is minus the energy, \( p_0 = -E_p/c \) (\( E_p > 0 \); for a free particle \( E_p/c = \sqrt{m^2c^2 + p^2} \)). The equality \( p^0 = E_p/c \) is not generally covariant, it is accidentally valid for Cartesian coordinates in flat Minkowski space.

For \( m^2 > 0 \) \( \lambda \) can be excluded from the condition

\[ \frac{\partial \mathcal{L}}{\partial \lambda} = \frac{1}{2} \left( \dot{x}^2 + \frac{m^2c^2}{\lambda^2} \right) = 0 \Rightarrow \mathcal{L}(x, \dot{x}) = -mc\sqrt{-\dot{x}^2}. \]  

**Remark 2** For \( m = 0 \) Eq. (9) implies the constraint \( \dot{x}^2 \approx 0 \) and only the original expression (8) for the Lagrangian remains meaningful.

For a two particle system we introduce a pair of generalized mass-shell constraints

\[ \varphi_a = \frac{1}{2} (p_a^2 + m_a^2 + \Phi) \approx 0, \quad a = 1, 2 \]  

satisfying the (strong) compatibility condition

\[ \{\varphi_1, \varphi_2\} = \left( p_2 \frac{\partial}{\partial x_2} - p_1 \frac{\partial}{\partial x_1} \right) \Phi = 0. \]  

Denote by \( P \) and \( w \) the CM momentum and the total energy:

\[ P = p_1 + p_2, \quad w^2 = -P^2c^2. \]  

We shall exploit the fact that the difference \( \varphi_1 - \varphi_2 \) of the constraints (10) is independent of the interaction \( \Phi \) to define the relative momentum

\[ p = \mu_2 p_1 - \mu_1 p_2, \quad \mu_1 + \mu_2 = 1 \]  

determining \( \mu_1 - \mu_2 \) from the strong equation

\[ 2Pp = 2\varphi_1 - 2\varphi_2 = m_1^2c^2 + p_1^2 - m_2^2c^2 - p_2^2 \Rightarrow \mu_1 - \mu_2 = \frac{m_2^2 - m_1^2}{w^2}c^4. \]  

(Thus, for unequal masses, the \( \mu_a \) depend on the Poincaré invariant total energy square, so that the relation (13) is actually nonlinear.) The constraint \( \varphi_1 - \varphi_2 \approx 0 \) together with (14) implies the orthogonality of \( p \) and \( P \) as an universal kinematical constraint, readily solved in the CM frame:

\[ pP \approx 0, \quad (P = (w/c, 0) \Rightarrow p = (0, \mathbf{p})). \]  

In order to solve the compatibility equation (11) we introduce the projection \( x_\perp \) of the relative coordinate \( x_{12} = x_1 - x_2 \) on the 3-space orthogonal to \( P \):

\[ x_\perp = x_{12} + c^2 \frac{P_{x_{12}}}{w^2} P. \]  

Its square,

\[ R^2 = x_\perp^2 = x_{12}^2 + c^2 \left( \frac{P_{x_{12}}}{w^2} \right)^2 \]  

provides an invariant measure of the distance between the two particles in the CM frame.

The general Poincaré invariant solution of (11) will be written in the form

\[ \Phi = \Phi (R, p^2, p_R; E_w), \quad \text{where} \quad R_{pR} = px_\perp. \]  

The effective particle energy \( E_w \) is singled out since, as we shall see, \( \Phi \) is a quadratic polynomial in \( E_w \) in the cases of interest. One should also assume that \( \Phi \to 0 \) for \( R \to \infty \), thus making the separation of the mass terms in (10) meaningful.

The Hamiltonian constraint which replaces the equality \( p^2 \approx b^2(w^2) \) (3) in the presence of interaction is given by

\[ H := \frac{1}{2\lambda} (\mu_2 \varphi_1 + \mu_1 \varphi_2) = \frac{1}{2\lambda} (p^2 + m_w^2 + \Phi - E_w^2) \approx 0. \]  

**Remark 3** The \( \mu_a \) defined in (13) and (14) have a simple expression in terms of the CM energies \( E_a \),
and approach their non-relativistic values \( \frac{w}{c^2} \) for \( \frac{w}{c^2} \rightarrow M:

\[
\mu_a = \frac{E_a}{w} = \frac{1}{2} \pm \frac{m_a^2 - m^2 c^4}{2w^2} c^4 \rightarrow \frac{m_a}{M}, \quad a = 1, 2. \tag{21}
\]

We note that there are precisely three ways to factorize \( b^2 \) into \((\frac{E}{c} - mc)(\frac{E}{c} + mc)\) for \( E \) equal to \( E_1, E_2 \) and \( E_w (4) \):

\[
b^2(w^2) = \frac{E^2}{c^2} - m_a^2 c^2 = \frac{E^2}{c^2} - m_w^2 c^2, \quad (a = 1, 2). \tag{22}
\]

They correspond to the \( \frac{4}{3}(3) \) subdivisions of the four zeros of the numerator of (3) into two pairs.

