Wilson loops in strongly coupled noncommutative gauge theories

Avinash Dhar \(^1\) and Yoshihisa Kitazawa \(^\dagger\)

Laboratory for Particle and Nuclear Physics, High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki 305-0801, JAPAN.

ABSTRACT

We discuss Wilson loop averages in 4-dimensional non-commutative superYang-Mills theory using the dual supergravity description. We postulate that the Wilson loops are located at the minimum length scale $R$ in the fifth radial coordinate. We find that they exhibit a crossover from Coulomb type of behaviour for large loops, for which non-commutativity is unimportant, to area law for small loops, for which non-commutativity effects are large. The string tension, which can be read off from the area law, is controlled by the non-commutativity scale. The crossover itself, however, appears to involve loops of size of order $R$ which is much larger than the non-commutativity scale. The existence of the area law in non-commutative super Yang-Mills theory which persists up to a large crossover length scale provides further evidence for connection to an underlying string theory.

\(^1\) On leave from Dept of Theoretical Phys, Tata Institute, Mumbai 400005, INDIA.
\(^\ast\) adhar@post.kek.jp
\(^\dagger\) kitazawa@post.kek.jp
1 Introduction

Non-commutative Yang-Mills (NCYM) theories arise as certain limits of closed string theories, in the presence of D-branes and background B-field. Recently these theories have attracted a lot of attention [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15], partly because non-commutative geometry naturally arises in string theory [16, 17, 18, 19, 20, 21, 22, 23] and partly because string theory simplifies considerably in the limit of large non-commutativity [24, 25, 26, 27, 28, 29] allowing a detailed study of a number of important issues.

One question of intrinsic interest in any gauge theory is the issue of the construction of local gauge-invariant operators since these form a complete set of observables of the theory. This question has been addressed in several recent works. In [4] a set of gauge-invariant operators in NCYM theories was constructed, and these were further studied from different perspectives in [9, 12, 13, 14, 15]. Roughly speaking, these gauge-invariant observables can be written as Fourier transforms of open Wilson lines. In the operator formalism they are given by the following expression

\[ W_k[C] = \text{Tr} \left( P \exp \left\{ i \int_C \, d\sigma \, \partial_\sigma y^\mu(\sigma) A_\mu(\hat{x} + y(\sigma)) \right\} e^{ik\hat{x}} \right), \quad (1.1) \]

where

\[ [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (1.2) \]

and the trace in (1.1) is over both the gauge group as well as the operator Hilbert space. These open Wilson lines are gauge-invariant in NCYM theories, unlike in ordinary YM theories, provided the straight line joining the end points of the path \( C \), given by \( y^\mu(\sigma) \), is fixed to be the vector

\[ l^\mu = k_\mu \theta^{\mu\nu}. \quad (1.3) \]

The path \( C \) is otherwise completely arbitrary. When \( k \) vanishes, the two ends of the path \( C \) must meet and we have a closed Wilson loop.

Recently, multi-point correlation functions of the Wilson line operators have been computed in NCYM theory by Gross, Hashimoto and Itzhaki [14]. These authors have found a universal exponential suppression of the normalized correlators at high momenta, reminiscent of the
behaviour of form factors in high energy fixed angle scattering of string theory. There are other hints of relation between NCYM theory and string theory. It has, in fact, been suggested that 4-dimensional $\mathcal{N} = 4$ NCSYM is dual to superstring theory in appropriate backgrounds with the non-commutativity scale as its string scale [30, 31, 4, 5, 12].

In this paper we provide further evidence of this string connection of NCYM theory. We study the behaviour of closed Wilson loops in NCSYM using the dual supergravity description [30, 31]. In our investigation, we postulate that the Wilson loops are located at the length scale $R$ in the fifth radial coordinate where the warp factor peaks. It is certainly very different from the standard procedure of ordinary AdS/CFT correspondence. In that case the Wilson loops are located at the infinity of the fifth coordinate where the warp factor diverges.

