Complex Gravity and Noncommutative Geometry\footnote{Talk given at the String 2000 meeting, July 10-15 2000, University of Michigan, Ann Arbor, USA}

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ABSTRACT

The presence of a constant background antisymmetric tensor for open strings or D-branes forces the space-time coordinates to be noncommutative. An immediate consequence of this is that all fields get complexified. By applying this idea to gravity one discovers that the metric becomes complex. Complex gravity is constructed by gauging the symmetry $U(1,D-1)$. The resulting action gives one specific form of nonsymmetric gravity. In contrast to other theories of nonsymmetric gravity the action is both unique and gauge invariant. It is argued that for this theory to be consistent one must prove the existence of generalized diffeomorphism invariance. The results are easily generalized to noncommutative spaces.
At planckian energies, the manifold structure of space-time will not hold, and a new geometrical setting is needed. At present there are two possible candidates to describe space-time at high energies, one is string theory and the other is noncommutative geometry \[1\]. Recently these two approaches came together when it was realized that the presence of constant background \(B\)-field for open strings or D-branes implies that the coordinates of space-time become noncommuting \([2],[3],[4],[5],[6],[7],[8]\)). This result is expected to generalize to the case of a non-constant \(B\)-field. The resulting geometrical space is expected to be noncommuting and curved. The question I will address in this talk is how to describe the dynamics of the gravitational field in such spaces.

One possibility is to use the tools of noncommutative geometry of Alain Connes as specified by the spectral triple \((\mathcal{A}, \mathcal{H}, D)\) where \(\mathcal{A}\) is an associative algebra with a * product and identity, \(\mathcal{H}\) a Hilbert space and \(D\) a self-adjoint operator on \(\mathcal{H}\) such that \([D,a], a \in \mathcal{A}\) defines a bounded operator on \(\mathcal{H}\) \[9\]. In this setting it is possible to develop the noncommutative analogue of Riemannian geometry. A good example of the realization of noncommutative geometry is the data encoded in superconformal field theory \[1\]. The operator \(D\) encodes the metric, differential calculus, integration and dynamics. For simple noncommutative spaces such as the noncommutative space defined by the standard model all information about the bosonic and fermionic action is encoded in the spectrum of the Dirac operator. This is known as the spectral action principle \[10\]. The difficulty in this approach is that in order to make progress one must know the Dirac operator. Enough information must be available about \(D\) to define geometrical quantities. In the problem at hand it is not easy to guess what \(D\) should one start with. The strategy I will adopt is to first gather information about noncommutative spaces with constant background \(B\)-fields.

Open strings or D-branes in presence of constant background \(B\)-field can be realized by deforming the algebra of functions on the classical world volume. The operator product expansion for vertex operators is identified with the star (Moyal) product of functions on noncommutative spaces \([11],[12]\)). In this respect it was shown that noncommutative U(N) Yang-Mills theory does arise in string theory.
The star product is defined by

\[ f(x) \ast g(x) = e^{i2\theta_{\mu\nu} \frac{\partial}{\partial \zeta} \frac{\partial}{\partial \eta}} f(x + \zeta) g(x + \eta) |_{\zeta = \eta = 0} \]

This definition forces the gauge fields to become complex. Indeed the noncommutative Yang-Mills action is invariant under the gauge transformations

\[ A_\mu^g = g \ast A_\mu \ast g_{\ast}^{-1} - \partial_\mu g \ast g_{\ast}^{-1} \]

where \( g_{\ast}^{-1} \) is the inverse of \( g \) with respect to the star product:

\[ g \ast g_{\ast}^{-1} = g_{\ast}^{-1} \ast g = 1 \]

The contributions of the terms \( i\theta_{\mu\nu} \) in the star product forces the gauge fields to be complex. Only conditions such as \( A_\mu^A = -A_\mu \) could be preserved under gauge transformations provided that \( g \) is unitary: \( g^\dagger \ast g = g \ast g^\dagger = 1 \). It is not possible to restrict \( A_\mu \) to be real or imaginary to get the orthogonal or symplectic gauge groups as these properties are not preserved by the star product ([8],[13]). For open strings in constant background B-field the effective metric is ([14],[8])

\[ g_{\mu\nu} = (G_{\mu\nu} + 2\pi \alpha' B_{\mu\nu})_{S}^{-1} \]
\[ \theta_{\mu\nu} = (G_{\mu\nu} + 2\pi \alpha' B_{\mu\nu})_{A}^{-1} \]
\[ g_{\mu\nu} = G_{\mu\nu} - (2\pi \alpha')^2 (BG^{-1} B)_{\mu\nu} \]

One can imagine a general setting where the closed string theory metric arise as an effective metric coming from open strings, or where the D-branes become dynamical. Under such circumstances one can get an effective metric of the form

\[ g_{\mu\nu} = e_{\mu}^a \ast e_{\nu a} \]

Because of \( \theta \) contributions the metric must become complex. This also seems inevitable as the star product appears in the operator product expansion of the string vertex operators. We are therefore led to investigate whether the metric can become complex.