This provides a fresh justification for the expressions (2) and (3) for the relativistic reduced mass and effective particle energy. We shall present yet another argument in favor of these expressions starting with a Lorentz covariant concept of a relative velocity (cf. [15] Sec.16). To simplify writing we choose units for which \( c = 1 \). The 4-velocities \( u_a \) of the two particles and the CM velocity \( U \) are proportional to the corresponding momenta:

\[
p_a = m_a u_a, \quad a = 1, 2; \quad P = wU, \quad m_a U u_a \approx -E_a, \quad U^2 = -1. \tag{23}
\]

The CM energies \( E_1 \) and \( E_2 (20) \) are thus related to the inner products of the velocities.

We note that the constraints (23) are equivalent to the kinematical constraint (15).

The 4-momentum of an effective particle with CM energy \( \epsilon \), \( u = (u_0, \mathbf{u}) \), \( \epsilon_0^{CM} = -\epsilon \) is introduced by defining first the relative 3-velocity \( \mathbf{u} \) as follows.

Let \( \Lambda = \Lambda(U) \) be the pure Lorentz transformation that carries the CM 4-velocity \( U \) into its rest frame. The conditions \( \Lambda^\mu \nu U_\nu = \delta_0^\mu \) and positive definiteness determine the (symmetric) Lorentzian matrix \( \Lambda \) uniquely:

\[
\Lambda = \begin{pmatrix} U^0 & -U_j \\ -U_i & \delta^i_j + \frac{U^i U_j}{1+U^0} \end{pmatrix}. \tag{24}
\]

We then find that the space parts of \( \Lambda u_1 \) and \( \Lambda u_2 \) are proportional to the same 3-vector \( \mathbf{u} \):

\[
\Lambda u_1 = \frac{m_2}{w} \mathbf{u}, \quad \Lambda u_2 = -\frac{m_1}{w} \mathbf{u}, \tag{25}
\]

which we shall identify with the relative 3-velocity:

\[
\mathbf{u} = \frac{(u_0^2 + U u_2) \mathbf{u}_1 - (u_0^2 + U u_1) \mathbf{u}_2}{1 + U^0}. \tag{26}
\]

(Note that a similar procedure reproduces the non-relativistic relative velocity \( \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \). Indeed, a Galilean transformation that sends the CM velocity \( \mathbf{v} = \frac{m_1}{m} \mathbf{v}_1 + \frac{m_2}{m} \mathbf{v}_2 \) to zero gives \( \mathbf{v}_1^{CM} = \mathbf{v}_1 - \mathbf{v} = \frac{m_1}{m} \mathbf{v}, \quad \mathbf{v}_2^{CM} = \mathbf{v}_2 - \mathbf{v} = -\frac{m_2}{m} \mathbf{v} \). In particular, in the CM frame we find

\[
\mathbf{p} = m_1 \mathbf{u}_1 + m_1 \mathbf{u}_2 = 0 \Rightarrow \\
\mathbf{u} = \frac{E_2}{m_2} \mathbf{u}_1 - \frac{E_1}{m_1} \mathbf{u}_2 = \frac{w}{m_2} \mathbf{u}_1 - \frac{w}{m_1} \mathbf{u}_2. \tag{27}
\]

The time component \( u_0 \) of the effective particle 4-momentum is determined by the condition:

\[
- \mathbf{u} = -u_0^{CM} = \epsilon. \tag{28}
\]

Here \( \epsilon \) is given by (5); for free particles it can be interpreted as the common value of the energy component \(-u_{10}\) in the rest frame of \( \mathbf{u}_2 \) and of \(-u_{20}\) in the rest frame of \( \mathbf{u}_1 \) which is a Lorentz invariant; in general, the following strong equation takes place

\[
u_1 u_2 + \frac{m_1}{2m_2} (u_1^2 + 1) + \frac{m_2}{2m_1} (u_2^2 + 1) = -\epsilon = \frac{m_1^2 + m_2^2 - w^2}{2m_1 m_2}. \tag{29}
\]

We identify, as usual, the space part of the effective particle 3-momentum \( \mathbf{p} \) in the CM frame with the common value of \( \mathbf{p}_1^{CM} \) and \( -\mathbf{p}_2^{CM} \) and define the proportionality coefficient between \( \mathbf{p} \) and the relative 3-velocity \( \mathbf{u} \) as the relativistic reduced mass \( m_w \).

\[
(\mathbf{p}_{eff} =) \quad \mathbf{p} = \mathbf{p}_1^{CM} = -\mathbf{p}_2 = m_1 \mathbf{u}_1^{CM} = m_w \mathbf{u} \tag{30}
\]

where \( \mathbf{p}_1^{CM} \) can also be expressed in terms of the components of \( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) in an arbitrary frame:

\[
\mathbf{p}_1^{CM} = \Lambda \mathbf{p}_1 = \frac{(E_2 + p_1^0) \mathbf{p}_1 - (E_1 + p_1^0) \mathbf{p}_2}{w + p_0}. \tag{31}
\]

(The relation (30) is consistent with the non-relativistic limit in which \( \mathbf{p}_1^{CM} = m_1 \mathbf{v}_1 = -m_2 \mathbf{v}_2 = \mu \mathbf{v} \).)