Let us recall why such a choice is reasonable in ordinary field theory. The fifth coordinate has been often interpreted as the renormalization scale. In such an interpretation, we need to rescale the warp factor by $\alpha^2$ if we change the length scale by $\alpha$. Such a logic leads to the conclusion that the local theory can only be located at where the warp factor diverges. In NCYM case, the warp factor in noncommutative dimensions has a maximum at the length scale $R$ in the fifth radial coordinate. It seems that all we can do to approach the bare theory is put the Wilson loops at the minimum length scale $R$. The warp factor can also be interpreted as the effective string tension. It is reassuring to find that the effective string tension is equal to noncommutativity scale in our choice.

The organization of this paper is as follows. We investigate the behaviour of closed Wilson loops in Sec.2. We find that while sufficiently large Wilson loops show a Coulomb law behaviour, as confirmed earlier in [30, 37], there is a crossover to an area law behaviour for small loops. The crossover is controlled by the scale that enters the supergravity solution, which is much larger than the non-commutativity scale. However, the string tension, which can be read off from the area law, is controlled by the noncommutativity scale. We end with a summary and discussion of our results in Sec.3.

\footnote{Other examples that discuss non-commutative AdS/CFT correspondence are [32, 33, 34, 35, 36]}
In this section we will first summarize the supergravity solution which is dual to NCSYM theory [30, 31]. We will then discuss calculation of Wilson loop averages using this solution.

2.1 Background solution

The space-time directions of the NCSYM theory are labeled by $x_0, x_1, x_2$ and $x_3$. The only non-zero B field component is in the space directions $x_2, x_3$, resulting in non-commutativity only in these directions. We give below only the string metric and B field, since the other fields will not be relevant for the calculations that follow.

$$ds^2 = \alpha' \sqrt{\lambda} \left[u^2(-dx_0^2 + dx_1^2) + u^2 h(dx_2^2 + dx_3^2) + \frac{du^2}{u^2} + d\Omega_5^2 \right],$$

$$h^{-1} = 1 + u^4 R^4, \quad \lambda = 4\pi gN, \quad R^4 = \lambda \theta^2,$$

$$B_{23} = \alpha' \sqrt{\lambda} R^2 u^4 h.$$(2.1-2.3)

Here $\theta = \theta^{23}$, the latter being as defined in (1.2). The metric reduces to the standard AdS solution for $uR << 1$. The scale $R$ that enters here is different from the non-commutativity scale defined by $\theta$ and, in fact, is much larger than the latter because the 't Hooft coupling $\lambda$ must be large for a consistent supergravity solution. The fact that the effects of non-commutativity persist at scales much larger than that defined by $\theta$ has been interpreted in the literature [30, 31] as renormalization of the non-commutativity scale due to strong coupling effects. However, as we shall see below, the scale $R$ is perhaps more correctly interpreted as the scale which controls the crossover behaviour of Wilson loops from Coulomb to area law.

2.2 Wilson loop averages

In the partially non-commuting case described by the supergravity solution (2.1-2.3) there can be three distinct kinds of Wilson loops. Those that lie entirely in the commuting $x_0 - x_1$ plane, those that lie entirely in the non-commuting $x_2 - x_3$ plane, and those that lie in mixed commuting and non-commuting directions. Wilson loops of the first kind always lead to Coulomb law, just like the AdS/CFT case involving ordinary 4-dimensional SYM theory.
Wilson loops of the second and third kind show a more interesting behaviour and it is these type of loops that we will now consider.

