On the structure of open-closed topological field theory in two-dimensions

C. I. Lazaroiu*

C. N. Yang Institute for Theoretical Physics
SUNY at Stony Brook
NY11794-3840, U.S.A.

ABSTRACT

I discuss the general formalism of two-dimensional topological field theories defined on open-closed oriented Riemann surfaces, starting from an extension of Segal’s geometric axioms. Exploiting the topological sewing constraints allows for the identification of the algebraic structure governing such systems. I give a careful treatment of bulk-boundary and boundary-bulk correspondences, which are responsible for the relation between the closed and open sectors. The fact that these correspondences need not be injective nor surjective has interesting implications for the problem of classifying ‘boundary conditions’. In particular, I give a clear geometric derivation of the (topological) boundary state formalism and point out some of its limitations. Finally, I formulate the problem of classifying (on-shell) boundary extensions of a given closed topological field theory in purely algebraic terms and discuss reducibility of boundary extensions.

* calin@insti.physics.sunysb.edu
# Contents

1 Introduction 3

2 Axiomatics 4
   2.1 Surfaces, state spaces and products 4
   2.2 Axioms 7
      2.2.1 Degree 7
      2.2.2 Sewing 7
      2.2.3 Permutation symmetry (equivariance) 8
      2.2.4 Normalization 9

3 Basic products and their properties 9
   3.1 The closed (bulk) product 10
   3.2 The open (boundary) product 11
   3.3 The bulk-boundary maps 11
   3.4 Topological metrics 12
   3.5 Reduction to correlators 14

4 Immediate properties of the topological products 15
   4.1 Units 15
   4.2 Topological traces (=topological one-point functions) 16
   4.3 Consequences of sewing constraints 17
      4.3.1 Bulk crossing symmetry 18
      4.3.2 Boundary crossing symmetry 19
      4.3.3 Consequences for traces 19
      4.3.4 The total boundary state space 20
      4.3.5 Bulk-boundary crossing symmetry 21
      4.3.6 Bulk modular invariance 22
      4.3.7 Modular invariance on the cylinder 22
   4.4 Summary 28

5 Decompositions and irreducible boundary theories 30
   5.1 Category-theoretic interpretation 34

6 Conclusions and directions for further research 35
1 Introduction

The central importance of open-closed strings has become progressively clear since the discovery of D-branes. It is now generally accepted that a deeper understanding of open-closed string theory holds the key to deciphering not only D-brane dynamics but also some of the basic structures involved in non-perturbative proposals for string theory such as M-theory. On the other hand, recent studies of open-closed string theory on Calabi-Yau manifolds hold the promise of providing new insight into the phenomena of mirror symmetry and topology change, as well as harmonizing the mathematical program of homological mirror symmetry [15] with modern developments in D-brane physics [1, 5, 10, 12, 9, 24].

It is somewhat surprising to notice that, in spite of its central importance, our understanding of open-closed string theory is quite incomplete when compared with the relatively well-developed framework available for the closed case. While considerable progress has been made in providing systematic constructions [31, 8, 35, 33], the current approach is largely based on the boundary state formalism [23], which is sometimes claimed to reduce most open-string questions to problems formulated in the bulk. While this is certainly correct for some problems, this approach is in fact rather incomplete and cannot fully replace a systematic analysis of open-closed conformal/string theory in its native domain, namely through direct constructions motivated by the geometry of open-closed Riemann surfaces and two-dimensional field theory dynamics. A clear approach to this problem seems especially important for studies of extended moduli spaces, which forms the core of the homological mirror symmetry program. Indeed, general points in the extended moduli space do not admit any standard geometric description, and a clear definition of the systems under study is necessary in the absence of any intuitive considerations.

The aim of this paper is to provide such an analysis for the simplified case of topological open-closed field theories in two dimensions. Beyond being technically simpler, such systems are bound to play a central role in current efforts to analyze D-brane dynamics in curved backgrounds. In particular, understanding their structure is crucial for studies of open-closed extensions of mirror symmetry.

By analogy with the closed case, open-closed topological strings can be built by coupling an open-closed topological field theory to topological gravity defined on open-closed Riemann surfaces (a generalization of the usual ‘closed’ two-dimensional topological gravity of [30]). A detailed understanding requires a close look at each of these building blocks. In this paper I consider the first element only, namely the formalism...
of open-closed topological field theories. These are distinguished from their gravitational counterpart in that they do not contain a ‘dynamical’ metric - no integration over worldsheet metrics is necessary in order to achieve diffeomorphism invariance. I consider the abstract framework of such systems along the lines of [19, 28, 29] (see [22] for a review). As in the closed case [20, 21, 18], one can exploit the topology of bounded Riemann surfaces and the relevant axioms in order to encode all information about such theories into a finite set of characteristic (‘structure’) constants. These are subject to a set of conditions stemming from the topological sewing constraints, and I analyze these in order to extract the mathematical object they define. After making contact with the usual description in terms of correlators, I discuss how (a topological version of) the boundary state formalism can be recovered in this approach, and point out some of its conceptual limitations. I also give an abstract definition of a boundary extension of a topological bulk theory and shortly discuss the problem of irreducible versus reducible boundary extensions of a bulk theory. Finally, I discuss a rather obvious category-theoretic interpretation of boundary data and point out that this physically-motivated structure underlies recent work on on D-brane categories [7].

The formalism of the present paper is restricted to open-closed topological field theories on oriented Riemann surfaces. The unoriented case requires a slightly modified approach, which will not be discussed here. Some of the results derived below are probably familiar to topological field theory experts, though a clear, general and systematic derivation does not seem to have been given before. The expert reader may be interested in the detailed analysis of bulk-boundary and boundary-bulk maps and the topological version of the (generalized) Cardy constraint discussed in Section 4, as well as the discussion of reducibility and the category-theoretic interpretation of Section 5. He may also be interested in our treatment of boundary-condition changing sectors. The mathematical structure governing open-closed topological field theories is summarized in Subsection 4.4. I tried to make this and Section 5 accessible to a mathematical audience, and to this end they collect some results derived in the rest of the paper in an attempt to make the presentation self-contained. This paper is ‘foundational’ and as such it does not contain examples. The example of topological sigma models in the presence of (many) D-branes will be treated in detail in [34].

2 Axiomatics

2.1 Surfaces, state spaces and products

The framework of open-closed topological field theories in two dimensions (‘boundary’ topological field theories) can be formulated through an extension of the geometric category approach of [19, 28] to the case of bounded Riemann surfaces. In this paper, we restrict to the case of oriented strings, and hence consider oriented Riemann surfaces only. Since we allow for general boundary conditions (i.e. we define our theory in the presence of D-branes), each open string boundary will carry a label (decoration) $a$,
which indicates the associated boundary sector. Our Riemann surfaces carry two types of boundaries. First, one has ‘closed’ and ‘open’ string boundaries. The former are oriented circles $C$, while the latter are oriented segments $I$. The ‘open string boundaries’ $I$ carry boundary sector labels $a, b$ at their ends, which we indicate by writing $C_a$ or $I_{ba}$ (in the latter case, the convention is that $I$ is oriented from $a$ to $b$). Second, one has ‘boundary sector’ boundaries, which are oriented open or closed curves $\gamma_a$ carrying a single label $a$. These are those bounding curves of $\Sigma$ on which the ‘boundary conditions’ are imposed. Since we deal with a topological field theory, we consider all objects up to orientation-preserving diffeomorphisms, which is to say that their parameterizations do not matter.

We shall declare a string boundary $C$ or $I_{ba}$ to be ‘incoming’ if its orientation agrees with that of $\Sigma$ and ‘outgoing’ otherwise. Physically, such boundaries are associated with incoming/outgoing strings. A topological field theory ‘living’ on such surfaces defines a bulk state space $H$ (obtained through quantization on the infinite cylinder) and a collection of boundary state spaces $H_{ba}$ (obtained through quantization on an infinite strip carrying boundary conditions $a$ and $b$). Our convention for the latter is that $H_{ba}$ corresponds to the state space of the oriented open string stretching from $a$ to $b$ (in this order). The bulk and boundary state spaces $H, H_{ba}$ are $\mathbb{Z}_2$-graded:

$$H = H^0 \oplus H^1,$$
$$H_{ba} = H^0_{ba} \oplus H^1_{ba},$$

where the $\mathbb{Z}_2$-degree of a state can be identified with its Grassmannality. That is, states belonging to $H^0, H^0_{ba}$ are ‘bosonic’ (and, when such a description is available, generated by Grassmann-even worldsheet fields), while states in $H^1, H^1_{ba}$ are ‘fermionic’ (and generated by Grassmann-odd fields). If a state $|\phi\rangle$ has pure degree $\hat{n}$, we shall write $\text{deg}|\phi\rangle = ||\phi\rangle| = \hat{n}$.

A surface $\Sigma$, together with an enumeration of its incoming and outgoing string boundary components, defines a map $\Phi_\Sigma$ (called the associated **product**) between the associated incoming and outgoing state spaces. These are defined through $H_{\text{in}} := \bigotimes_{i=1}^m H_{\Gamma_{\text{in}}^i}, H_{\text{out}} := \bigotimes_{j=1}^m H_{\Gamma_{\text{out}}^j}$, where $\Gamma_{\text{in}}^i, \Gamma_{\text{out}}^j$ are the (enumerated) incoming and outgoing string boundary components of $\Sigma$. Let us recall the path integral definition for completeness. In the path integral formalism, one associates a configuration space with each string boundary of $\Sigma$. In our situation, one has a bulk configuration

---

2When a nonlinear sigma model description of some sort is available, the indices $a$ label various boundary conditions/choices of Chan-Paton data.

3For topological sigma models, the $\mathbb{Z}_2$-grading is induced by a $\mathbb{Z}$-grading associated with the worldsheet $U(1)$ charge: states of even $U(1)$ charge are Grassmann even and states of odd $U(1)$ charge are Grassmann odd. A general model does not possess a worldsheet $U(1)$ symmetry, since it need not be obtained by twisting an $N = 2$ superconformal field theory. However, the $\mathbb{Z}_2$-degree is always defined.