Assume that we start with the \( U(1, D-1) \) gauge fields \( \omega_{\mu b}^a \). The \( U(1, D-1) \) group of transformations is defined as the set of matrix transformations leaving the quadratic form

\[ (Z^a)^\dagger \eta_{ab} Z^b \]
invariant, where $Z^a$ are $D$ complex fields and

$$\eta_b^a = \text{diag} (-1, 1, \cdots, 1)$$

with $D-1$ positive entries. The gauge fields $\omega^a_{\mu b}$ must then satisfy the condition

$$\left(\omega^a_{\mu b}\right)^\dagger = -\eta_b^c \omega^c_{\mu d} \eta_a^d$$

The curvature associated with this gauge field is

$$R^a_{\mu \nu b} = \partial_\mu \omega^a_{\nu b} - \partial_\nu \omega^a_{\mu b} + \omega^a_{\mu c} \omega^c_{\nu b} - \omega^a_{\nu c} \omega^c_{\mu b}$$

Under gauge transformations we have

$$\tilde{\omega}^a_{\mu b} = M^a_c \omega^c_{\mu d} M^{-1}_{b d} - M^a_c \partial_\mu M^{-1}_{b c}$$

where the matrices $M$ are subject to the condition:

$$(M^a_c)^\dagger \eta_b^a M^b_d = \eta_c^d$$

The curvature then transforms as

$$\tilde{R}^a_{\mu \nu b} = M^a_c R^c_{\mu \nu d} M^{-1}_{b d}$$

Next we introduce the complex vielbein $e^a_\mu$ and its inverse $e^a_\mu$ defined by

$$e^a_\mu e^a_\nu = \delta^\nu_\mu, \quad e^a_\mu e^b_\nu = \delta^b_\nu$$

which transform as

$$\tilde{e}^a_\mu = M^a_b e^b_\mu, \quad \tilde{e}^\mu_a = M^{-1}_{b a}$$

It is also useful to define the complex conjugates

$$e^a_\mu \equiv \left(e^a_\mu\right)^\dagger, \quad e^{\mu a} \equiv \left(e^{\mu a}\right)^\dagger$$

With this, it is not difficult to see that

$$e^a_\mu R^a_{\mu \nu b} \eta_b^c e^{\nu c}$$

is hermitian and $U(1, D-1)$ invariant. The metric is defined by

$$g_{\mu \nu} = \left(e^a_\mu\right)^\dagger \eta_b^a e^b_\nu$$
satisfy the property \( g^\dagger_{\mu\nu} = g_{\nu\mu} \). When the metric is decomposed into its real and imaginary parts:

\[
g_{\mu\nu} = G_{\mu\nu} + iB_{\mu\nu}
\]

the hermiticity property then implies the symmetries

\[
G_{\mu\nu} = G_{\nu\mu}, \quad B_{\mu\nu} = -B_{\nu\mu}
\]

The gauge invariant Hermitian action is given by

\[
I = \int d^D x \sqrt{|e|} e^a_\mu R^{a}_{\mu\nu} b^{a}_{\mu\nu} \sqrt{|e|}
\]

where \( e = \det \left( e^a_\mu \right) \). One goes to the second order formalism by integrating out the spin connection and substituting for it its value in terms of the vielbein. The resulting action depends only on the fields \( g_{\mu\nu} \). It is worthwhile to stress that the above action, unlike others proposed to describe nonsymmetric gravity [15] is unique, except for the measure, and unambiguous. Similar ideas have been proposed in the past based on gauging the groups \( O(D,D) \) [16] and \( GL(D) \) [17], in relation to string duality, but the results obtained there are different from what is presented here. The ordering of the terms in writing the action is done in a way that generalizes to the noncommutative case. The idea of a hermitian metric was first forwarded by Einstein and Strauss [18], which resulted in a nonsymmetric action for gravity, with two possible contractions of the Riemann tensor.

The infinitesimal gauge transformations for \( e_\mu^a \) is \( \delta e_\mu^a = \Lambda^a_b e_\mu^b \), which can be decomposed into real and imaginary parts by writing \( e_\mu^a = e_0^a_{\mu} + i e_1^a_{\mu} \), and \( \Lambda^a_b = \Lambda^a_{0b} + i \Lambda^a_{1b} \).