In the case of ordinary 4-dimensional SYM theory in the strong coupling limit, Wilson loop averages are postulated to be given by minimizing the relevant Nambu-Goto area action \cite{38, 39}. In the present case of NCSYM theory, there is also a non-zero B field and, for the generic Wilson loop, the B field term also contributes in the string action. Thus, the relevant action to consider here is the area action together with the B field term:

\[
S = \frac{1}{2\pi\alpha'} \int d\tau d\sigma \left[ \sqrt{-\det g + B_{\mu\nu} \partial_{\tau} x^\mu \partial_{\sigma} x^\nu} \right],
\]

where

\[
g_{\alpha\beta} = G_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu, \quad \alpha, \beta = \tau, \sigma
\]

is the induced metric on the world-sheet. As in \cite{37}, we will consider the string configuration

\[
x_0 = \tau, \ x_3 = \sigma, \ x_2 = v\tau, \ u = u(\sigma).
\]

Thus, the string is moving in the \(x_2\) direction with velocity \(v\). As a result, on any 4-dimensional fixed \(u\) slice, the contour of the loop is not along coordinate directions. The loops are, however, rectangular with either one time-like and one space-like direction or both directions space-like, depending on what we want. Thus, from the above string configuration we can either get mixed loops with one commuting and one non-commuting direction, or the space-like loops lying entirely in the non-commuting \(x_2 - x_3\) plane.

For the string configuration (2.6) the action (2.4) is

\[
S = \frac{T\sqrt{\lambda}}{2\pi} \int d\sigma \left[ \sqrt{(1 - v^2 h)(u'^2 + u^4 h) + v R^2 u^4 h} \right],
\]

where \(u' = \partial_\sigma u\). At an extremum of this action the quantity

\[
u^4 h \left( \frac{\sqrt{1 - v^2 h}}{\sqrt{u'^2 + u^4 h}} + v R^2 \right)
\]

is independent of \(\sigma\). This gives the solution

\[
u' = \frac{b u^2 \sqrt{u^4 - u_0^4}}{u_0^2 (b + c u^4 / u_0^4)},
\]
where the constants \(b\) and \(c\) are given by

\[
\begin{align*}
    b &= \sqrt{1 - v^2h_0 + vR^2u_0^2}\sqrt{h_0}, \\
    c &= R^4u_0^4\sqrt{1 - v^2h_0 - vR^2u_0^2}\sqrt{h_0}.
\end{align*}
\]

(2.9) (2.10)

The constant \(u_0\) is the turning point in \(u\) where \(u'\) vanishes and \(h_0\) is the value of \(h\) at \(u = u_0\). From the solution (2.8) we obtain

\[
x_3(u) = \frac{1}{bu_0} \int_1^{u/u_0} dy \frac{b + cy^4}{y^2\sqrt{y^4 - 1}}.
\]

(2.11)

For \(v = 0\), \(b = 1\) and \(c = R^4u_0^4\). This is the case considered in [30]. In this case \(x_3(u)\) grows linearly with \(u\) at large values. However, for the special value \(v = R^2u_0^2\), \(c\) vanishes [37] and then \(x_3\) remains finite as \(u\) becomes large.

We will work here with generic values of \(v\). This is because from the viewpoint of supergravity calculations, \(x_3(u)\) has no direct physical significance. The physical quantity is \(x_3(u)/uR\), which measures proper distances along this direction at large \(u\), and this remains finite as \(u \to \infty\). In fact, at large \(u\) the metric in (2.1) has the scaling isometry

\[
x_{0,1} \to \beta^{-1}x_{0,1}, \ x_{2,3} \to \beta x_{2,3}, \ u \to \beta u.
\]

(2.12)

This also shows that \(x_3(u)\) has no direct physical relevance at large \(u\). However, the physical quantity \(x_3(u)/uR\) is invariant under the above scaling isometry.