The relativistic reduced mass \( m_w \) defined by (30) is given, as anticipated, by (2). The effective particle 4-momentum \( \mathbf{p}_{eff} \) is expressed in terms of the relative momentum \( \mathbf{p} \) (13) and the CM 4-velocity \( U \) by

\[
\mathbf{p}_{eff} = E_w U + \mathbf{p}, \quad \text{i.e.} \quad p_{eff}^{CM} = (-E_w, \mathbf{p}), \quad p^{CM} = (0, \mathbf{p}). \tag{32}
\]

For positive mass particles it is convenient to write the Hamiltonian constraint (19) in terms of a dimensionless 4-momentum, corresponding to setting \( \Lambda = \lambda m_w^2 c^4 \) (= \( \lambda m_w^2 \)):

\[
\hat{H} = \frac{1}{2\lambda} (u_0^2 + \Phi + 1 - \epsilon^2) \approx 0, \tag{31}
\]

\[
\hat{\Phi} = \frac{\Phi}{m_w^2 c^4} = \hat{\Phi} (r, u^2, u_r; \epsilon) \tag{32}
\]
where the radial normalized momentum \( u_r \) is dimensionless while \( r \) has dimension of an action: \( r = R m_u c, \) \( p_r = m_u c u_r. \)

The interaction function \( \Phi \) in (31) is chosen to reproduce the known interaction of a test particle in an external field. For the combined electromagnetic and gravitational interaction of two particles of charges \( e_1 \) and \( e_2 \) we shall write:

\[
H = \frac{1}{2\lambda} \left( 1 + \left( 1 - 2\frac{\alpha G}{r} \right) \lambda \right) u_r^2 + \frac{J^2}{r^2},
\]

\[= \left( \frac{e + e_2}{r^2} \right)^2 \approx 0 \]

for \( r > 2\alpha G \) (33)

where \( J^2 \) is the square of the total angular momentum while \( \alpha G \) (denoted in [8] by \( \alpha \)) is the gravitational coupling measured in units of action:

\[
J = r \times u, \quad r = m_w e x_{CM}^c, \quad ru_r = ru, \quad r^2 = r^2, \quad J^2 = J^2; \quad \alpha G = \frac{m_1 m_2}{c}.
\]

This Hamiltonian constraint corresponds to the dimensionless interaction function

\[
\Phi = -2\frac{\alpha G}{r - 2\alpha G} c^2 - \frac{\hbar \alpha}{r - 2\alpha G} \epsilon^2 - \frac{\hbar^2 \alpha^2}{r (r - 2\alpha G)} - 2\frac{\alpha G}{r} u_r^2,
\]

\[\alpha = \frac{-e_1 e_2}{c \hbar} \]

(35)

(\( \alpha \) is positive for oppositely charged particles, studied in Sec.3 below; it coincides with the fine structure constant \( \alpha^{-1} = 137.036 \), for \( e_1 = -e_2 \) equal to the electron charge.) It is remarkable that the Poincaré invariant constraints in flat phase space admit an interpretation in terms of an effective particle moving along geodesics in a Schwarzschild space-time.

An advantage of our choice of \( \Lambda \) and of the variables \( r, u \) and \( \epsilon \) (instead of \( R = x_{CM}^c, p_{CM}^c \) and \( E_z \)) yielding the dimensionless Hamiltonian constraint (33) is the quadratical dependence of \( H \) in the (single !) energy parameter \( \epsilon \) (instead of the two \( w \)-dependent quantities \( E_w \) and \( m_w \) in (19). This allows to write down a Lagrangian for the (interacting) two-particle system using the standard Legendre transform

\[
\mathcal{L}(r, \dot{r}, \dot{\phi}, \phi; \epsilon, \lambda) = u \dot{r} - \epsilon \dot{r} - H, \quad u \dot{r} = u_r \dot{r} + J \dot{\phi},
\]

\[\epsilon = \lambda \left( 1 - 2\frac{\alpha G}{r} \right) \frac{\dot{r}}{\lambda}, \quad u_r = \frac{\lambda \dot{r}}{1 - 2\frac{\alpha G}{r}}, \quad J = \lambda r^2 \dot{\phi} \]

yielding

\[
\mathcal{L} = \frac{\lambda}{2} \left( \frac{\dot{r}^2}{1 - 2\frac{\alpha G}{r}} + r^2 \dot{\phi}^2 - \left( 1 - 2\frac{\alpha G}{r} \right) \dot{r}^2 \right) + \frac{\hbar \alpha}{r} - \frac{1}{2\lambda},
\]

or, varying in \( \lambda \) and excluding it from the resulting constraint,

\[
\mathcal{L} = \frac{\hbar \alpha}{r} - \lambda^{-1} = \left( \frac{1}{2} \left( 1 - 2\frac{\alpha G}{r} \right) \dot{r}^2 - \frac{\dot{r}^2}{1 - 2\frac{\alpha G}{r}} - r^2 \dot{\phi}^2 \right)^2.
\]

Here we have used angular momentum conservation which implies that the effective particle moves in a plane orthogonal to \( \mathbf{J} \):

\[
r = r (\cos \varphi, \sin \varphi, 0) \quad \text{for} \quad \mathbf{J} = (0, 0, J).
\]

We shall study the case of electromagnetic and gravitational interaction (corresponding to \( \alpha_G = 0 \) and to \( \hbar a = 0 \), respectively) in Sec.3 and 4 below solving in each case the resulting equations of motion. The result for the first relativistic ("post-Newtonian") approximation agrees with (more complicated) traditional calculations. Higher order corrections require taking into account retardation effects which can be also done within the present 1-body approach, as discussed in Sec.5. The Wheeler-Feynman non-local action for electrically charged particles [16] (see also [17]) seems to provide a systematic treatement of retardation effects (to all orders) but its inclusion in the present framework is not obvious.