4In practice, one often obtains a realization of these axioms through cohomological field theories (such as the A/B models); in this case, the two-dimensional metric enters as an explicit parameter and becomes irrelevant only after taking the cohomology of a nilpotent operator $Q$. The path integral derivation of sewing constraints does not directly apply to such models, through it is easy to show that they satisfy our axioms
space $V$ (the space of configurations of worldsheet fields restricted to a string bounding circle $C$) and open configuration spaces $V_{ba}$ (the space of field configurations on a string bounding interval, subject to the boundary conditions labeled by $a$ and $b$ at its two ends). Both bulk and boundary configuration spaces are (infinite-dimensional) supermanifolds. Next, one defines $H, H_{ba}$ as the spaces of functionals over configuration spaces (functions defined on the supermanifolds $V$ and $V_{ba}$). Their $\mathbb{Z}_2$-grading is induced by Taylor expansion with respect to odd coordinates on $V, V_{ba}$.

Given an enumeration of incoming/outgoing boundaries of $\Sigma$, define the incoming and outgoing configuration spaces by $V_{in} = \times_{i=1}^{m} V_{\Gamma_{in}^{i}}$ and $V_{out} = \times_{j=1}^{n} V_{\Gamma_{out}^{j}}$. The enumeration of incoming/outgoing string boundaries does matter in these definitions, since the configuration spaces $V, V_{ba}$ contain Grassmann-odd elements. Picking $\phi_{in} \in V_{in}$, and $\phi_{out} \in V_{out}$, we next consider the (Euclidean) path integral over field configurations $\phi$ on $\Sigma$ subject to the boundary conditions $\phi_{|\Gamma_{in}^{i}} = \phi_{in}^{i}$ and $\phi_{|\Gamma_{out}^{j}} = \phi_{out}^{j}$ on the string boundaries, and to the boundary conditions indexed by the label $a$ on each other boundary $\gamma_{a}$:

$$K_{\Sigma}(\phi_{out}, \phi_{in}) = \int_{\phi_{|\Gamma_{out}^{j}} = \phi_{out}^{j}, \phi_{|\Gamma_{in}^{i}} = \phi_{in}^{i}} D[\phi] e^{-S[\phi]} .$$  \hspace{1cm} (3)

This gives a function $K_{\Sigma}$ defined on $V_{out} \times V_{in}$. It allows us to define the map $\Phi_{\Sigma}$ from $H_{in} = \otimes_{\Gamma_{in}} H_{in}^{i}$ to $H_{out} = \otimes_{\Gamma_{out}} H_{out}^{j}$ as follows. For each incoming state $\eta \in H_{in}$, we define the associated outgoing state $\eta' = \phi_{\Sigma}(\eta)$ to be the function(al) on $V_{out}$ given by the following equation:

$$\eta'(\phi_{out}) = \int D[\phi_{in}] K_{\Sigma}(\phi_{out}, \phi_{in}) \eta(\phi_{in}) .$$  \hspace{1cm} (4)

Here $D[\phi_{in}]$ is the path integral measure on boundary configurations.

In the particular case when all boundaries of $\Sigma$ are incoming, the outgoing space associated with $\Sigma$ is $H_{out} = \mathbb{C}$ and the map $\Phi_{\Sigma}$ is a complex-valued linear functional defined on the incoming space. This is the correlator defined by $\Sigma$, and will also be denoted by $\langle \ldots \rangle_{\Sigma}$.

---

5Because we study topological field theories, Grassmann-odd worldsheet/boundary configurations are not spinors from the worldsheet point of view. This is important when considering sewing operations which produce nontrivial closed curves, in which case the path integral gives objects of the type $\text{Tr}((-1)^{F}\{\ldots\})$. The factor $(-1)^{F}$ is due to the fact that odd configurations are always periodic along such cycles. This is familiar from the case of twisted sigma models, where the G-odd fields are related to Ramond sector fermions of the untwisted model.
2.2 Axioms

2.2.1 Degree

Topological products are subject to a **degree axiom**, which requires that all products with a single output are maps of degree zero, while the maps with two inputs and no output (the topological metrics, see below) have definite (but model-dependent) degree.

2.2.2 Sewing

The topological surfaces $\Sigma$ can be composed by sewing at their closed or open string boundary components. Sewing is allowed only between two closed string boundaries or two open string boundaries, and the orientations and endpoint labels of the sewn boundaries must match. Since we deal with a topological field theory, parameterizations at the boundaries do not matter, and hence there is no twist-sewing operation. Sewing defines an associative composition on the collection of topological open-closed Riemann surfaces, which endows it with the structure of a category. In this category, the objects are direct products of closed and open string boundaries, i.e. oriented topological circles and oriented segments with endpoint decorations. The morphisms are the Riemann surfaces themselves—mapping incoming into outgoing boundary components, while sewing gives the morphism compositions (figure 1). These compositions are clearly associative. Since the objects in our category are given by direct products, they come with a choice of ordering on their components, and hence the Riemann surfaces connecting them are endowed distinguished enumerations of the incoming and outgoing string boundaries. What we have is a generalization of (the topological version of) Segal’s geometric category [19].

![Figure 1. A typical open-closed Riemann surface](image)

The **sewing axiom** is the requirement that the correspondence $\Sigma \rightarrow \Phi_\Sigma$ be a functor from the geometric category to the linear category defined by tensor products
of the spaces $\mathcal{H}, \mathcal{H}_{bd}$ together with linear maps between such products. This requires that sewing of two Riemann surfaces $\Sigma$ and $\Sigma'$ corresponds to composition of the associated maps $\Phi_{\Sigma}$ and $\Phi_{\Sigma'}$:

$$\Sigma \times \Sigma' \rightarrow \Phi_{\Sigma} \circ \Phi_{\Sigma'}.$$  

(5)

The sewing axiom can be ‘derived’ from elementary properties of the path integral in the standard manner.

2.2.3 Permutation symmetry (equivariance)

One also requires that the correspondence $\Sigma \rightarrow \Phi_{\Sigma}$ be \textbf{graded equivariant} with respect to arbitrary permutations of closed string boundaries and cyclic permutations of open string boundaries. For closed string boundaries, such permutations correspond to the associated action on the tensor product components of $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$, and the map $\Phi_{\Sigma_{\text{perm}}}$ determined by the ‘permuted’ surface should be related to $\Phi_{\Sigma}$ by composing with these linear operations at its ends. The latter permutations act and with signs dictated by the degree of the permuted elements (permuting two G-odd states gives a minus sign etc.). For open boundaries, the condition is imposed on diagrams whose open string boundaries are all incoming or all outgoing. This corresponds to graded cyclic symmetry of open amplitudes. Note that only cyclic permutations are allowed, even for the case when all open string boundaries carry the same label$^6$ (figure 2).

---

$^6$This fact is familiar in boundary conformal field theory. In that case, open amplitudes are invariant under arbitrary permutations of the boundary insertions (assuming all such boundaries carry the same label) only if such insertions are ‘mutually local’ (see the second reference of [8]). This happens, for example, for those open boundary correlators which can be continued to the bulk.
Figure 2. Permutations allowed in the equivariance axiom. For closed boundaries, any permutation is allowed among incoming or outgoing data. For open boundaries, equivariance requires cyclic symmetry of amplitudes. This condition refers only to diagrams having only incoming or only outgoing open boundaries with the topology of a segment. Only cyclic permutations are allowed in the second diagram, even for the case \( a = b = c = d = e \).

### 2.2.4 Normalization

Finally, we have to impose a normalization constraint. This requires that the linear maps defined by the surfaces in figure 3 are the identity operators of the corresponding state spaces, and encodes triviality of topological propagators.\(^7\)

\[\begin{array}{c}
\quad
\end{array}\]

Figure 3. The surfaces entering the normalization axiom.

### 3 Basic products and their properties

Any oriented open-closed Riemann surface (with any choice of orientation of its string boundaries) can be obtained by sewing some combination of the five basic surfaces shown in figure 4. This is the analogue of the well-known pants decomposition of closed Riemann surfaces. For want of a better name, we shall call the three surfaces shown in figure 4 (a, b, c) by the names of closed pants, open pants and open-closed conduits. Beyond these, we also need two exceptional surfaces, namely the cylinder and the half-strip with certain string boundary orientations, which are shown in figure 4 (d,e). Some of these surfaces are endowed with boundary decorations, as shown in the figure.

\(^7\)Cohomological field theories satisfy our axioms only after taking BRST cohomology. For these models, the bulk and boundary Hamiltonians \( H \) and \( H_{ba} \) are BRST exact, and they induce trivial propagators in BRST cohomology. However, off-shell propagation in such models is nontrivial.
Figure 4. The basic open-closed Riemann surfaces are considered with the indicated boundary orientations. Note that sewing with one of the surfaces \((d, e)\) allows us to revert the orientation of any outer leg of the surfaces \((a, b, c)\).

The sewing axiom allows us to decompose an arbitrary topological product into the products defined by the basic surfaces. Hence the entire information about our theory is encoded by some basic data, which we now consider in turn.

### 3.1 The closed (bulk) product

This is the degree zero bilinear product \(C : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}\) defined by the surface in figure 4 (a). Since closed Riemann surfaces form a closed subclass under sewing, we can immediately identify this with the basic product of the associated closed topological field theory. It defines a bulk state-operator correspondence \(g\), as follows. To each state \(|\phi\rangle \in \mathcal{H}\), we associate the operator \(\hat{\phi} := g(|\phi\rangle)\) from \(\mathcal{H}\) to \(\mathcal{H}\) given by:

\[
\hat{\phi}(|\phi'\rangle) := C(|\phi\rangle, |\phi'\rangle) .
\]

This parallels the bulk state-operator correspondence of closed conformal field theories, albeit in a simplified fashion.

Since the vector space \(\mathcal{H}\) is typically finite-dimensional, we can choose a finite basis \(|\phi_i\rangle\) and define the coefficients \(C_{ij}^k\) through the expansions:

\[
C(|\phi_i\rangle, |\phi_j\rangle) = \sum_k C_{ij}^k |\phi_k\rangle .
\]

These are the well-known bulk structure constants familiar from closed topological field theory. Via the state-operator correspondence, the product \(C\) corresponds to usual composition:

\[
g(|\phi_i\rangle) \circ g(|\phi_j\rangle) = g(C(|\phi_i\rangle, |\phi_j\rangle)) .
\]
This follows from the definition of $g$, by using associativity of the bulk product $C$, to be discussed in Section 5.