Another way of saying the same thing is the following. Let us take the boundary to be the 4-dimenional slice of the metric (2.1) at \(u = \Lambda\). The metric on this 4-dimensional slice is given by

\[
ds_4^2 = \alpha'\sqrt{\Lambda}\left[\Lambda^2(-dx_0^2 + dx_1^2) + \frac{\Lambda^2}{(1 + R^4\Lambda^4)}(dx_2^2 + dx_3^2)\right].
\]

(2.13)

For finite coordinate distances in the non-commuting directions, proper distances in these directions become very small, even smaller than string scale, at large \(\Lambda\). Since we are neglecting stringy corrections in the present semiclassical analysis, we cannot trust results obtained in this case. However, for coordinate distances growing linearly with \(\Lambda\), proper distances remain finite at large \(\Lambda\) and so the present approximation is reliable. This suggests that we should consider Wilson loops which have a large horizontal length. An alternative possibility is that the “boundary” might be located at a finite value of \(\Lambda\). As we shall see in the following, there are actually strong reasons for considering the latter possibility.
Let us now consider the case of rectangular Wilson loops with a time-like direction. At any constant $u$ slice, the length of the loop in the time-like direction equals $\sqrt{1-v^2 h}$. For this to be real and non-zero, we require $(1-v^2 h) > 0$. This is satisfied for all values of $u$ if we take $|v| < 1$. In this subsection we will restrict ourselves to this range of values of $v$.

Now, let $L$ be the length between the two ends of the string in the $x_3$ direction on the 4-dimensinal slice at $u = \Lambda$. From the solution (2.8) we get,

$$
\frac{L}{2} = \frac{1}{bu_0} \int_{1}^{\Lambda/u_0} dy \frac{b + cy^4}{y^2 \sqrt{y^4 - 1}},
$$

and the action evaluated on the solution is

$$
S = \frac{T \sqrt{\lambda}}{\pi} (b + c) u_0 \sqrt{h_0} \int_{1}^{\Lambda/u_0} dy \frac{y^2}{\sqrt{y^4 - 1}}.
$$

We should mention that for $|v| < 1$ while $b$ is always positive, $c$ is positive only for $v < R^2 u_0^2$ and changes sign for $v > R^2 u_0^2$. If $R u_0 < 1$, this is within the range of values of $v$ being considered here. In this case the denominator in (2.8) vanishes at $u = |b/c|^{\frac{1}{2}}$. Beyond this point the string bends back and crosses itself at some larger value of $u$. It would be interesting to understand the physics of such self-intersecting surfaces. Here we will avoid such configurations by restricting ourselves to $v < R^2 u_0^2$ for $R u_0 < 1$. If $R u_0 > 1$, these configurations do not occur because of the restriction to $|v| < 1$.

Let us now analyse the expressions (2.14) and (2.15). Two different cases arise which we will now discuss in turn.

**case(i):** $u_0 \ll \Lambda$

In this case (2.14) and (2.15) give

$$
\frac{L}{2} = \frac{1}{bu_0} \left[ \frac{\Lambda}{u_0} + (b - c) \frac{\pi \sqrt{2\pi}}{\Gamma(\frac{1}{4})^2} + \cdots \right],
$$

$$
S = \frac{T \sqrt{\lambda}}{\pi} (b + c) u_0 \sqrt{h_0} \left[ \frac{\Lambda}{u_0} - \frac{\pi \sqrt{2\pi}}{\Gamma(\frac{1}{4})^2} + \cdots \right].
$$

---

2The value $v = 1$ is singular in the sense that the action vanishes on the classical solution.
The dots in these expressions refer to terms that are suppressed by powers of $\Lambda$. Now, if the location $u = \Lambda$ of the boundary is held fixed, then for an appropriately small value of $Ru_0 (<< 1)$ the second term in (2.16) can be made much larger than the first term. To be concrete, let us take $v = 0$. Then, $b = 1$ and $c = R^4 u_0^4$. The expression for $L$ then becomes

$$L = \frac{1}{u_0} \left[ R^4 u_0^3 \Lambda + (1 - R^4 u_0^4) \frac{\pi \sqrt{2\pi}}{\Gamma(\frac{1}{4})^2} + \cdots \right].$$

(2.18)

We see that the second term dominates for $Ru_0 << (R\Lambda)^{-1/3}$. In this case, after subtracting the W-boson mass contribution, we recover the Coulomb interaction familiar from AdS/CFT duality of ordinary SYM theory:

$$S \approx -\frac{4\pi^2 \sqrt{\Lambda}}{\Gamma(\frac{1}{4})^4} \frac{T}{L}$$

(2.19)

This is as expected since loops with large values of $L$ should not see the effects of non-commutativity.