From the gauge transformations of \( e_0^a_{\mu} \) and \( e_1^a_{\mu} \) one can easily show that the gauge parameters \( \Lambda^a_{0b} \) and \( \Lambda^a_{1b} \) can be chosen to make \( e_0^a_{\mu} \) symmetric in \( \mu \) and \( a \) and \( e_1^a_{\mu} \) antisymmetric in \( \mu \) and \( a \). This is equivalent to the statement that the Lagrangian should be completely expressible in terms of \( G_{\mu\nu} \) and \( B_{\mu\nu} \) only, after eliminating \( \omega^a_{\mu b} \) through its equations of motion. In reality we have

\[
G_{\mu\nu} = e_0^a_{\mu} e_0^b_{\nu} \eta_{ab} + e_1^a_{\mu} e_1^b_{\nu} \eta_{ab}
\]

\[
B_{\mu\nu} = e_0^a_{\mu} e_1^b_{\nu} \eta_{ab} - e_1^a_{\mu} e_0^b_{\nu} \eta_{ab}
\]

In this special gauge, where we define \( g_0^{\mu\nu} = e_0^a_{\mu} e_0^b_{\nu} \eta_{ab} \), \( g_0^{\mu\nu} g_0^{\nu\lambda} = \delta^\lambda_\mu \), and use \( e_0^a_{\mu} \) to raise and lower indices we get

\[
B_{\mu\nu} = -2 e_1^{1\mu\nu}
\]
G_{\mu \nu} = g_{0 \mu \nu} - \frac{1}{4} B_{\mu \kappa} B_{\nu \lambda} g_0^{\kappa \lambda}

The last formula appears in the metric of the effective action in open string theory [14].

We can express the Lagrangian in terms of $e_a^\mu$ only by solving the $\omega_{\mu \nu}^a$ equations of motion

$$e_a^\nu e^\nu_b \omega_{\nu \mu}^c + e_b^\nu e^\nu_c \omega_{\nu \mu}^b - e_b^\nu e^\nu_c \omega_{\nu \mu}^b - e_b^\nu e^\nu_c \omega_{\nu \mu}^b = \frac{1}{\sqrt{G}} \partial_{\nu} \left( \sqrt{G} (e_\nu^\mu e_\nu^\mu - e_\nu^\mu e_\nu^\mu) \right) = X_{\mu \nu}^a$$

where $X_{\mu \nu}^a$ satisfy $(X_{\mu \nu}^a)^\dagger = -X_{\mu \nu}^a$. One has to be very careful in working with a nonsymmetric metric

$$g_{\mu \nu} = e_a^\mu e_a^\nu, \quad e^\mu_a = e^\mu_a e_a^\nu, \quad g_{\mu \nu} g^{\nu \rho} = \delta_\mu^\rho$$

but $g_{\mu \nu} g^{\mu \rho} \neq \delta_\mu^\rho$. Care also should be taken when raising and lowering indices with the metric.

Before solving the $\omega$ equations, we point out that the trace part of $\omega_{\mu \nu}^a$ (corresponding to the $U(1)$ part in $U(D)$) must decouple from the other gauge fields. It is thus undetermined and decouples from the Lagrangian after substituting its equation of motion. It imposes a condition on the $e_\mu^a$

$$\frac{1}{\sqrt{G}} \partial_{\nu} \left( \sqrt{G} (e_\nu^\mu e_\nu^\mu - e_\nu^\mu e_\nu^\mu) \right) \equiv X^{\mu \alpha}_a = 0$$

We can therefore assume, without any loss in generality, that $\omega_{\mu \nu}^a$ is traceless ($\omega_{\mu \alpha}^a = 0$).

The $\omega-$equation gives

$$\omega_{\kappa \rho}^\mu + \omega_{\rho \kappa}^\mu = \frac{1}{8} \delta_\kappa^\mu \left( 3X^{\mu \rho}_\kappa - X^{\mu \rho}_\kappa \right) + \frac{1}{8} \delta_\rho^\mu \left( -X^{\mu \alpha}_\kappa + 3X^{\mu \kappa}_\alpha \right) - X^{\mu \kappa}_\alpha \equiv Y^{\mu \alpha}_{\rho \kappa}$$