### 3.2 The open (boundary) product

The open pants of figure 4(b) define a degree zero bilinear product $B(cba) : \mathcal{H}_{cb} \times \mathcal{H}_{ba} \to \mathcal{H}_{ca}$. We introduce a boundary state-operator correspondence $g(cba)$ which associates to each state $|\psi\rangle$ of $\mathcal{H}_{cb}$ an operator $\hat{\psi}^{(a)} := g(cba)(|\psi\rangle)$ from $\mathcal{H}_{ba}$ to $\mathcal{H}_{ca}$:

$$\hat{\psi}^{(a)}(|\psi'\rangle) := B(cba)(|\psi\rangle,|\psi'\rangle) \in \mathcal{H}_{ca},$$  \hspace{1cm} \text{(9)}$

where $|\psi'\rangle \in \mathcal{H}_{ba}$. Choosing bases $|\psi_{ba}\rangle$ for all spaces $\mathcal{H}_{ba}$, we can define boundary structure constants $B_{\beta\alpha}^\gamma (cba)$ via:

$$B(cba)(|\psi_{cb}\rangle,|\psi_{ba}\rangle) = \sum_{\gamma} B_{\beta\alpha}^\gamma (cba) |\psi_{ca}\rangle .$$  \hspace{1cm} \text{(10)}$

Associativity of the boundary product (to be discussed in Section 5) implies that the boundary state-operator correspondence takes the boundary product into usual operator compositions:

$$g(cbe)(\psi_2) \circ g(bae)(\psi_1) = g(B(cba)(\psi_2,\psi_1)) ,$$  \hspace{1cm} \text{(11)}$

for $\psi_1 \in \mathcal{H}_{ba}$ and $\psi_2 \in \mathcal{H}_{cb}$.

As we shall see in more detail below, the role of the ‘diagonal’ spaces $\mathcal{H}_a := \mathcal{H}_{aa}$ is slightly different from that of the ‘off-diagonal’ spaces $\mathcal{H}_{ba}$ with $b \neq a$. Following standard terminology, the operators $\psi^a_a := \psi^a_a$ will be called ‘boundary operators in the sector $a$’, while the operators $\psi^b_a$ with $b \neq a$ will be called ‘topological boundary condition changing operators’. They are the topological counterparts of CFT operators bearing the same names.

### 3.3 The bulk-boundary maps

The surface of figure 4 (c) defines degree zero bulk-boundary maps, which we denote by $e(a) : \mathcal{H} \to \mathcal{H}_a$. These take the closed (bulk) state space $\mathcal{H}$ into each ‘diagonal’ boundary state space $\mathcal{H}_a = \mathcal{H}_{aa}$. There is generally no such map into the ‘off-diagonal’ spaces $\mathcal{H}_{ba}$ ($b \neq a$). Expressing this map in the bases $\phi_i$ and $\psi^a_a$ allows us to define bulk-boundary coefficients $e_i^a(a)$ for each boundary condition $a$:

$$e(a)(|\phi_i\rangle) = \sum_{a} e_i^a(a)|\psi^a_a\rangle .$$  \hspace{1cm} \text{(12)}$

---

\[8\] In a nonlinear sigma model, these are realized through restriction of the bulk fields to the string boundary components of the Riemann surface $\Sigma$. No such interpretation exists for systems which do not admit a sigma model description.
The bulk and boundary state–operator correspondences translate this into the bulk-boundary expansions. These are the maps $E(ab) = g(aab) \circ e(a) \circ g^{-1}$ between the bulk and boundary operator spaces. We have:

$$E(ab)(\phi_i) = \sum_{\alpha} e_i^\alpha(a)(\psi^\alpha_b)^{(b)} .$$  \hspace{1cm}(13)

This relation is the topological counterpart of the bulk-boundary expansion in boundary conformal field theories [23]. In the conformal case, it is usually written without explicit indication of (the conformal analogue of) the map $E(ab)$. Following the same convention, we could rewrite it in the more familiar form:

$$\phi_i = \sum_{\alpha} e_i^\alpha(a)\psi^\alpha_0 .$$  \hspace{1cm}(14)

However, the maps $e(a), E(ab)$ need not be injective nor surjective, hence care must be used when interpreting formal relations such as (14).

3.4 Topological metrics

The last two surfaces of figure 4 define complex bilinear maps $\eta : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ and $\rho(ab) : \mathcal{H}_{ab} \times \mathcal{H}_{ba} \to \mathbb{C}$, the bulk and boundary topological metrics. The equivariance axioms shows that these have the graded symmetry properties:

$$\eta(\phi_1, \phi_2) = (-1)^{|\phi_1||\phi_2|} \eta(\phi_2, \phi_1)$$  \hspace{1cm}(15)

$$\rho(ab)(\psi_1, \psi_2) = (-1)^{|\psi_1||\psi_2|} \rho(ba)(\psi_2, \psi_1) .$$  \hspace{1cm}(16)

The metrics are invariant with respect to the bulk and boundary products, respectively, as can be seen from figure 5:

$$\eta(C(\phi_1, \phi_2), \phi_3) = \eta(\phi_1, C(\phi_2, \phi_3))$$  \hspace{1cm}(17)

$$\rho(ac)(B(abc)(\psi_1^{ab}, \psi_2^{bc}, \psi_3^{ca})) = \rho(ab)(\psi_1^{ab}, B(bca)(\psi_2^{bc}, \psi_3^{ca})) .$$  \hspace{1cm}(18)

Explicit examples of noninjectivity/nonsurjectivity are provided by Calabi-Yau sigma models [34].
Figure 5. Invariance of the topological metrics with respect to the bulk and boundary products.

Moreover, figure 6 shows that the metrics are non-degenerate as bilinear forms\textsuperscript{10}:

\begin{align}
\eta(\phi_1, \phi_2) &= 0 \quad \text{for all } \phi_1 \Rightarrow \phi_2 = 0 \quad (19) \\
\rho(ab)(\psi_1^{ab}, \psi_2^{ba}) &= 0 \quad \text{for all } \psi_1^{ab} \Rightarrow \psi_2^{ba} = 0 \quad (20) \\
\rho(ab)(\psi_1^{ab}, \psi_2^{ba}) &= 0 \quad \text{for all } \psi_2^{ba} \Rightarrow \psi_1^{ab} = 0 \quad . \quad (21)
\end{align}

Figure 6. Graphical construction of ‘inverses’ for the topological metrics. The two-legged surfaces below the cuts define states $q \in \mathcal{H} \otimes \mathcal{H}$ and $p(ab) \in \mathcal{H}_{ab} \otimes \mathcal{H}_{ba}$. The figure shows that $(\eta \otimes id_{\mathcal{H}})(\phi \otimes q) = q$ and $(\rho(ab) \otimes id_{\mathcal{H}_{ba}})(\psi_{ab} \otimes p) = \psi_{ab}$. This implies that the topological metrics are non-degenerate.

Hence the topological metrics allow us to identify $\mathcal{H}^\ast$ with $\mathcal{H}$ and $\mathcal{H}_{ab}^\ast$ with $\mathcal{H}_{ba}$, where $^\ast$ indicates the linear dual of a vector space. Defining the metric coefficients via:

\begin{align}
\eta_{ij} &:= \eta(|\phi_i\rangle, |\phi_j\rangle) \quad (22) \\
\rho_{\alpha\beta}(ab) &:= \rho(ab)(|\psi_{ab}^{\alpha}\rangle, |\psi_{ba}^{\beta}\rangle) \quad , \quad (23)
\end{align}

we introduce functionals $\langle \phi_i | \in \mathcal{H}^\ast$ and $\langle \psi_{ab}^{\alpha} | \in \mathcal{H}_{ab}^\ast$ through the conditions:

\begin{align}
\langle \phi_i | (|\phi_j\rangle) &:= \langle \phi_i | \phi_j \rangle = \eta_{ij} \quad (24) \\
\langle \psi_{ab}^{\alpha} | (|\psi_{ab}^{\beta}\rangle) &:= \langle \psi_{ab}^{\alpha} | \psi_{ab}^{\beta} \rangle = \rho_{\alpha\beta}(ab) \quad (25)
\end{align}

and we have the completeness relation:

\begin{equation}
\sum_{i,j} |\phi_i\rangle \eta^{ij} \langle \phi_j | = id_{\mathcal{H}} \quad , \quad (26)
\end{equation}

where $\eta^{ij}$ is the matrix inverse to $\eta_{ij}$.

For the boundary sector, we proceed similarly by defining a map $F(ab) : \mathcal{H}_{ab} \rightarrow \mathcal{H}_{ba}$ through the equation:

\begin{equation}
F := \sum_{\alpha,\beta} |\psi_{ab}^{\alpha}\rangle \rho_{\alpha\beta}(ab) \langle \psi_{ab}^{\beta} | \quad , \quad (27)
\end{equation}

\textsuperscript{10}We assume that our state spaces are all finite dimensional, which is the usual case in practice.
with $\rho^{\alpha\beta}(ab)$ the inverse of $\rho_{\alpha\beta}(ab)$. This takes $|\psi^{\alpha b}_a\rangle$ into $|\psi^{\beta a}_b\rangle$ for all $\alpha$, and thus gives an isomorphism between $\mathcal{H}_{ab}$ and $\mathcal{H}_{ba}$. Identifying these two spaces via the isomorphism $F$, we can treat them as identical, in which case $F$ can be viewed as the identity operator of the space $\mathcal{H}_{ab} \approx \mathcal{H}_{ba}$. Then (27) can be understood as the completeness relation for the basis $|\psi^{\alpha b}_a\rangle \approx |\psi^{\beta a}_b\rangle$ in this vector space.

Note that the bulk and boundary topological metrics are not related in any simple fashion. In particular, the boundary topological metric is not the ‘boundary restriction’ of the bulk metric$^{11}$. That is, one need not have $\rho(a)(e(a)(\phi_1), e(b)(\phi_2)) = \eta(\phi_1, \phi_2)$, as can be seen from the geometry of the associated Riemann surfaces$^{12}$.