An interesting aspect of the expression (2.18) for $L$ is that it attains a minimum value as a function of $u_0$. This happens at $Ru_0 \sim (R\Lambda)^{-1/3}$ for which $L \sim R(R\Lambda)^{1/3}$. For values of $L$ above this minimum, there are two different values of $u_0$ for the same value of $L$. This is a problem since we expect a Wilson loop with a definite value of $L$ to exhibit a unique physical behaviour\(^3\). If the boundary is placed at large $\Lambda$, then a way out of this problem is to choose the branch corresponding to the smaller value of $u_0$ as physical. This is because this branch is continuously connected to the limit in which the expected ordinary AdS/CFT behaviour is obtained.

There is, however, another possibility, namely there may be a barrier, $R\Lambda \leq 1$. In fact, we do find such a barrier for the purely space-like Wilson loops in the next subsection. If we do place the boundary at $R\Lambda \sim 1$, then according to the formula (2.18) the minimum value of $L$ is attained when $Ru_0 \sim R\Lambda \sim 1$. We still have the Coulomb law (2.19) for $Ru_0 << 1$. However, when $u_0 \sim \Lambda$ the approximation under which (2.16) and (2.17) were calculated breaks down and so we must proceed differently. This is what we do in case(ii) below. Before proceeding to that case, however, we mention that the metric in (2.13) gives proper distances of order string scale when coordinate distances in the non-commuting directions are of order

\(^3\)For large values of $u_0$, the Wilson loop satisfies an area law as can easily be deduced from (2.16) and (2.17).
the non-commutativity scale if $R\Lambda \sim 1$. For $R\Lambda >> 1$ coordinate distances much larger than the non-commutativity scale are needed to get proper distances of order the string scale. We believe that this observation provides an important clue to having the possible barrier $R\Lambda \leq 1$.

**case (ii): $u_0 < \Lambda$**

As we have discussed above, this case becomes relevant only if there is a barrier $R\Lambda \leq 1$. A natural choice is to take $R\Lambda \sim 1$ since at this point the metric in the non-commuting directions has a maximum and so this 4-dimensional slice can accommodate the maximum number of degrees of freedom. Also, this choice of “boundary” is consistent with the standard AdS/CFT correspondence since $\Lambda$ goes to infinity in the commuting limit $R \to 0$. As mentioned above, with the “boundary” at $R\Lambda \sim 1$ we still get Coulomb law behaviour for $Ru_0 << 1$. However, for values of $u_0$ close to $\Lambda = R^{-1}$ the behaviour is different. In this case we may evaluate the integrals (2.14) and (2.15) in powers of $(\Lambda/u_0 - 1)$. We then get,

\[
\frac{L}{2} = b + c \frac{\Lambda}{bu_0} \sqrt{\frac{\Lambda}{u_0} - 1 + \cdots},
\]

and

\[
S = \frac{T\sqrt{\Lambda}}{\pi} (b + c)u_0 \sqrt{h_0} \sqrt{\frac{\Lambda}{u_0} - 1 + \cdots}
\]

where the dots refer to terms higher order in powers of $(\Lambda/u_0 - 1)$. Let us, for the sake of definiteness, again take $v = 0$. Then, from (2.20) and (2.21) we get

\[
S \approx \frac{TL}{2\sqrt{2}\pi \theta},
\]

where we have used $Ru_0 \sim 1$. We see that in this case there is a crossover to area law. The tension is controlled by the non-commutativity parameter.