We can rewrite this equation after contracting with $e_{\mu \alpha} e_\alpha^\nu$ to get

$$\omega_{\kappa \rho}^\mu + e_\mu^\kappa e_{\rho \kappa} e_\alpha^\nu \omega_{\nu \alpha}^\mu = g_{\sigma \mu} Y^{\mu \alpha}_{\rho \kappa} = Y^{\sigma \alpha}_{\rho \kappa}$$

By writing $\omega_{\rho \kappa}^\alpha = \omega_{\rho \kappa}^\alpha e_\alpha^\nu$ we get

$$\left( \delta_\kappa^\alpha \delta_\rho^\beta \delta_\gamma^\delta + g_\beta^\alpha g_\sigma^\mu \delta_\rho^\delta \delta_\kappa^\delta \right) \omega_{\alpha \beta \gamma} = Y_{\sigma \rho \kappa}$$

To solve this equation we have to invert the tensor

$$M_{\kappa \rho \sigma}^{\alpha \beta \gamma} = \delta_\kappa^\alpha \delta_\rho^\beta \delta_\gamma^\delta + g_\beta^\alpha g_\sigma^\mu \delta_\rho^\delta \delta_\kappa^\delta$$
In the conventional case when all fields are real, the metric $g_{\mu\nu}$ is symmetric and $g^{\beta\mu}g_{\sigma\mu} = \delta^\beta_\sigma$ so that the inverse of $M^{\alpha\beta\gamma}_{\kappa\rho\sigma}$ is simple. In the present case, because of the nonsymmetry of $g_{\mu\nu}$ this is fairly complicated and could only be solved by a perturbative expansion. Writing $g_{\mu\nu} = G_{\mu\nu} + iB_{\mu\nu}$, and defining $G^{\mu\nu}G_{\nu\rho} = \delta^\mu_\rho$ implies that

$$g^{\alpha\alpha}g_{\nu\alpha} = \delta_\nu^\alpha + L^\mu_\nu$$

$$L^\mu_\nu = iG^{\mu\rho}B_{\rho\nu} - 2G^{\mu\rho}B_{\rho\sigma}G^{\sigma\alpha}B_{\alpha\nu} + O(B^3)$$

The inverse of $M^{\alpha\beta\gamma}_{\kappa\rho\sigma}$ defined by

$$N^{\sigma\rho\kappa}_{\alpha\beta\gamma}M^\alpha^\beta^\gamma_{\rho\sigma} = \delta^\mu_\alpha \delta^\nu_\beta \delta^\gamma_\kappa$$

is evaluated to give

$$N^{\sigma\rho\kappa}_{\alpha\beta\gamma} = \frac{1}{2} \left( \delta^\sigma_\gamma \delta^\rho_\beta \delta^\kappa_\alpha + \delta^\sigma_\alpha \delta^\rho_\beta \delta^\kappa_\gamma - \delta^\sigma_\alpha \delta^\rho_\gamma \delta^\kappa_\beta \right)$$

$$- \frac{1}{4} \left( \delta^\kappa_\gamma \delta^\rho_\alpha L^\beta_\gamma + \delta^\kappa_\beta \delta^\rho_\alpha L^\gamma_\beta - \delta^\kappa_\beta \delta^\rho_\gamma L^\gamma_\alpha \right)$$

$$+ \frac{1}{4} \left( L^\gamma_\beta \delta^\rho_\alpha \delta^\gamma_\alpha + L^\gamma_\alpha \delta^\rho_\alpha \delta^\gamma_\beta - L^\gamma_\alpha \delta^\rho_\beta \delta^\gamma_\beta \right)$$

$$- \frac{1}{4} \left( \delta^\kappa_\alpha L^\beta_\gamma \delta^\rho_\beta + \delta^\kappa_\beta L^\gamma_\alpha \delta^\rho_\gamma - \delta^\kappa_\beta L^\gamma_\beta \delta^\rho_\alpha \right) + O(L^2)$$

This enables us to write

$$\omega_{\alpha\beta\gamma} = N^{\sigma\rho\kappa}_{\alpha\beta\gamma}Y_{\rho\sigma\kappa}$$

It is clear that the leading term reproduces the Einstein-Hilbert action plus contributions proportional to $B_{\mu\nu}$ and higher order terms. We can check that in the flat approximation for gravity with $G_{\mu\nu}$ taken to be $\delta_{\mu\nu}$, the $B_{\mu\nu}$ field gets the correct kinetic terms. First we write

$$e^a_\mu = \delta^a_\mu - \frac{i}{2}B_{\mu a}, \quad e^a_\mu = \delta^a_\mu + \frac{i}{2}B_{\mu a}$$