### 3.5 Reduction to correlators

Given a surface $\Sigma$ (without a choice of orientation for its boundary components $\Gamma_i$), one can use it to define various products $\Phi_{\Sigma, O}$ associated to the possible orientations $O$ of $\Gamma_i$. A ‘canonical’ choice is to consider incoming boundaries only, in which case one obtains the correlator $\langle . . \rangle_{\Sigma}$. This can be related to the other products defined by $\Sigma$ with the help of the topological metrics. For the example, let $\langle . . \rangle_{\Sigma}$ be the correlator defined by the surface and boundary orientations shown in figure 7. Then cutting very close to the outgoing boundary gives:

$$\langle \psi^{ab}_1 \psi^{bc}_2 \psi^{ca}_3 \rangle_{\Sigma} = \rho(ac)(m(\psi^{ab}_1, \psi^{bc}_2, \psi^{ca}_3)) ,$$

where $m$ is the product defined by the three-pronged surface determined by the cut.

Due to non-degeneracy of the topological metrics, we conclude that all products $\Phi_{\Sigma}$ can be determined from the knowledge of correlators. Similarly, they can all be

---

11 In a nonlinear sigma model, the bulk-boundary map $e$ is typically given by ‘restriction to the boundary’. However, the action of the model will generally contain a nonzero boundary term, such as a boundary coupling to a gauge connection. The boundary metric is given by a path integral on the strip (upper half plane punctured at the origin), and hence depends on the boundary action. The bulk metric is given by a path integral on a cylinder (complex plane punctured at the origin), and depends only on the bulk action.

12 We use the notation $\rho(a) := \rho(aa)$ for the diagonal boundary sectors.
determined from knowledge of the topological metrics and of products with a single output.

Using the state-operator correspondence, we can identify correlators with the topological vevs of the associated operator products. This recovers the usual formalism.

4 Immediate properties of the topological products

4.1 Units

The surfaces of figure 8 define degree zero linear maps from the field \( \mathbb{C} \) of complex numbers into the spaces \( \mathcal{H} \) and \( \mathcal{H}_a \). Evaluating these maps at the complex identity 1 ∈ \( \mathbb{C} \) defines special degree zero states which we denote by \(|0\rangle \in \mathcal{H} \) and \(|0_a\rangle \in \mathcal{H}_a \). These states play the role of ‘topological vacua’ in their respective spaces, as can be seen by considering the surfaces shown in figure 9 below. This figure shows that \( C(|0\rangle, |\phi_j\rangle) = |\phi_j\rangle, \) \( C(|\phi_i\rangle, |0\rangle) = |\phi_i\rangle \) and \( B(aab)(|0_a\rangle, |\psi_{ab}\rangle) = |\psi_{ab}\rangle, \) \( B(abb)(|\psi_{ab}\rangle, |0_a\rangle) = |\psi_{ab}\rangle, \)

i.e.:

\[
C_{0j} = \delta_{j}^{k}, \quad C_{i0} = \delta_{i}^{k} \quad \text{and} \quad B_{0j}(aab) = \delta_{j}^{\gamma}, \quad B_{0\alpha}(abb) = \delta_{\alpha}^{\gamma} .
\]

(29)

It follows that the states \(|0\rangle \) (in \( \mathcal{H} \)) and \(|0_a\rangle \) (in \( \mathcal{H}_a \)) associated with these states are neutral elements (units) with respect to the bulk and boundary operator products. Note that there is no natural definition of a topological vacuum in a boundary condition changing sector \( \mathcal{H}_{ab} \) with \( a \neq b \).

Figure 8. Defining surfaces for the topological vacua. Each of these surfaces contains a single string boundary, namely the circle/segment on their right. The boundary topological vacua arise from a path integral with boundary condition \( a \) on the non-string boundary. This gives a functional on the space of open string states supported on the string boundary, which is the segment \( I_{aa} \) to the right.

\[\text{This involves the observation that} \quad g(|\phi\rangle|0\rangle = C(|\phi\rangle, |0\rangle) = |\phi\rangle \quad \text{and} \quad g(baa)(|\psi_{ba}\rangle|0_a\rangle) = B(baa)(|\psi_{ba}\rangle, |0_a\rangle) = |\psi_{ba}\rangle, \quad \text{where} \ |0\rangle \quad \text{and} \quad |0_a\rangle \quad \text{are the topological vacua discussed below, and the definition of dual states given in eqs. (22).}\\]
Investigation of figure 10 shows that the boundary vacua are related to the bulk vacuum through the maps $e(a)$:

$$e(a)|0\rangle = |0_a\rangle$$  \hspace{1cm} (30)\]

Figure 10. Relation between the boundary and bulk topological vacua. Sewing the cap to the ‘conduit’ of figure 4 (c) gives a surface which is topologically equivalent with the half strip.

4.2 Topological traces (=topological one-point functions)

Considering the surfaces of figure 8 with the opposite string boundary orientations gives linear maps $Tr$, $Tr_a$ from the spaces $\mathcal{H}, \mathcal{H}_a$ to the field of complex numbers. Figure 11 shows that the bulk and (diagonal) boundary topological metrics can be expressed in terms of products and traces:

$$\eta(\phi_1, \phi_2) = Tr(C(\phi_1, \phi_2))$$ \hspace{1cm} (31)\]

$$\rho(ab)(\psi_1^{ab}, \psi_2^{ba}) = Tr_a(B(aba)(\psi_1^{ab}, \psi_2^{ba})) .$$ \hspace{1cm} (32)\]

Figure 11 shows that the bulk and (diagonal) boundary topological metrics can be expressed in terms of products and traces:
In particular, we have:

\[ \eta(|0\rangle, \phi) = Tr(\phi) \quad (33) \]
\[ \rho(ab)(|0_a\rangle, \psi^a) = Tr_a(\psi^a) \quad . \quad (34) \]
\[ (35) \]

Hence \( Tr, Tr_a \) are the linear functionals on \( \mathcal{H}, \mathcal{H}_a \) dual to the topological vacua \( |0\rangle, |0_a\rangle \) with respect to the topological metrics \( \eta, \rho(a) \).

**4.3 Consequences of sewing constraints**

We saw that the entire information about an open-closed topological field theory is encoded by the three classes of products \( C, B, e \) and the topological metrics \( \eta, \rho \). As in boundary conformal field theory, consistency of topological amplitudes under different decompositions of the same Riemann surface into the basic surfaces of figure 4 imposes constraints on this data. The sewing constraints in the conformal case have been analyzed in detail in [27], and since we deal with a topological field theory (which is, in particular, conformally invariant), we can apply some of those results. The main observation of [27] is that all sewing constraints are satisfied provided that the five basic conditions described in figure 12 are obeyed. The first two conditions in this figure are the basic sewing constraints of the closed case (bulk crossing duality and bulk modular covariance), while the remaining conditions (shown in figure 12 (c,d,e,f)) encode boundary, open-open-closed and closed-open-open crossing duality and a supplementary constraint relating the bulk and boundary sector. Let us analyze the consequences of these conditions on our basic data \( b, c, e, \eta, \rho \). In fact, it turns out that the constraint of figure 12(b) is not required in the topological case (it reduces to a tautology for topological field theories). We have included it in our discussion since we want to stress similarity with the analysis of [27].
Figure 12. Graphical depiction of the sewing constraints. The constraint (b) is void for topological field theories, as explained later in this section.

### 4.3.1 Bulk crossing symmetry

As in closed topological field theory, the sewing constraint described by figure 12 (a) amounts to the statement that the product $C$ must be associative. On the other hand, the equivariance axiom requires that this product is graded commutative:

$$C(\phi_1, \phi_2) = (-1)^{|\phi_1||\phi_2|}C(\phi_2, \phi_1).$$  \hspace{1cm} (36)

Since the $|0\rangle$ is a unit, we conclude that $c$ defines a structure of commutative and associative ring with a unit on the bulk state space $\mathcal{H}$. Since the product is bilinear, this ring is in fact an algebra over the field of complex numbers. Moreover, we know that the bulk topological metric is invariant with respect to the product $C$. An graded associative, graded commutative $\mathbb{C}$-algebra with unit, endowed with a non-degenerate, graded-symmetric, invariant bilinear form is called a *Frobenius (super)algebra* [16, 20, ...]
Hence we immediately recover the well-known fact that the bulk data \((\mathcal{H}, c, \eta)\) define a Frobenius algebra.

### 4.3.2 Boundary crossing symmetry

The constraint described by figure 12 (b) can be written:

\[
B(acd)(B(abc)(\psi_1, \psi_2), \psi_3) = B(abd)(\psi_1, B(bcd)(\psi_2, \psi_3)) ,
\]

which is a ‘decorated’ associativity condition. On the other hand, equivariance requires that the triple correlator on the decorated disk is (graded) cyclically symmetric:

\[
\langle \psi_1^ab \psi_2^bc \psi_3^ca \rangle_{abc} = (-1)^{|\psi_1|(|\psi_2|+|\psi_3|)} \langle \psi_2^bc \psi_3^ca \psi_1^ab \rangle_{cba} = (-1)^{|\psi_3|(|\psi_1|+|\psi_2|)} \langle \psi_3^ca \psi_1^ab \psi_2^bc \rangle_{cab}
\]

Hence the boundary product is cyclic in the following sense:

\[
\rho(ac)(B(abc)(\psi_1, \psi_2), \psi_3) = (-1)^{|\psi_1|(|\psi_2|+|\psi_3|)} \rho(ba)(B(bca)(\psi_2, \psi_3), \psi_1) = (-1)^{|\psi_3|(|\psi_1|+|\psi_2|)} \rho(cb)(B(cab)(\psi_3, \psi_1), \psi_2) .
\]

### 4.3.3 Consequences for traces

Associativity of the bulk and boundary products imply that relations (31) generalize to all tree-level bulk and boundary amplitudes:

\[
\langle \phi_1, \ldots, \phi_n \rangle_{0,0} = Tr(\phi_1 \cdot \ldots \cdot \phi_n) \quad (42)
\]

\[
\langle \psi_1^{a_1 a_2}, \ldots, \psi_n^{a_{n-1} a_n} \psi_1^{a_n a_1} \rangle_{0,1}^{a_1 \ldots a_n} = Tr_{a_1}(\psi_1^{a_1 a_2} \cdot \ldots \cdot \psi_n^{a_n a_1}) .
\]