### III. THE BOUND STATE PROBLEM FOR TWO OPPOSITELY CHARGED RELATIVISTIC PARTICLES

We start with the Hamiltonian constraint (33) in the absence of gravitational forces (i.e. with \( G = 0 = \alpha_G \)):

\[
2\lambda H = u_r^2 + \frac{J^2}{r^2} + 1 - \left( \epsilon + \frac{e^2}{r} \right)^2 \approx 0,
\]

\[e^2 \equiv \hbar \alpha = -\frac{e_1 e_2}{c}.
\]

(39)

The canonical Poisson bracket relations \( \{ x_{CM}^c, p_{CM}^c \} = \delta_{\alpha \beta} \delta_{\mu \nu} \) imply the following non-zero brackets for the radial and angular variables relevant for the planar motion:

\[
\{ r, u_r \} = 1 = \{ \phi, J \}.
\]

(40)

The equations of motion derived from (39), (40) read:

\[
\dot{r} = \frac{\partial H}{\partial u_r}, \quad \dot{\phi} = \frac{\partial H}{\partial J} = \frac{J}{\lambda r^2} \quad \text{and} \quad \dot{J} = \frac{\partial H}{\partial \phi} = 0.
\]

(41)

The equation for the effective particle trajectory obtained by dividing \( \dot{r} \) by \( \dot{\phi} \) is independent of \( \lambda \):

\[
- \frac{d}{d\phi} \left( \frac{J}{r} \right) = u_r = \frac{\sqrt{2\alpha G J}}{r} - (1 - \alpha_G^2) \left( \frac{J}{r} \right)^2 - \beta
\]

(42)

where all variables – starting with \( \lambda \) – are dimensionless, \( \alpha_j \) plays the role of a "classical fine structure constant":

\[
\alpha_j = \frac{e^2}{J} \left( \frac{\epsilon_j e_{2j}}{c J} \right) = \frac{\hbar \alpha}{J}.
\]

(43)

The bounded motion corresponds to the case when the expression under the square root has two positive zeros in \( \frac{J}{r} \). This implies
so that \[ 0 < \beta \equiv 1 - e^2 \leq \alpha_j^2, \quad (44) \]

\[ 0 < -b^2(w^2) \leq \alpha_j^2 m_w^2. \]

In fact, introducing the dimensionless inverse radius variable \[ y = \frac{1 - \alpha_j^2}{\epsilon} \frac{J}{r}, \quad (45) \]
we find
\[
\left( \frac{dy}{d\phi} \right)^2 = (1 - \alpha_j^2) \left( \frac{1 - \alpha_j^2}{\alpha_j^2} \frac{\beta}{1 - \beta} + 2y - y^2 \right). \quad (46)
\]

The discriminant of the quadratic expression in \( y \) in the right hand side is positive whenever Eq. (44) takes place. (The positivity requirement for the two real roots in \( \frac{J}{r} \) implies that \( 1 - e^2 \) and \( 1 - \alpha_j^2 \) are both positive.) We can then rewrite Eq. (42) in the form
\[
\frac{dy}{\sqrt{(y_1 - y)(y - y_2)}} = -\sqrt{1 - \alpha_j^2} d\phi \quad (47)
\]
with \[ y_{1.2} = 1 \pm \epsilon(\beta, J) \]
where \( \epsilon(\beta, J) \) plays the role of eccentricity:
\[
e^2(\beta, J) = 1 - \frac{1 - \alpha_j^2}{\alpha_j^2} \frac{\beta}{1 - \beta}, \quad 0 \leq \epsilon(\epsilon, J) < 1, \quad (48)
\]
the last inequality being valid in the domain (44), i.e. for \( 1 < \alpha_j^2 + e^2, \quad 0 < \alpha_j < 1, \quad 0 \leq \epsilon < 1 \) in the \((\epsilon, J, \beta)\) plane.
Integrating Eq. (47) with initial condition \( y(\phi = 0) = y_1 \) (i.e., the orbit passes through the perihelion for \( \phi = 0 \)) we obtain
\[
y = 1 + \epsilon(\beta, J) \cos \left( \sqrt{1 - \alpha_j^2} \phi \right). \quad (49)
\]

To compare this result with the familiar non-relativistic elliptic orbit we first observe that for \( w \) given by (1) \( \epsilon(5) \) is expressed in terms of the dimensionless measure \( \epsilon \) of the binding energy and the ratio \( \nu \) between the reduced and the total mass,
\[
\epsilon = \frac{E}{\mu c^2} \left( \mu = \frac{m_1 m_2}{M} \right), \quad \nu = \frac{\mu}{M} \quad (|\epsilon| \ll 1), \quad (50)
\]
as follows
\[
\epsilon = 1 + \epsilon + \frac{\nu}{2} \epsilon^2 \quad (51)
\]
so that
\[
\beta = 1 - e^2 = -\epsilon \left( 2 + (\nu + 1) \epsilon \right) + O(\epsilon^3). \quad (52)
\]

In the non-relativistic limit Eqs. (45) and (49) yield an elliptic trajectory
\[
y_{NR} = \frac{J}{\alpha_j r} = 1 + \sqrt{1 - 2\frac{\epsilon}{\alpha_j^2} \cos \phi}. \quad (53)
\]

The relativistic orbit (49), on the other hand, is not closed (except for \( \epsilon(\beta, J) = 0 \)). The perihelion shift \( \delta \phi \) is given by
\[
\delta \phi = 2\pi \left( 1 - \alpha_j^2 \right)^{1/2} - 1 = \pi \alpha_j^2 + O(\alpha_j^4) \quad (54)
\]
for a circular orbit we have
\[
e^2(\beta, J) = 0 \implies \beta = \alpha_j^2, \quad \text{i.e.,} \quad -\epsilon = f(\alpha_j^2, \nu)
\]
where, however, a more complicated metric was introduced, computed in a quantum field theoretic framework.