In summary, then, the picture that has emerged, for the mixed type of loops that we have considered in this subsection, is that there is a crossover from Coulomb type of behaviour for large Wilson loops to area law for small Wilson loops, provided the “boundary” is placed at $R\Lambda \sim 1$. The tension involved in the area law is determined by the non-commutativity scale,

\footnote{Note, however, that $u_0$ cannot get too close to $\Lambda$, otherwise $L$ will become too small for the present analysis to be reliable. An analysis of the expected sigma-model corrections to the leading semiclassical result shows that this restricts $L$ to be much larger than the non-commutativity scale. This translates into the restriction $(\Lambda/u_0 - 1) >> 1/\sqrt{\Lambda}$. Since $\lambda$ is large this condition is easily satisfied.}
while the scale involved in the crossover itself is quite likely the scale that enters the dual supergravity solution.

### 2.4 Rectangular Wilson loops with both directions non-commuting

Rectangular Wilson loops with both directions non-commuting (and so space-like) are obtained for negative values of \((1 - v^2 h)\). These are possible only for \(|v| > 1\). Actually, for such values of \(v\), \((v^2 h - 1)\) vanishes for \(R u = (v^2 - 1)^{\frac{1}{4}}\). This value is large for large values of \(|v|\), and in the limit can be pushed to infinity. We will take this limit in the following. Now, for positive values of \((v^2 h - 1)\), the first term in the action (2.7) becomes imaginary and what we have done effectively amounts to a Euclidean continuation, the Euclidean action being

\[
S_E = \frac{T \sqrt{\lambda}}{2\pi} \int d\sigma \left[ \sqrt{(v^2 h - 1)(u^2 + u^4 h)} - ivR^2 u^4 h \right].
\]  

(2.23)

Taking the large \(v\) limit and writing \(vT \equiv L_2\), the action becomes

\[
S_E = \frac{L_2 \sqrt{\lambda}}{2\pi} \int d\sigma \left[ \sqrt{h(u^2 + u^4 h)} - iR^2 u^4 h \right].
\]  

(2.24)

One could also have obtained the above action by starting from the Euclidean version of the action in (2.4) and choosing the string configuration \(x_2 = \tau, \ x_3 = \sigma, \ u = u(\sigma), \ x_0 = x_1 = 0\). This action also appears in the dual description of rectangular Wilson loops in two mutually non-commuting directions in the fully non-commuting Euclidean NCSYM theory [30, 5].

The Euclidean action (2.24) is complex because of the contribution of the B field. This gives rise to a phase in the path integral which measures the magnetic flux passing through the string world-sheet. In the following we will ignore this contribution to the action and work only with the real part. Some a posteriori justification for this will be given at the end.

At an extremum of the real part of the above action the quantity

\[
u^4 h \frac{\sqrt{h}}{\sqrt{u^2 + u^4 h}}\]

is independent of \(\sigma\). This gives the solution

\[
u' = \frac{u^2 h^{3/2}}{u_0^2} \sqrt{u^4 - u_0^4 \sqrt{1 - R^8 u_0^4 u^4}}.
\]  

(2.25)
We see that $u' = 0$ at $u = u_0, 1/R^2 u_0$. This reflects the symmetry of the background metric under $u \rightarrow 1/R^2 u$ in the non-commuting directions. Because of this symmetry, it is sufficient for us to restrict ourselves to $R u_0 \leq 1$. Then, we see from (2.25) that starting from $u = u_0$, the string turns back at $u = 1/R^2 u_0$. It never goes all the way to infinitely large $u$, though for very small values of $R u_0$ it does go through a very large region.