It is clear from (31) that the traces and topological metrics determine each other, provided that the bulk/boundary products are known. Hence one can view the traces as ‘derived’ concepts, if we treat the topological metrics as ‘fundamental’. Cyclicity of topological metrics (eqs. (40)) implies that the traces are graded cyclic:

\[
Tr(\phi_1 \phi_2 \phi_3) = (-1)^{|\phi_1|(|\phi_2|+|\phi_3|)} Tr(\phi_2 \phi_3 \phi_1) \quad (44)
\]

\[
Tr_{a}(\psi_1^{ab} \psi_2^{bc} \psi_3^{ca}) = (-1)^{|\phi_1|(|\phi_2|+|\psi_3^{ca}|)} Tr(\psi_2^{bc} \psi_3^{ca} \psi_1^{ab}) .
\]

which in particular justifies their name. Cyclicity of the traces also follows by applying the equivariance axiom to three-point correlators. In particular, we see that the bulk trace obeys a stronger constraint, which allows us to commute arbitrary entries:

\[
Tr(\phi_1 \phi_2) = (-1)^{|\phi_1||\phi_2|} Tr(\phi_2 \phi_1) .
\]

This generally is not allowed for the boundary traces \(Tr_{a}\).
Also note that non-degeneracy of the metrics is equivalent with the following properties of the traces:

\[ \text{Tr}(\phi_1 \phi_2) = 0 \text{ for all } \phi_2 \Rightarrow \phi_1 = 0 \quad (48) \]
\[ \text{Tr}_a(\psi^{ab}_1 \psi^{ba}_2) = 0 \text{ for all } \psi^{ba}_2 \Rightarrow \psi^{ab}_1 = 0 \quad (49) \]
\[ \text{Tr}_a(\psi^{ab}_1 \psi^{ba}_2) = 0 \text{ for all } \psi^{ab}_1 \Rightarrow \psi^{ba}_2 = 0 \quad . \quad (50) \]

The maps \( \text{Tr}, \text{Tr}_a \) need not be related to the usual traces \( tr_{\mathcal{H}_{ab}} \) on the spaces \( \mathcal{H}_{ab} \). For the operators \( \psi^{a}_\alpha \), we have \( \psi^{a}_\alpha |\psi^{\beta}_\beta\rangle = \mathcal{B}(a)(|\psi^{a}_\alpha\rangle , |\psi^{\beta}_\beta\rangle) = \mathcal{B}^a_{\alpha\beta}(a)|\psi^{a}_\alpha\rangle \) (where \( \mathcal{B}(a) := B(aaa) \)). Hence:

\[ tr_{\mathcal{H}_a}(\psi^{a}_\alpha) = \mathcal{B}^a_{\alpha\beta}(a) \quad (51) \]

and

\[ \text{Tr}_a(\psi^{a}_\alpha) = \text{Tr}(\psi^{a}_\alpha 1_a) = \rho_{a0}(a) = \text{Tr}(1_a \psi^{a}_\alpha) = \rho_{0a}(a) \quad . \quad (52) \]

In the topological case, the one-point functions \( \text{Tr}(\phi_i) = \eta_{i0} = \eta_{0i} \) and \( \text{Tr}_a(\psi^{a}_\alpha) = \rho_{a0}(a) = \rho_{0a}(a) \) need not be zero for nonzero \( i, \alpha \). Let us compare this situation with the case of boundary conformal field theories. For such systems, one has a unique state of conformal dimension \( h = 0 \), the conformal vacuum. For a boundary state \( \psi^a \) of nonzero dimension, conformal invariance requires \( \text{Tr}_a(\psi^a) = \langle \psi^{aa}\rangle_{0,1} = 0 \), so that only the boundary vacua \( |0\rangle \) can have nontrivial one-point functions. In the topological case, this constraint cannot be applied for 0-form observables, since these transform trivially under the diffeomorphism group, and in particular under its conformal subgroup.

In the conformal case, it is well-known that \( (1_{a})_{0,1} \) can have different values for different boundary conditions \( a \); in general, one cannot normalize these to have the same value. The same is true for topological theories.

### 4.3.4 The total boundary state space

The properties of boundary correlators can be given a more transparent form as follows. Consider the ‘total open state space’ \( \mathcal{H}_o := \bigoplus_{ab} \mathcal{H}_{ab} \). On this space, we introduce a ‘total boundary product’ \( B \) defined as follows. If \( \psi^{ab}_1 \in \mathcal{H}_{ab} \) and \( \psi^{bc}_2 \in \mathcal{H}_{bc} \), then \( B(\psi^{ab}_1, \psi^{bc}_2) := B(abc)(\psi^{ab}_1, \psi^{bc}_2) \). Then we extend \( B \) to a bilinear map defined for arbitrary elements \( \psi_1, \psi_2 \) of \( \mathcal{H}_o \). We also define a total boundary topological metric \( \rho \) on \( \mathcal{H}_o \) through the requirement of bilinearity and the condition that it reduces to \( \rho(ab) \) on ‘pure’ boundary states \( \psi_1 \in \mathcal{H}_{ab}, \psi_2 \in \mathcal{H}_{ba} \) (the metric is defined to be zero on states \( \psi_1 \in \mathcal{H}_{a1b}, \psi_2 \in \mathcal{H}_{ab2} \) which do not satisfy the constraints \( a_1 = b_2, a_2 = b_1 \)). Non-degeneracy of all \( \rho(ab) \) is equivalent with non-degeneracy of \( \rho \). It is not hard to see that the two boundary sewing constraints are equivalent with the conditions:

\[ B(B(\psi_1, \psi_2), \psi_3) = B(\psi_1, B(\psi_2, \psi_3)) \quad (53) \]
\[ \rho(B(\psi_1, \psi_2), \psi_3) = (-1)^{|\psi_1||\psi_2|+|\psi_3|}\rho(B(\psi_2, \psi_3), \psi_1) \quad . \quad (54) \]
\[ = (-1)^{|\psi_2|+|\psi_3|}\rho(B(\psi_3, \psi_1), \psi_2) \quad . \quad (55) \]
The first of these equations shows that $b$ is associative, while the second is the requirement that the ‘total boundary correlator’ $\langle \psi_1, \psi_2, \psi_3 \rangle = Tr(\psi_1 \psi_2 \psi_3)$ (defined on $\mathcal{H}_o$ in the obvious fashion\textsuperscript{14}) be cyclically symmetric.

Moreover, it is easy to see that the ‘total boundary vacuum’ $|0\rangle_o := \sum_a |0\rangle_o$ is a unit for the total boundary product. We conclude that $(\mathcal{H}_0, B, Tr)$ forms a graded associative algebra with unit (over the complex field), endowed with a non-degenerate graded cyclic trace. This structure is familiar from open string field theory [32, 14], where it appears in a slightly different context. Since the boundary metric $\rho$ is invariant under the boundary product:

$$\rho(B(\psi_1, \psi_2), \psi_3) = \rho(\psi_1, B(\psi_2, \psi_3)) \ ,$$

we also conclude that $(\mathcal{H}_o, B, \rho)$ is a ‘non-commutative Frobenius algebra’.

### 4.3.5 Bulk-boundary crossing symmetry

The constraints of figures 13 (d) and (e) can be formulated as follows:

$$B(abb)(\psi, e(b)(\phi)) = (-1)^{|\phi||\psi|} B(aab)(e(a)(\phi), \psi) \quad (57)$$
$$B(aaa)(e(a)(\phi_1), e(a)(\phi_2)) = e(a)(C(\phi_1, \phi_2)) \ . \quad (58)$$

To simplify their analysis, define the total bulk boundary map $e : \mathcal{H} \to \mathcal{H}_o$ by $e := \oplus_a e(a)$ . The image of this map is contained in the ‘diagonal’ subspace $\mathcal{H}_d := \oplus_a \mathcal{H}_{aa}$. Then conditions (57) can be rewritten as:

$$B(\psi, e(\phi)) = (-1)^{|\phi||\psi|} B(e(\phi), \psi) \quad (59)$$
$$B(e(\phi_1), e(\phi_2)) = e(C(\phi_1, \phi_2)) \ . \quad (60)$$

The second equation shows that $e$ is a morphism from the bulk ring $(\mathcal{H}, C)$ to the boundary ring $(\mathcal{H}_o, B)$. This morphism preserves units, since $e(a)|0\rangle = |0\rangle_o$ (cf. (30)).

If we define a multiplication $\mathcal{H} \times \mathcal{H}_o \to \mathcal{H}_o$ by:

$$\phi \psi := B(e(\phi), \psi) \ , \quad (61)$$

then the boundary ring $(\mathcal{H}_o, B)$ becomes a (graded) algebra over the bulk ring $(\mathcal{H}, C)$. To finish the check of graded algebra properties, we have to show that:

$$B(\psi_1, B(e(\phi), \psi_2)) = (-1)^{|\psi_1||\phi|} B(B(e(\phi), \psi_1), \psi_2) = (-1)^{|\phi||\psi_1|} B(e(\phi), B(\psi_1, \psi_2)) \quad (62)$$

and

$$B(e(C(\phi_1, \phi_2)), \psi) = B(e(\phi_1), B(e(\phi_2), \psi)) \ . \quad (63)$$

\textsuperscript{14}This correlator is defined to be zero on ‘pure’ boundary states $\psi_i \in \mathcal{H}_{ab}$, which fail to satisfy the requirement $b_1 = a_2, b_2 = a_3, b_3 = a_1$. The total boundary trace is defined through $Tr := \sum_a Tr_a$, and satisfies $\rho(\psi_1, \psi_2) = Tr(B(\psi_1, \psi_2))$. 

21
These properties follow from the constraints (59) and associativity of the boundary product $B$:

$$B(\psi_1, B(e(\phi), \psi_2)) = B(B(\psi_1, e(\phi)), \psi_2)$$  \hspace{1cm} (64)

$$B(B(\psi_1, e(\phi)), \psi_2) = (-1)^{|\phi||\psi_1|} B(B(e(\phi), \psi_1), \psi_2) = (-1)^{|\phi||\psi_1|} B(e(\phi), B(\psi_1, \psi_2))$$  \hspace{1cm} (65)

and:

$$B(e(C(\phi_1, \phi_2)), \psi) = B(B(e(\phi_1), e(\phi_2)), \psi) = B(e(\phi_1), B(e(\phi_2), \psi)) .$$  \hspace{1cm} (66)

We conclude that the (associative but generally not graded commutative) boundary ring $(\mathcal{H}_0, B)$ has the structure of a graded algebra with unit over the (associative and graded commutative) bulk ring $(\mathcal{H}, C)$. The former is endowed with a non-degenerate complex-valued graded cyclic trace.