The $\omega_{\mu a}$ equation implies the constraint

$$X^\mu_a = \partial_\nu (e^\nu_a \epsilon^{\mu a} - e^a_\nu \epsilon^{\mu a}) = 0$$

This gives the gauge fixing condition $\partial^\nu B_{\mu\nu} = 0$. We then evaluate

$$\omega_{\mu\nu\rho} = -\frac{i}{2} (\partial_\mu B_{\nu\rho} + \partial_\nu B_{\mu\rho})$$
When the $\omega_{\mu\nu\rho}$ is substituted back into the Lagrangian, and after integration by parts one gets

$$L = \omega_{\mu\nu\rho} \omega^{\nu\rho\mu} - \omega^{\nu\rho}_{\mu} \omega_{\nu\rho} = -\frac{1}{4} B_{\mu\nu} \partial^2 B^{\mu\nu}$$

This is identical to the usual expression $\frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}$, where $H_{\mu\nu\rho} = \partial_{\mu} B_{\nu\rho} + \partial_{\nu} B_{\rho\mu} + \partial_{\rho} B_{\mu\nu}$. The later developments of nonsymmetric gravity showed that the occurence of the trace part of the spin-connection in a linear form would result in the propagation of ghosts in the field $B_{\mu\nu}$ [19]. This can be traced to the fact that there is no gauge symmetry associated with the field $B_{\mu\nu}$.

For the theory to become consistent one must show that the action above has an additional gauge symmetry, which generalizes diffeomorphism invariance to complex diffeomorphism. This would protect the field $B_{\mu\nu}$ from having nonphysical degrees of freedom. It is therefore essential to identify whether there are additional symmetries present in the above proposed action. This is presently under investigation.

Having shown that it is possible to formulate a theory of gravity with nonsymmetric complex metric, based on the idea of gauge invariance of the group $U(1, D-1)$ it is not difficult to generalize the steps that led us to the action for complex gravity to spaces where coordinates do not commute, or equivalently, where the usual products are replaced with star products.

First the gauge fields are subject to the gauge transformations

$$\tilde{\omega}_{\mu b}^a = M_{\mu}^a * \omega_{\mu b}^a \equiv M^{-1}_{c b} * M_{\mu}^c - M_{b}^c * \partial_{\mu} M^{-1}_{c b}$$

where $M^{-1}_{c b}$ is the inverse of $M_{c b}$ with respect to the star product. The curvature is now

$$R_{\mu\nu b}^a = \partial_{\mu} \omega_{\nu b}^a - \partial_{\nu} \omega_{\mu b}^a + \omega_{\mu c}^a * \omega_{\nu b}^c - \omega_{\nu c}^a * \omega_{\mu b}^c$$

which transforms according to

$$\tilde{R}_{\mu\nu b}^a = M_{c}^a * R_{\mu\nu d}^c * M_{s b}^{-1}$$

Next we introduce the vielbeins $e_{\mu}^a$ and their inverse defined by

$$e_{\mu}^a * e_{a}^\mu = \delta_{\mu}^\nu, \quad e_{\mu}^a * e_{s b}^\mu = \delta_{b}^\nu$$

which transform to

$$\tilde{e}_{\mu}^a = M_{b}^a * e_{b}^\mu, \quad \tilde{e}_{s a}^\mu = \tilde{e}_{b}^\mu * M^{-1}_{s a}$$
The complex conjugates for the vielbeins are defined by
\[ e_{\mu a} \equiv (e_{\mu}^a)^\dagger, \quad e^\mu_{\ast a} \equiv (e_{\ast a}^\mu)^\dagger. \]

Finally we define the metric \( g_{\mu \nu} = (e_{\mu}^a)^\dagger \eta_{ab}^* e_{\nu b}. \) The \( U(1, D-1) \) gauge invariant Hermitian action is
\[ I = \int d^Dx \left( \sqrt{e}^\dagger * e_{\nu a}^\mu * R_{\mu \nu b} \eta_{a b} \gamma^k * e_{\nu c}^\mu * \gamma^c \sqrt{e} \right) \]

where \( e = \det (e_{\mu}^a). \) This action differs from the one considered in the commutative case by higher derivatives terms proportional to \( \theta^{\mu \nu}. \) It would be very interesting to see whether these terms could be reabsorbed by redefining the field \( B_{\mu \nu}, \) or whether the Lagrangian reduces to a function of \( G_{\mu \nu} \) and \( B_{\mu \nu} \) and their derivatives only.

The connection of this action to the gravity action derived for noncommutative spaces based on spectral triples ([20],[21],[22]) remains to be made. In order to do this one must understand the structure of Dirac operators for spaces with deformed star products.

References