IV. GRAVITATIONAL 2-BODY PROBLEM

The Hamiltonian constraint for the gravitational interaction of two (point) particles of arbitrary masses \( m_1, m_2 \), obtained from (33) for \( e_1 e_2 = 0 \), can be interpreted as the condition that the effective particle 4-momentum \( u = (\epsilon, \mathbf{u}) \) has unit mass in a Schwarzschild metric whose "radius" \( 2\alpha_j \) is expressed in terms of the dimensionless measure \( \epsilon \) of the binding energy and the ratio \( \nu \) between the reduced and the total mass,
\[
\epsilon = \frac{E}{\mu c^2} \left( \mu = \frac{m_1 m_2}{M} \right), \quad \nu = \frac{\mu}{M} \quad (|\epsilon| \ll 1), \quad (50)
\]
as follows
\[
\epsilon = 1 + \epsilon + \frac{\nu}{2} \epsilon^2 \quad (51)
\]
so that
\[
\beta = 1 - e^2 = -\epsilon \left( 2 + (\nu + 1) \epsilon \right) + O(\epsilon^3). \quad (52)
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y_{NR} = \frac{J}{\alpha_j r} = 1 + \sqrt{1 + 2\frac{\epsilon}{\alpha_j^2} \cos \phi}. \quad (53)
\]
As observed in [8] the classical Schwarzschild metric gives a better approximation – in accord with our general prescription of Sec.2.) Here the metric is expressed in terms of the radial variable \( r \) by

\[
g_{00} = 2\frac{\alpha_c}{r} - 1 = 1 - \frac{1}{g_{00}}(<0),
\]

\[
g^{ij}u_iu_j = \left(1 - 2\frac{\alpha_c}{r}\right)u_r^2 + \frac{J^2}{r^2}
\]

where \( r, u_r, J \) and \( \alpha_c \) are given by (34).

Proceeding to the Hamiltonian equations of motion we introduce the \((J\) dependent) dimensionless coupling parameter

\[
\rho = \frac{2\alpha_c}{J} = \frac{R_w m_w c}{J} = \frac{2m_1 m_2 G}{c J}
\]

\((R_w = 2\frac{\alpha_c}{J} c \) being the energy dependent "Schwarzschild radius"). As we shall see shortly, \( \rho^2 < \frac{1}{3} \) for the bounded motion: by contrast, the counterpart \( \alpha_c / c \) of the electromagnetic fine structure constant is rather big: for \( m_1 = m_2 = M_\odot \) (the solar mass) it is of the order of \( 10^{76} \). (The parameter \( \rho \) coincides with 2/j of [8].)

The Hamiltonian constraint can be written in terms of \( \rho \) and \( \frac{J}{r} \) as

\[
H = \frac{1}{2\lambda} \left( 1 + \left(1 - \frac{J}{r}\right) u_r^2 + \frac{J^2}{r^2} - \frac{\epsilon^2}{1 - \rho \frac{J}{r}} \right) \approx 0.\]

The Poisson brackets (40) remain unchanged and we deduce as before the equations of motion

\[
\dot{r} = \frac{\partial H}{\partial u_r} = \left(1 - \frac{J}{r}\right) u_r \lambda, \quad \dot{\phi} = \frac{\partial H}{\partial J} = J \frac{\lambda r^2}{\epsilon}.
\]

Introducing again a dimensionless variable proportional to the inverse radius (cf. (45)),

\[
y = \frac{J}{r}
\]

we obtain the following \((\lambda\)-independent) differential equation for the effective particle trajectory:

\[
\frac{dy}{d\phi} = (1 - \rho y) u_r = (\rho y^3 - y^2 + \rho y - \beta)^{1/2},
\]

\[
\beta = 1 - \epsilon^2.
\]

The energy independent coefficient \( \rho = \rho_j \) will play the role of dimensionless expansion parameter (replacing the commonly used \( \frac{1}{\epsilon} \)).

Eq. (63) can be solved in terms of Jacobi elliptic functions (cf. [19] Sec.VII.8). To begin with, we assume that all three zeros \( y_0, y_1, y_2 \) of the cubic polynomial under the square root are positive reals

\[
P_3(y) := \rho y^3 - y^2 + \rho y - \beta = \rho(y - y_0)(y - y_1)(y - y_2),
\]

\[
0 < y_2 \leq y_1 < y_0.
\]