4.3.6 Bulk modular invariance

It is easy to see that this condition does not give any further constraints on the bulk quantities $c$ and $\eta$ — for a topological field theory, this reduces to the tautology $tr(e(., \phi)) = tr(e(., \hat{\phi}))$. In the conformal case, the two surfaces obtained by cutting the torus along homologically inequivalent cycles are not equivalent, since they have different complex structures. For topological field theories, however, one can smoothly deform them into one another, which is why the constraint is void.

4.3.7 Modular invariance on the cylinder

In contrast with bulk modular invariance, modular invariance on the cylinder gives a non-trivial constraint on the theory. The reason is that the sewing condition depicted in figure 12 (f) involves both closed and open cuts of the surface, and thus relates bulk and boundary data. This constraint can be written:

$$\langle \psi_1, \psi_2 \rangle_{0,2}^{ab} = \langle f(a)(\psi_1), f(b)(\psi_2) \rangle_{0,0}$$  \hspace{1cm} (67)

where we used subscripts $\{g, h\}$ to indicate the genus and number of boundaries of the surfaces involved.

The map $f(a) : \mathcal{H}_a \to \mathcal{H}$ appearing in this equation is obtained by considering the surface of figure 12 (c) with its opposite orientation (see figure 13). In particular, the bulk state $\langle a \rangle_B := f(a) \langle 0_a \rangle$ is described by the surface of figure 14. Geometrically, the surface of figure 14 translates between non-string bounding circles $C_a$ and closed string circle boundaries $C$. 

![Diagram](image-url)
The map $f(a)$ is related to $e(a)$ as follows. If $\sigma$ is the surface of figure 4 (c) (with both string boundaries taken to be incoming), then the associated correlator $\langle \phi \psi \rangle_\sigma$ can be expressed in either of the following two ways:

$$
\rho(a)(e(a)\phi, \psi) = \eta(\phi, f(a)\psi) \quad .
$$

This tells us that $e(a)$ and $f(a)$ are adjoint to each other with respect to the topological metrics on their defining spaces.

One can rewrite relation (67) as follows:

$$
tr_{\mathcal{H}_{ab}}((-1)^F \Phi_{ab}(\psi_1, \psi_2)) = \eta(f(a)(\psi_1), f(b)(\psi_2)) \quad ,
$$

where $F$ is the ‘fermion number’ (i.e. $F$ counts the degree of states), the map $\Phi_{ab}(\psi_1, \psi_2) : \mathcal{H}_{ab} \rightarrow \mathcal{H}_{ab}$ is defined through:

$$
\psi \rightarrow B(ab\bar{b})(B(aab)(\psi_1, \psi), \psi_2) \quad ,
$$

and $tr_{\mathcal{H}_{ab}}$ is the usual trace in the vector space $\mathcal{H}_{ab}$.

To make this more specific, let us expand:

$$
e(a)\phi_i = e^a_i(a)\psi^a_\alpha \quad ,
$$

$$
f(a)\psi^a_\alpha = f^i_\alpha(a)\phi_i \quad .
$$

Then (68) (applied to $\phi_i$ and $\psi^a_\alpha$) gives:

$$
f^j_\alpha(a)\eta_{ji} = e^\beta_i(a)\rho_{\beta\alpha}(a) \quad .
$$

Defining $f_{i\alpha}(a) := f^j_\alpha(a)\eta_{ji}$ and $e_{i\alpha}(a) := e^\beta_i(a)\rho_{\beta\alpha}(a)$, we obtain:

$$
f_{i\alpha}(a) = e_{i\alpha}(a) \Leftrightarrow f^j_\alpha(a) = \eta^{ij}\rho_{\beta\alpha}e^\beta_j(a) \quad .
$$
In matrix form:

\[
\hat{\eta} \hat{f}(a) = \hat{e}^T(a) \hat{\rho}(a)
\]  

(75)

where \( \hat{f}(a)_{i\alpha} := f^i_\alpha(a), \hat{e}(a)_{\alpha j} := e^j_\alpha(a) \) and \( \hat{\rho}(a)_{\alpha \beta} := \rho_{\alpha \beta}(a), \hat{\eta}(a)_{ij} := \rho_{ij}(a) \).

Applying (69) to the states \( \psi_1 := \psi^a_\alpha \) and \( \psi_2 := \psi^b_\beta \) gives:

\[
\eta_{ij} f^i_\alpha(a) f^j_\beta(b) = (-1)^{|\sigma|} B^\gamma_{\alpha \beta}(aab) B^\alpha_{\gamma \beta}(abb) .
\]

(76)

In this equation, we assumed that the basis \( \psi^{ab}_\sigma \) is formed of states of definite degrees \( |\psi^{ab}_\sigma| = |\sigma| \). Combining with expression (74) for \( f \), we can rewrite the constraint (69) as follows:

\[
\eta^{ij} e_{i\alpha}(a) e_{j\beta}(b) = (-1)^{|\sigma|} B^\gamma_{\alpha \beta}(aab) B^\alpha_{\gamma \beta}(abb) .
\]

(77)

This can be recognized as a topological version of the generalized Cardy constraint (see below).

Finally, we formulate the bulk-boundary sewing conditions in terms of the total boundary space \( \mathcal{H}_o \). We start by defining the total bulk-boundary map \( e := \oplus_a e(a) : \mathcal{H}_c \rightarrow \mathcal{H}_o \) and the total boundary-bulk map \( f := \sum_a f(a) : \mathcal{H}_{ad} \rightarrow \mathcal{H}_c \). Here \( \mathcal{H}_{ad} := \oplus_a \mathcal{H}_a \) is the ‘diagonal’ part of \( \mathcal{H}_o \).

The adjointness relations (68) imply that \( e \) and \( f \) are adjoint with respect to the bulk metric and the total boundary metric:

\[
\eta(\phi, f(\psi)) = \rho(e(\phi), \psi) \quad \text{for all } \phi \in \mathcal{H}_c \text{ and } \psi \in \mathcal{H}_o ,
\]

(78)

while the topological Cardy constraint becomes:

\[
\eta(f(\psi_1), f(\psi_2)) = tr_{\mathcal{H}_c}((-1)^F \Phi(\psi_1, \psi_2)) \quad \text{for all } \psi_1, \psi_2 \in \mathcal{H}_o ,
\]

(79)

where \( \Phi(\psi_1, \psi_2) \) is the endomorphism of \( \mathcal{H}_o \) defined through:

\[
\Phi(\psi_1, \psi_2) = \oplus_{ab} \Phi_{ab}(\psi^a_1, \psi^b_2) = [\psi \in \mathcal{H} \rightarrow B(B(\psi_1, \psi), \psi_2)],
\]

(80)

for all pairs of ‘diagonal’ states \( \psi_k = \oplus_a \psi^a_k \in \mathcal{H}_o \) \((k = 1, 2)\), of components \( \psi^a_k \in \mathcal{H}_{aa} \).

**Boundary states** In boundary conformal field theory, a relation similar with (67) is used to define boundary states. Following this tradition, we call \( |\psi^a_{\sigma B}\rangle := f(a)|\psi\rangle \in \mathcal{H} \) the **topological boundary state** associated with the open string state \( |\psi\rangle \in \mathcal{H}_a \) (this is an extension of the terminology used in the conformal case, as we shall see in a moment). Application of the sewing constraints allows one to reduce certain questions about the ‘diagonal’ boundary sectors \( \mathcal{H}_{aa} \) to problems for the associated boundary states. In particular, the correlator associated to the surface \( \sigma \) of figure 4 (c) can be written:

\[
\langle \phi, \psi \rangle^a_{\sigma} = \langle \phi, f(a)(\psi) \rangle_{0,0} = \eta(\phi, \psi^a_{\sigma B}) = \langle e(a)(\phi), \psi \rangle^a_{0,1} = \rho(a) (e(a)(\phi), \psi) .
\]

(81)

---

15By generalized we mean that we apply the modular constraint of figure 12 (f) for arbitrary incoming states \( |\psi_1\rangle \) and \( |\psi_2\rangle \), and not only for \( |\psi_1\rangle = |0_a\rangle \) and \( |\psi_2\rangle = |0_b\rangle \) as is customary in the conformal field theory literature.
The adjointness relation (68) tells us that we can compute tree-level bulk-boundary two-point functions $\langle \phi \psi^a \rangle$ either as $\langle e(a)(\phi)\psi^a \rangle_a$ (i.e. by pulling the bulk state to the boundary and computing a correlator on the disk) or as $\langle \phi f(a)(\psi^a) \rangle = \langle \phi \psi^a_B \rangle_{1,0}$, i.e. by pushing the state $\psi$ to its bulk image (its associated boundary state) and computing a correlation function on the sphere. Even though these descriptions are equivalent at the level of two-point functions, such a ‘duality’ is not very powerful unless the maps $e$ and $f$ have appropriate surjectivity/injectivity properties.

**Limitations of the boundary state approach** What is the precise power of the boundary state approach? To answer this question, let us investigate the properties of the maps $e(a)$ and $f(a)$. Consideration of figure 15 shows that $f(a)e(a)|\phi\rangle = C(f(a)|0_a\rangle,|\phi\rangle) = C(|a_B\rangle,|\phi\rangle)$, where $|a_B\rangle := f(a)|0_a\rangle$. The best we can hope for is to have $fe = \sum_a f(a)e(a) = id_\mathcal{H}$. In that case, $e = \oplus_a e(a)$ would be injective and $f := \oplus_a f(a)$ would be surjective. This can be achieved provided that $\sum_a f(a)|0_a\rangle = \sum_a |a_B\rangle = |0\rangle$, which is a sort of completeness constraint for the set of boundary conditions (D-branes) present in the theory. Thus, one can expect a simplification in the case when ‘all’ topological D-branes have been included as a background, a situation which suggests that in a certain sense the bulk data can be recovered from knowledge of all possible boundary data.