Let us now consider the contour of the Wilson loop at a 4-dimensional constant $u$ slice at $u = \Lambda$. As in the case of $u_0$, because of the symmetry $u \rightarrow 1/R^2 u$, it is sufficient to restrict to $R \Lambda \leq 1$. In fact, in the following we will take $R \Lambda \sim 1$. Now, the length, $L_3$, between the two ends of the string on this slice can be obtained from (2.25) and is given by

\[
\frac{L_3}{2} = \frac{1}{u_0} \int_1^{\Lambda/u_0} dy \frac{(1 + R^4 u_0^4 y^4)^{3/2}}{y^2 \sqrt{(y^4 - 1)(1 - R^8 u_0^8 y^4)}}. \quad (2.26)
\]

The action evaluated on the solution is given by

\[
S_E = \frac{L_2 \sqrt{\Lambda}}{\pi} \int_1^{\Lambda/u_0} dy \frac{y^2}{\sqrt{(1 + R^4 u_0^4 y^4)(y^4 - 1)(1 - R^8 u_0^8 y^4)}}. \quad (2.27)
\]

We now discuss the two cases that arise.

**case(i):** $u_0 \ll \Lambda$

It is clear from the expressions in (2.25) and (2.26) that in this case we can ignore the explicit dependence of the integrands on $u_0$. Thus, in this case one recovers the usual AdS/CFT results, except for one difference. In the case of ordinary AdS/CFT, the perimeter term is large because it is proportional to a large cut-off. Here, on the other hand, the perimeter term is finite because the cut-off cannot be larger than $R^{-1}$. We will make important use of this observation in the concluding section.

**case(ii):** $u_0 \sim \Lambda$

For $R u_0 \lesssim 1$, we need to keep the explicit dependence on $u_0$ in the integrands of (2.25) and (2.26). The integrals can be done by expanding in powers of $(\Lambda/u_0 - 1)$. We get,

\[
\frac{L_3}{2} \approx \sqrt{2} R \left[ \frac{\pi}{2} - \sin^{-1}(1 - \frac{\epsilon}{\delta}) + \cdots \right], \quad (2.28)
\]

\[
S_E \approx \frac{L_2 \sqrt{\Lambda}}{\pi} \frac{1}{\sqrt{2} R} \left[ \frac{\pi}{2} - \sin^{-1}(1 - \frac{\epsilon}{\delta}) + \cdots \right], \quad (2.29)
\]
where $\epsilon \equiv \Lambda/u_0 - 1$, $\delta \equiv 1 - Ru_0$ and the dots refer to terms higher order in $\epsilon$ and $\delta$. Thus, we may write

$$S_E \approx \frac{L_2 L_3}{4\pi \theta}.$$  

(2.30)

So in this case we find area law. As before, the tension is determined purely by the non-commutativity scale. Note that the tension computed from (2.30) is identical to that computed from (2.21). This is because of the extra metric factor of $1/\sqrt{2}$ in the proper length at $R\Lambda = 1$. Note also that $L_3$ given by (2.29) is of order $R$. For smaller values of $L_3$ we need to take the other solution for which $u' = 0$ and the loop is spanned by a straight surface at $R\Lambda = 1$.

To summarize, then, Wilson loops lying entirely in the non-commuting plane show a crossover from Coulomb type of behaviour to area law as the loop size decreases through the length scale $R$ which characterizes the supergravity solution.

Finally, we will now provide some justification for neglecting the contribution of the B field term in the action. One can check from the solution (2.25) that the magnitude of this term relative to the action (2.27) is $R^2 u_0^2 \theta_0$. So it is suppressed for $Ru_0 \ll 1$. For $Ru_0 < 1$, $u$ is essentially constant and obviously the largest weight in the functional integral is for the worldsheet with minimal area, which is essentially a straight surface on the slice at $\Lambda \sim 1/R$. We expect that quantum fluctuations around this solution are suppressed, but we have not checked that. It is relevant to add here that the B field term is actually absent for Wilson loops lying in mutually commuting directions in the fully non-commuting Euclidean NCSYM theory.