The finite (bound state) motion belongs to the range \( y_2 \leq y \leq y_1 \) for which \( P_3(y) \) is non-negative. (The infinitesimal interval \( y > y_0 \), in which \( P_3(y) > 0 \) as well, corresponds to falling on a centre.) The necessary and sufficient conditions for \( \rho \) and \( \beta \) for which all zeros of \( P_3 \) are positive are the positivity of \( \beta \),

\[
0 < \beta = (1 - \epsilon^2) < 1, \quad \text{i.e.,} \quad (m_1^2 + m_2^2) < \frac{w^2}{c^4} < M^2
\]

and the non-negativity of the discriminant:

\[
(0 < \rho ) \rho^2 \leq \frac{1}{3}, \quad 27 \left( \beta \rho^2 + \frac{1}{3} \left(\frac{2}{9} - \rho^2 \right)^2 \right) \leq \left(\frac{1}{3} - \rho^2 \right)^3.
\]

We begin our discussion with the case of vanishing discriminant that gives rise to a circular orbit with

\[
y_1 = y_2 =: y_c = \frac{1 - \sqrt{1 - 3\rho^2}}{3\rho} = \frac{\rho^2}{2} + \frac{3}{8} \rho^3 + \frac{9}{16} \rho^5 + O(\rho^7),
\]

\[
y_0 = \frac{1}{\rho} - 2y_c
\]

obtained by solving the quadratic equation \( \frac{dP_3}{dy} = 0 \) for

\[
\beta = (\rho y_c^2 - y_c + \rho y_c) = \frac{2}{27 \rho^2} \left( (1 - 3\rho^2)^{3/2} - 1 + \frac{9}{2} \rho^2 \right) = \frac{\rho^2}{4} + \frac{\rho^4}{8} + \frac{9}{64} \rho^6 + O(\rho^8).
\]

A measure of the frequency on a circular orbit is

\[
\omega = \frac{\text{d} \phi}{\text{d} t} = \frac{\phi}{\epsilon c^2} = \frac{y_c}{c J} (1 - \rho y_c)
\]

where we have used (60), (61) and (62).

The last (innermost) stable circular orbit, LSO (whose significance stems from the fact that gravitational radiation damping tends to circularize binary orbits [9] – see also [8] for a discussion and references) corresponds to the values

\[
\rho = \rho^* = \frac{1}{\sqrt{3}} = y_c^* = y_0^*, \quad \beta^* = \frac{1}{9}, \quad \epsilon^* = \frac{\sqrt{5}}{3},
\]

\[
w^* = M \sqrt{1 - 2\nu \left(1 - \frac{\sqrt{5}}{3}\right)}, \quad m_w^* = \frac{\mu}{\sqrt{1 - 2\nu \left(1 - \frac{\sqrt{5}}{3}\right)}}
\]

\[
(\alpha_c \omega^*)^{2/3} = \frac{1}{6},
\]

for which both sides of the last inequality (66) vanish and all three zeros of \( P_3 \) coincide. These values correspond to a limit point of local minima of what is called the "effective potential" \( V(r, J) \) that enters the expression for \( \epsilon^* \) obtain from (60):

6
\[ \epsilon^2 \left( 1 - \left( \frac{dr}{dt} \right)^2 \right) = V(r, J) := \left( 1 - \rho \frac{J}{r} \right) \left( 1 + \frac{J^2}{r^2} \right) = (71) \]

\[ \epsilon^2 - P_3(y). \]

We have
\[ \frac{d}{dr} V(r, J) = \frac{J}{r} \left( \frac{dP_3(y)}{dy} \right) \bigg|_{y = \frac{3}{2}} = \frac{y^2}{3} \left( 3y^2 - 2y + \rho \right) = 0. \quad (72) \]

It is the smaller zero of the second factor, \( y_c \), (67), that corresponds to a minimum of \( V \). The distance \( r_c = \frac{1}{y_c} \) decreases when \( \rho = \rho_1 \) increases and attains its minimal value for the maximal possible value \( \rho^* \) (70) of \( \rho \).

The dimensionless binding energy \( \varepsilon = \frac{5}{6} (\frac{\omega}{\kappa} - M) = -(m_w - \mu) / \nu \) of the LSO, evaluated from (51) and (70) is:
\[ \varepsilon^* = - \frac{1}{\nu} \left( 1 - \sqrt{1 - 2\rho(1 - \varepsilon^*)} \right) = - (1 - \varepsilon^*) \left( 1 + \frac{1 - \varepsilon^*}{\nu} \right) + \nu \left( (\nu^2 \varepsilon^*)^3 \right), \quad (73) \]
\[ 1 - \varepsilon^* = 1 - \frac{\sqrt{8}}{3} \sim \frac{2}{33}. \]

The increase of \( |\varepsilon^*| = - \varepsilon^* \) (compared to its Schwarzschild value \( |\varepsilon^*| = 1 - \varepsilon^* \)) by the factor \( \left( 1 + \frac{1 - \varepsilon^*}{\nu} + \ldots \right) \) is coupled to a similar increase of the relativistic reduced mass
\[ m_w = \frac{m_1 m_2}{M(1 + \nu \varepsilon^*)} = \frac{\mu}{1 - \nu |\varepsilon^*|} \quad (74) \]
for \( \xi^* = \frac{\omega^*}{c} = M(1 + \nu \varepsilon^*). \)

The expansion parameter \( x = (\alpha \omega)^{2/3} \) of [13] takes for the LSO its Schwarzschild value \( x^* = \frac{5}{6} \) – see (70). Returning to the “true radius” \( R = \frac{m_w}{m_1 \varepsilon^*} \) (measured in units of length) of LSO we find, in accord with [13], that it is smaller than its Schwarzschild value \( R^*_S \)
\[ R^* = R^*_S \frac{\omega^*}{c^2} = R^*_S (1 + \nu \varepsilon^*) \sim R^*_S \left( 1 - \nu \left( 1 - \frac{\sqrt{8}}{3} \right) \right). \quad (75) \]