![Figure 15. Geometric description of the composition $f(a)e(a)$. This figure shows that $f(a)e(a)|\phi\rangle = C(f(a)|0_a\rangle,|\phi\rangle)$.](image)

Now consider the composition $e(b)f(a)$, which is shown in figure 16. This composition is also nontrivial, and nothing can be said about it from general considerations. Provided that $e(a)f(a) = id_\mathcal{H}_{a}$ in a given model, one could conclude that the boundary-bulk map is injective and the bulk-boundary map is surjective. Since the former is responsible for associating a boundary state to an open string state, this would assure us that the boundary state formalism gives precise information on all open string states (i.e. we do not lose information on the spaces $\mathcal{H}_{a}$ when taking their images through $f(a)$). Unfortunately, this is generally not the case, as one can see in the particular example of Calabi-Yau topological sigma models [34].

In general, the maps $e$ and $f$ are neither injective nor surjective. In particular, the boundary algebra $(\mathcal{H}_o,B)$ will generally have torsion as a module over the bulk algebra $(\mathcal{H},C)$. 

![Figure 16.](image)
We saw above that the map \( e \) \ is a ring morphism from the bulk to the boundary algebra. It is natural to ask whether \( f \) has a similar property. This is clearly not the case \(^{16}\), since the quoted property of \( e \) is a consequence of the constraint depicted in figure 12 (e), and no similar condition holds true for \( f \). The closest candidate would be figure 12 (d), which cannot be interpreted in such a manner since one cannot continuously pull the closed string tube in that figure along the open string strips in order to produce two instances of the map \( f \).

**On boundary states as D-branes**

Let us add a few remarks on the way boundary states are used in traditional boundary conformal field theory. In standard treatments, one is interested in computing couplings of the boundary vacua \( |0_a\rangle \), and we do the same in our topological field theories. This amounts to considering only couplings of bulk states with the semiclassical D-brane state associated with the boundary condition labeled by \( a \). The associated boundary state \( |a\rangle_B = |0_a\rangle_B \in \mathcal{H} \) is then called the boundary state defined by the D-brane \( D_a \). Then one writes the constraint (69) for incoming states given by the boundary vacua \( |0_a\rangle, |0_b\rangle \), in which case it reduces to:

\[
\eta(|a\rangle_B, |b\rangle_B) = \eta_{ij} f^i_0(a) f^j_0(b) = tr_{\mathcal{H}_{ab}} (-1)^F .
\]  

This can be recognized as a topological version of the standard Cardy relation (note that \( |a\rangle_B = \sum_i f_i^0(a) |\phi_i\rangle \)). The quantity in the right hand side is the Witten index of the boundary sector \( \mathcal{H}_{ab} \).

It is more useful, however, to view the ‘quantum’ D-brane \( D_a \) as being defined by the entire open string state space \( \mathcal{H}_{aa} \), since this includes all of the associated open string excitations. In fact, this is necessary for logical consistency since inclusion of D-branes should completely characterize our theory together with the bulk data. Since the boundary vacua \( |0_a\rangle \) (and even more so the boundary states \( |a\rangle_B \)) do not suffice to so characterize the theory, we must conclude that there is more information in D-brane physics than can be possibly encoded by the boundary vacua \( |0_a\rangle \). It follows that the technique of boundary states represents at best a first step towards a complete characterization of boundary conformal/topological field theories. We should also note that the literature on boundary states is almost exclusively concerned with the states \( |a\rangle_B \), which encode information only about the boundary vacua.

Non-injectivity of \( f(a) \) (if present) sets sharp limitations for the boundary state approach. In fact, this formalism is further weakened by the following problem. Suppose that one is given a state \( |\phi\rangle \) in the bulk space \( \mathcal{H} \). One would like to know whether it corresponds to an open string state \( |\psi^a\rangle \in \mathcal{H}_{aa} \) for some \( a \), and, if so, whether this state

---

\(^{16}\)One can use relation (68) to show that \( f \) is a ring morphism provided that \( ef = id \). Unfortunately, this is generally not true, as can be seen by considering the geometric interpretation of this composition (see below).
is unique. This is precisely the approach followed by many studies of open conformal field theory through the boundary state formalism — instead of trying to determine $H_{aa}$ directly, one tries to find a set of bulk states $|\phi\rangle$ which satisfy certain constraints (such as conformal invariance and Cardy’s constraints in the case of rational conformal field theories). This approach is especially useful in the case of abstract models (such as Gepner models) for which a worldsheet description of the associated boundary conditions is difficult. Unfortunately, a solution of such constraints does not necessarily correspond to a true boundary state, since we are not assured that $|\phi\rangle$ lies in the image of any $f(a)$. To establish this for any given candidate, one needs to provide a full construction of a boundary extension of the bulk theory, and show that the candidate boundary state indeed is image through $f(a)$ of some open string state (say, of the boundary vacuum $|0_a\rangle$).

Moreover, the answer to this question could be highly ambiguous, since one does not know a priori what are the allowed boundary sectors $a$, nor does one know what the maps $f(a)$ are. Even if one knew these maps, it is very possible to have $|\phi\rangle = \sum_a f(a)|\psi_a\rangle$ for some nonzero states $|\psi_a\rangle \in H_{aa}$, in which case $|\phi\rangle$ is not associated with any single state belonging to one of the spaces $H_{aa}$. Since the full collection of possible boundary sectors $a$ is generally large, it is likely that this problem is quite widespread.

The observations above raise some obvious questions about the extent to which recent work on open-closed Calabi-Yau compactifications (which is largely based on the boundary state approach) gives us unambiguous information about the associated D-branes, and how seriously one can take the geometric interpretation of those states as type A/B D-branes wrapped over specific cycles in the large radius limit 17.

It is apparent that a complete understanding of open-closed extensions of a closed topological/conformal field theory should go beyond the boundary state approach. At the very least, any attempt at classifying such extensions must explicitly include the freedom allowed by the choice of the maps $e(a)$, i.e. must take into consideration the bulk-boundary operator products as part of the extension problem, rather than as an afterthought. In this respect, recent progress has been made in [33], where bulk-boundary products are discussed in certain rational conformal field theory situations.

What is an abstract ‘boundary condition’? In an abstract system, one lacks a direct construction of the boundary theory through boundary conditions and boundary couplings in the action. If one is interested in classifying all open-closed theories

---

17Many such identifications of D-branes with particular cycles have been proposed in recent work [5, 6]. Such proposals are based largely on cohomological arguments, and seem to neglect most of the issues we just mentioned, as well as the more basic geometric fact that specifying a cohomology class does not fix a (cycle,bundle) pair or a complex of such with any reasonable degree of uniqueness (i.e. it does not fix them up to the natural ambiguity of such data in topological sigma models, which is specified by ‘gauge transformations’ of a certain kind). I should however mention that there are situations where symmetry arguments can be used to reduce, though not completely eliminate, this ambiguity. In any case, it should be clear that the issue of such identifications is far from having been settled.
compatible with given bulk data, one would like to have a conceptual definition of the ‘boundary part’ of such a system. The traditional approach to this problem is to try to isolate the boundary data through use of the boundary state formalism, and to roughly identify the boundary states \(|a\rangle_B\) with abstract ‘boundary configurations’.

As mentioned above, this approach encounters certain conceptual difficulties, the most obvious of which is that it does not take into account the bulk-boundary map \(f(a)\).

The latter is essentially ‘restriction to the boundary’ in the standard case of nonlinear sigma models (with ‘geometric’ boundary conditions). Therefore, its specification is crucial for any consistent definition of boundary data. I believe that the only generally meaningful procedure is to define ‘boundary data’ as the entire system \((\mathcal{H}_o, B, \rho, e)\).

This reduces the task of classifying boundary theories associated to given bulk data to the problem of finding all non-commutative Frobenius algebras over \((\mathcal{H}, C, \eta)\) which obey the topological Cardy constraint. In such generality, this problem can be expected to have solutions which do not fit into a geometric ‘boundary condition’ approach—for example, a nonlinear sigma model or Landau-Ginzburg model at the conformal point could possess more open-closed extensions than predicted by the classical boundary condition approach [2, 11]. Whether such an abstract boundary extension has a ‘classical’ boundary condition interpretation or not is a model-dependent question which is largely irrelevant once a complete solution of the problem is known.

4.4 Summary

We showed that a (two-dimensional) topological open-closed field theory is equivalent with the following data:

1. A Frobenius (super) algebra \((\mathcal{H}, C, \eta)\) over the complex numbers. We recall that this is an associative, graded commutative algebra \((\mathcal{H}, C)\) with unit \(|0\rangle\), endowed with a graded-symmetric nondegenerate bilinear form \(\eta\) (the bulk topological metric). This metric has a definite degree \(deg(\eta)\) (i.e. \(\eta(\phi_1, \phi_2) = 0\) unless \(|\phi_1| + |\phi_2| = deg(\eta)\)), which is model-dependent.

2. A collection of finite-dimensional vector spaces \((\mathcal{H}_{ab})_{a,b \in \Lambda}\) indexed by a set \(\Lambda\) (taken to be finite, for ease of exposition), together with degree zero bilinear maps \(B(abc) : \mathcal{H}_{ab} \times \mathcal{H}_{bc} \to \mathcal{H}_{ac}\) and bilinear maps \(\rho(ab) : \mathcal{H}_{ab} \times \mathcal{H}_{ba} \to \mathbb{C}\) of definite (but model-dependent) degree, with the following properties:

   (2.1) \(B(abc)(B(abd)(\psi_1, \psi_2), \psi_3) = B(abd)(\psi_1, B(dbc)(\psi_2, \psi_3))\).

   (2.2) \(B(abb)(\psi, |0_a\rangle) = B(aab)(|0_a\rangle, \psi) = \psi\) for some elements \(|0_a\rangle \in \mathcal{H}_a\).

   (2.3) \(\rho(ab)\) are nondegenerate and satisfy the ‘graded-commutativity’ relations:

\[
\rho(ab)(\psi_1, \psi_2) = (-1)^{|\psi_1||\psi_2|}\rho(ba)(\psi_2, \psi_1) \ . \tag{83}
\]

and the cyclicity property:

\[
\rho(\psi_1, B(\psi_2, \psi_3)) = (-1)^{|\psi_1||\psi_2|+|\psi_3|}\rho(\psi_2, B(\psi_3, \psi_1)) = (-1)^{|\psi_3||\psi_1|+|\psi_2|}\rho(\psi_3, B(\psi_1, \psi_2)) \ ; \tag{84}
\]

and

\[
\rho(\psi_1, B(\psi_2, \psi_3)) = (-1)^{|\psi_1||\psi_2|+|\psi_3|}\rho(\psi_2, B(\psi_3, \psi_1)) = (-1)^{|\psi_3||\psi_1|+|\psi_2|}\rho(\psi_3, B(\psi_1, \psi_2)) \ ; \tag{85}
\]
(3) Degree zero linear maps \( e(a) : \mathcal{H} \to \mathcal{H}_a \) with the properties:
(3.1) \( e(a)[0] = [0_a] \)
(3.2) \( B(\alpha\beta\gamma)(e(a)\phi_1, e(a)\phi_2) = e(a)(C(\phi_1, \phi_2)) \)
(3.3) \( B(\alpha\beta\gamma)(\psi, e(b)\phi) = (-1)^{|\phi||\psi|} B(\alpha\beta\gamma)(e(a)\phi, \psi) \).