3 Summary and concluding remarks

We have performed a detailed analysis of Wilson loops in NCSYM theory in the strong coupling region using the conjectured dual supergravity description. We have seen that the Wilson loops exhibit a crossover from Coulomb type of behaviour for large Wilson loops, which essentially do not see any non-commutativity, to area law for small Wilson loops, for which non-commutativity is strong. Perhaps the most interesting aspect of our investigations is the likely existence of a length scale in NCSYM theory, where the crossover from Coulomb behaviour to area law takes place. The string tension that emerges from the area law is determined by the non-commutativity scale. However, the crossover scale itself is probably much larger than this. As
we have seen, the crossover scale is likely to be the scale $R$ that enters the dual supergravity description.

Our conclusions are based on the choice of the 4-dimensional slice of the dual metric at $u \sim R^{-1}$ as the location of the “boundary”. The standard choice of the location of the “boundary” at large $u$ gives Coulomb behaviour for Wilson loops of arbitrary size. In the case of commuting SYM theory, Coulomb behaviour at all scales is guaranteed by conformal symmetry. In the case of the NCSYM theory, there is no such symmetry. Therefore, in this case it is legitimate to find the area law, at least for those loops whose size is of order the non-commutativity scale, since one would expect non-commutativity effects to show up for such loops. Moreover, in the fully non-commuting case (in which all the four coordinates are non-commuting), the large $u$ region shrinks to a point, so it is hard to imagine how all the degrees of freedom of the system can be located there. Our choice of the location of the “boundary” predicts interesting nonperturbative behavior of NCYM, while being compatible with the expectation that large loops should behave like in the commutative SYM theory.

We stress here that our choice of putting the “boundary” at the length scale $R$ is not an ad-hoc choice. In ordinary field theory, we can probe arbitrarily short distance scale by colliding particles with very high momenta. So any short distance cut-off must be removed in field theory. In NCYM, the situation is completely different. The counterpart to a local operator with definite momentum is a Wilson line in NCYM. In [14], the high energy limit of two point functions of the Wilson lines is investigated. Through their work, the average distance between the Wilson lines can be estimated as $\langle \lambda \mid \theta k \mid / |k| \rangle^{1/2}$ in the weak coupling region. It is the minimum distance which can be probed by physical observables in the theory. In the uniformly large momentum region, it is $O(\sqrt{\lambda} \theta)$. It may imply that the minimum distance scale is indeed $R$ in the strong coupling region if we adopt the standard replacement rule $\lambda \rightarrow \sqrt{\lambda}$. Therefore we argue that our proposal is consistent with the existence of minimum distance of $R$ in NCYM at strong coupling.

In fact this investigation was partly motivated by the high energy behavior of the multi-point correlators of Wilson lines. The results we have obtained here are consistent with it. The comparison may be made between the normalized Wilson line correlators in NCYM and our geometric prediction. It may make sense to consider the normalized Wilson line operators...
since it is free from the short distance divergences. One can argue that in the extreme ultraviolet region, for generic momenta, the three-point and higher correlators should be related to large Wilson loops lying entirely in the non-commuting plane, with the length scale involved of order $|k\theta|$. Given this, we can make predictions for the extreme ultraviolet behaviour of correlation functions, using the results we have obtained here. The Wilson loop correlators typically have a very large area since the length scale involved is $|k\theta|$. So these should exhibit Coulomb behaviour. The Coulomb behaviour is now actually dominated by the finite perimeter term we found in case(i) of subsection 2.4. So, this would predict a perimeter suppression for three-point and higher correlators relative to the two-point function, which is qualitatively the result obtained in [14].

We therefore see the relevance of our proposal to the high energy behavior of the correlation functions of the Wilson lines in NCYM. Finally, we mention that one can derive Schwinger-Dyson equations [40, 41, 42] for NCSYM theories within the context of matrix models [43, 44]. It would be interesting to obtain the behaviour we have found here for Wilson loops as solutions to these equations.

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References


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