Accordingly the angular frequency \( \omega^* \) for the relativistic two-body LSO is bigger than its Schwarzschild value \( \omega_S^* \):
\[ \omega^* = \omega_S^* \frac{c^2}{\omega^*} = \frac{\omega_S^*}{1 + \nu \varepsilon^*}. \quad (76) \]

We now proceed to the general case in which the relation (68) between \( \beta \) and \( \rho^2 \) becomes an inequality
\[ 0 < \beta \leq \frac{2}{27 \rho_1^2} \left( (1 - \rho^2)^{3/2} - 1 + \frac{9}{2} \rho^2 \right) \leq \frac{1}{4} \rho^2 + \frac{1}{8} \rho^4 + \frac{9}{64} \rho^6 + \frac{45}{64} \rho^8 \leq \frac{3}{3} \rho^2 \quad (77) \]

(The coefficient to \( \rho^8 \) is chosen in such a way that for \( \rho^2 = \frac{4}{9} \) inequalities (77) become equalities.) We shall compute the zeros of \( P_3 \) as expansions in \( \rho^2 \) taking into account the fact that \( \beta \) is, according to (77), (at most) of order \( \rho^2 \).

For the largest root one finds the following expansion
\[ y_0 = \frac{1}{\rho} - \rho \left( 1 + \rho^2 - \beta + 2 \rho^4 - 3 \rho^2 \beta \right) + O(\rho^7). \quad (78) \]

The two smaller roots are then computed from the relations
\[ y_1 + y_2 = \frac{1}{\rho} - y_0 = \rho \left( 1 + \rho^2 - \beta + 2 \rho^4 - 3 \rho^2 \beta \pm \lambda \right) + O(\rho^7), \]
\[ \rho y_1 y_2 = \beta. \quad (79) \]

The result is
\[ y_{1,2} = \frac{\rho}{2} \left( 1 + \rho^2 - \beta + 2 \rho^4 - 3 \rho^2 \beta \pm \lambda \right) + O(\rho^7), \quad \lambda \geq 0. \quad (80) \]

\[ \rho^2 \lambda^2 = \rho^2 - 4 \beta + 2 \rho^4 - 6 \rho^2 \beta + 5 \rho^6 + 16 \rho^4 \beta + 5 \rho^2 \beta^2 + O(\rho^8). \quad (81) \]

The solution of Eq. (63) satisfying \( y(0) = y_1 \) is expressed in terms of the elliptic sine function (see [20]):
\[ y(\phi) - y_2 = sn^2 \left( K - \sqrt{\rho(y_0 - y_2)} \phi \right) \left( \frac{2}{k} \right). \quad (82) \]

The module square, \( k^2 \), of the elliptic functions is expressed as a ratio of differences of roots of \( P_3 \):
\[ k^2 = \frac{y_1 - y_2}{y_0 - y_2} = \frac{\lambda \rho^2}{1 - \frac{3}{2} \rho^2 + (1 + \rho^2 - \beta) + \frac{4}{2} \rho^2 \lambda + O(\rho^8)} = \frac{\lambda \rho^2}{1 + \frac{3}{2} \rho^2 + 4 \rho^4 - 5 \rho^2 \beta} - \frac{\rho^2}{2} \left( \rho^2 - 4 \beta + 5 \rho^4 - 18 \rho^2 \beta + O(\rho^8) \right); \quad (83) \]

\[ 4K(k^2) \]

is the real period of \( sn(x, K) \) and \( cn(x, k) \):
\[ sn(K, k) = 1, cn(K, k) = 0, \]
\[ K = \int_0^1 dx \frac{1}{\sqrt{(1-x^2)(1-k^2 x^2)}} = \frac{\pi}{2} F \left( \frac{1}{2}, \frac{1}{2}; 1; k^2 \right). \quad (84) \]

We now proceed to computing the perihelion shift \( \delta \phi \) of the effective particle. The change \( \Delta \phi \) of \( \phi \) for a full turn and the shift \( \delta \phi \) are given by
\[ \Delta \phi = 2 \int_{y_2}^{y_1} \frac{dy}{\sqrt{y}} = \frac{4K}{\sqrt{\rho(y_0 - y_2)}} = 2\pi + \delta \phi. \] (85)

This is an exact formula. For small \( \rho \) the elliptic module \( k \) is also small (according to (83)) and we can approximate \( 4K \); using (84), by

\[ 4K = 2\pi \left( 1 + \frac{k^2}{4} + \frac{9}{64} k^4 + O(k^6) \right) =
2\pi \left( 1 + \frac{\lambda}{4} \rho^2 \left( 1 + \frac{3}{2} \rho^2 \right) + \frac{\rho^2}{64} (\rho^2 - 4\beta) + O(\rho^6) \right). \] (86)

Combining this with the expansion

\[ (\rho(y_0 - y_2))^{-1/2} = \left( 1 - \frac{3}{2} \rho^2 (1 + \rho^2 - \beta) + \frac{\rho^2}{2} \lambda \right)^{-1/2} + O(\rho^6) =
1 + \frac{3}{4} \rho^2 \left( 1 + \frac{3}{4} \rho^2 - 2\beta \right) - \frac{\lambda \rho^2}{4} \left( 1 + \frac{9}{4} \rho^2 \right) + O(\rho^6) \] (87)

we end up with an expression for \( \delta \phi \) in which the odd powers of \( \lambda \) cancel out:

\[ \delta \phi = \frac{4K}{\sqrt{\rho(y_0 - y_2)}} - 2\pi =
\frac{3\pi}{2} \rho^2 \left( 1 + \frac{5}{16} (7\rho^2 - 4\beta) \right) + O(\rho^6). \] (88)

To compare with earlier calculations [10], [11], [12] one again uses the expansion (52) of \( \beta \) in terms of the dimensionless binding energy (50). The result clearly agrees with the first post-Newtonian approximation. The missing fourth order (in \( \rho \)) term \( \frac{9}{4} \rho^4 (\beta - \frac{5}{2} \rho^2) \) can be shown to correspond to retardation effects.

### V. CONCLUDING REMARKS

Formulae for particle trajectories in a relativistic 2-body system (including recoil effects) have been derived with the same ease as for a test particle problem in a Coulomb or Schwarzschild potential. The expression \( \epsilon = 1 + \frac{\epsilon}{E^2} + \frac{\epsilon^2}{\alpha} \) (51) for the CM energy per unit mass \( \epsilon = E/M \) in terms of the dimensionless measure \( \epsilon = \frac{E}{\mu c^2} \) of the binding energy (\( \epsilon < 0 \) for finite motion) has been deduced as a straightforward consequence of the relation (1) between the CM energy, the total mass \( M = m_1 + m_2 \) and \( E \):

\[ \frac{w}{Mc^2} = 1 + \nu \epsilon, \quad \nu = \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}. \] (89)

(This simple and natural derivation should be compared with the rather involved argument of Sec.4 of [8] yielding the same result. It thus provides a posteriori justification of our definition of a relativistic effective particle – on top of the arguments presented in Sec.2.)

A systematic way to compute higher order corrections has been worked out in the quantum case [2,3], and can, in principle, be applied to the classical limit as well. It is, however, desirable to have a consistent classical algorithm for calculating retardation effects.

It is known (and follows by comparing results of the previous sections with earlier calculations) that these first contribute to order \( \epsilon^2 \) (i.e. to \( \alpha^2 \) in the electromagnetic case or to \( \rho^4 (\sim \frac{\alpha^4}{\epsilon}) \) in the gravitational case). It turns out that it is quite feasible to include such corrections within the effective 1-body approach developed here. We shall indicate how to do this by modifying the effective particle Lagrangian.

It follows from Eq. (26.23) of [15] (or from Eq. (65.7) of [14]) that the retardation effect in electrodynamics (to order \( \frac{1}{\epsilon} \)) is accounted for by multiplying the interaction term (i.e., \( \frac{1}{\epsilon} \)) by the velocity dependent expression

\[ k_{EM} = 1 + \frac{1}{2} \left( \frac{v_1 v_2 + 1}{r^2} (v_1 + v_2) \right) \] (90)

(where we are using our dimensionless velocities \( v_1, v_2 \), corresponding to \( \frac{1}{2} \alpha v_1, \alpha = 1, 2, \) in the above references).

Similarly, for the gravitational case (following Sec.103 of [15]), one has to multiply the interaction term (i.e., \( -\frac{\alpha}{r} \)) in the Newtonian Lagrangian with the velocity dependent expression

\[ k_G = k_{EM} + 1 + \frac{1}{2} \left( 3 (v_1 - v_2)^2 - \frac{\alpha \epsilon}{r} \right). \] (91)

We recall that a simple minded application of the above procedure would give spurious first order effects that should be eliminated.

Noting that the CM velocities of the two particles are expressed in terms of the relative velocity \( v \) by

\[ v_1 = \frac{E_w}{E_1} v, \quad v_2 = -\frac{E_w}{E_2} v \] for \( v = m_a v_a, v = m_w v \) (92)

we find

\[ k_{EM} = 1 + \frac{1}{2} \frac{E^2}{E_1 E_2} (v^2 + v_2^2), \] (93)

\[ k_G = k_{EM} + 1 + \frac{1}{2} \left( \frac{w E_w}{E_1 E_2} v^2 - \frac{\alpha G}{r} \right). \]

In the above approximation one should replace the energy dependent factors with their non-relativistic limits

\[ \frac{E^2}{E_1 E_2} \rightarrow v, \quad \frac{w E_w}{E_1 E_2} \rightarrow 1 \] for \( \epsilon \rightarrow 0. \] (94)

We end up with the following simple expressions depending on the effective particle 3-velocity \( v \) and position \( r \) only:

\[ k_{EM} = 1 + \frac{1}{2} v (v^2 + v_2^2) \] (95)

\[ k_G = k_{EM} + \frac{1}{2} \left( 3 v^2 - \frac{\alpha G}{r} \right). \]
It is clear that these new terms will account for the differences between our result (88) for $\delta \phi$ (obtained neglecting retardation effects, i.e., under assumption $k_G \equiv 1$) and the result in [11] [12] (in order $\rho^4$, $\rho^6 \beta$). It is, however, challenging to develop the corresponding calculational scheme in order to be able to compare the third post-Newtonian approximation tackled in [13].

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