This data is such that:
(4) the topological Cardy constraint (69) is satisfied.

For reader’s convenience, let us explain how one can determine the basic data
\( \eta, \rho, c, b, e \) from computations of topological correlators. It is convenient to chose the
bases \( \phi_i \) and \( \psi_{\alpha\beta}^{ab} \) of \( \mathcal{H}, \mathcal{H}_{ab} \) to be homogeneous, i.e. such that \( \phi_i \) are elements of definite degree \(|i|\) and \( \psi_{\alpha\beta}^{ab} \) are elements of definite degree \(|\alpha|\) \(^{18}\).
Since \( \rho(ab) \) is nondegenerate and of definite degree, the spaces \( \mathcal{H}_{ab} \) and \( \mathcal{H}_{ba} \) are isomorphic (possibly after a shift of grading) as graded vector spaces and hence the bases \( \psi_{\alpha\beta}^{ab} \) and \( \psi_{\alpha\beta}^{ba} \) are indexed by the
same set of labels \( \alpha \).

Raising/lowering indices with the bulk and boundary topological
metrics, we define:

\[
C_{ijk} := C^l_{ij} \eta_{lk} \tag{86}
\]
\[
B_{\alpha\beta\gamma}(abc) := B^\delta_{\alpha\beta\gamma}(abc) \rho_{\delta\gamma}(ac) \tag{87}
\]
\[
e_{io}(a) := e^i_o(a) \rho_{io}(a) \, \tag{88}
\]

where we use the notations \( \rho_{\alpha\beta} := \rho_{\alpha\beta}(aa) \) etc. One has:

\[
C_{ijk} = \eta(C(\phi_i, \phi_j), \phi_k) = \langle \phi_i \phi_j \phi_k \rangle_{0,0} \tag{89}
\]

\[
B_{\alpha\beta\gamma}(abc) = \rho(ac)(B(\psi^{ab}_{\alpha} \psi^{bc}_{\beta} \psi^{ca}_{\gamma}), \psi^{ca}_{\gamma}) = \langle \psi^{ab}_{\alpha} \psi^{bc}_{\beta} \psi^{ca}_{\gamma} \rangle_{0,1} \tag{90}
\]

\[
e_{io}(a) = \rho(a)(e(a)\phi_i), \psi^{a}_{\alpha} = \langle \phi_i \psi^{a}_{\alpha} \rangle_{0,1} \tag{91}
\]

On the other hand, we have:

\[
\eta_{ij} = \langle \phi_i \phi_j \rangle_{0,0} \quad \rho_{\alpha\beta}(ab) = \langle \psi^{ab}_{\alpha} \psi^{ba}_{\beta} \rangle_{0,1} \tag{92}
\]

Hence all relevant data can be determined by computing:

(1) The two and 3-point functions on the sphere \( \eta_{ij} \) and \( C_{ijk} \)
(2) The boundary two and 3-point functions on the disk \( \rho_{\alpha\beta}(ab) \) and \( B_{\alpha\beta\gamma}(abc) \)
(3) The bulk-boundary two-point function on the disk \( e_{io}(a) = f_{io}(a) \)

In coordinates, the constraints on this data are as follows:

(a) \( \eta_{ij} \) and \( \rho_{\alpha\beta}(ab) \) are non-degenerate and we have:

\[
\eta_{ij} = (-1)^{|i||j|} \eta_{ji} \tag{93}
\]

\[
\rho_{\alpha\beta}(ab) = (-1)^{|\alpha||\beta|} \rho_{\beta\alpha}(ba) \tag{94}
\]

(b) \( C_{ijk} \) are graded symmetric:

\[
C_{ijk} = (-1)^{|i||j|} C_{jik} = (-1)^{(|i|+|j|)|k|} C_{kij} \quad \text{etc} \tag{96}
\]

\(^{18} |i| \) and \(|\alpha| \) should not be confused with absolute values !
and form the structure constants of an associative algebra with unit. We can chose this unit \( |0⟩ \) to be part of our basis: \( \phi_0 := |0⟩ \). In this case, we must have:

\[
C_{0j}^k = \delta_j^k \quad (97)
\]

\[
C_{i0}^k = \delta_i^k \quad . (98)
\]

Notice that \( \eta_{jk} = C_{0jk} \) with this choice of basis.

(c) \( B_{\alpha\beta\gamma}(abc) \) are graded cyclically symmetric:

\[
B_{\alpha\beta\gamma}(abc) = (-1)^{|\alpha||\beta|+|\gamma|} B_{\beta\gamma\alpha}(bca) = (-1)^{|\gamma|(|\alpha|+|\beta|)} B_{\gamma\alpha\beta}(cab) \quad (99)
\]

and satisfy the associativity property:

\[
B_{\alpha\beta\gamma}(abc)B_\delta^\gamma(acd) = B_\delta^\alpha(abd)B_\beta^\gamma(bcd) \quad . (100)
\]

They are also required to admit units \(|0_a⟩\) in the sense of Subsection 4.1. Choosing \( \psi_{a0}^a := |0_a⟩ \), we can formulate this as the constraints:

\[
B_{0\beta}^\gamma(aab) = \delta_\beta^\gamma \quad (101)
\]

\[
B_{a0}^\beta(abb) = \delta_\alpha^\beta \quad . (102)
\]

Notice that \( \rho_{\beta\gamma}(ab) = B_{0\beta\gamma}(aab) \) with this choice of basis.

(d) \( e_\alpha^i \) induce a graded algebra structure of \( \mathcal{H}_a \) over \( \mathcal{H} \), i.e. we have (see equations (57)) :

\[
B_{a\beta\gamma}(aab)e_\beta^\gamma(b) = (-1)^{|\alpha||\beta|} e_\beta^\gamma(a) B_{a\beta\gamma}(aab) \quad (103)
\]

\[
B_{a\beta\gamma}(aaa)e_\alpha^\beta(a) e_\beta^\gamma(a) = e_\gamma^\beta(a) C_{ij}^k \quad . (104)
\]

(e) The topological Cardy constraint (77) is satisfied.

As in closed topological field theory, the entire information is contained in tree-level amplitudes \(^{19}\). Hence one can invert the argument and define an (on-shell) topological open-closed field theory to be given by such a structure. In particular, on-shell deformations of open-closed topological models are governed by the deformation theory of such objects.

5 Decompositions and irreducible boundary theories

We saw that the boundary and bulk-boundary sewing constraints admit a simplified form when expressed in terms of the total boundary state space. In this section, I analyze the structure of open-closed topological field theories from this point of view.

\(^{19}\)This is of course not the case in topological string theory, i.e. after coupling to topological gravity.
Since the data summarized in Subsection 4.4. is quite complicated, it is useful to follow a formal approach and start with a few definitions.

**Definition 5.1.** A bulk algebra is a Frobenius (super) algebra \((H, C, \eta)\), whose unit we denote by \(|0\rangle\). Remember that we take the invariant, nondegenerate metric \(\eta\) to be graded-symmetric, as required by the equivariance axiom.

**Definition 5.2.** A boundary algebra is a triple \((H_o, B, \rho)\) with the properties:

(1) \(H_o\) is a (finite-dimensional) complex vector space, endowed with a \(\mathbb{Z}_2\)-grading \(H_o = H^0_o \oplus H^1_o\).

(2) \((H_o, B)\) is an associative graded algebra over the field of complex numbers. In particular, the product \(B: H_o \times H_o \to H_o\) is bilinear with respect to complex multiplication and has degree zero. This algebra is endowed with a unit which we denote by \(|0\rangle_o\). The product \(B\) need not be graded-commutative.

(3) \(\rho: H_o \times H_o \to \mathbb{C}\) is a nondegenerate bilinear form of degree zero, satisfying the properties:

(3.1.) graded symmetry:
\[
\rho(\psi_1, \psi_2) = (-1)^{|\psi_1||\psi_2|}\rho(\psi_2, \psi_1)
\]

(3.2) invariance:
\[
\rho(B(\psi_1, \psi_2), \psi_3) = \rho(\psi_1, B(\psi_2, \psi_3))
\]

(3.3) graded cyclicity:
\[
\rho(\psi_1, B(\psi_2, \psi_3)) = (-1)^{|\psi_1||\psi_2|+|\psi_3|}\rho(\psi_2, B(\psi_3, \psi_1)) = (-1)^{|\psi_3||\psi_1|+|\psi_2|}\rho(\psi_3, B(\psi_1, \psi_2))
\]

**Definition 5.3.** A boundary extension of a bulk algebra \((H, C, \eta)\) is a boundary algebra \((H_o, B, \rho)\) together with a degree zero linear map \(e: H \to H_o\) having the properties:

(1) \(e\) endows \((H_o, B)\) with the structure of a graded unital algebra over the bulk ring \((H, C)\), i.e. we have:

(1.1) \(e|0\rangle = |0\rangle_o\)

(1.2) \(B(e(\phi), \psi) = (-1)^{|\phi||\psi|}B(\psi, e(\phi))\)

(1.3) \(B(e(\phi_1), e(\phi_2)) = e(C(\phi_1, \phi_2))\).

(2) The topological Cardy constraint:
\[
\eta(f(\psi_1), f(\psi_2)) = tr[(-1)^F\Phi(\psi_1, \psi_2)]
\]

is satisfied. Here \(f\) is the adjoint of \(e\) with respect to the metrics \(\eta, \rho\):
\[
\eta(f(\psi), \phi) = \rho(\psi, e(\phi))
\]

while \(\Phi(\psi_1, \psi_2)\) is the endomorphism of \(H_o\) defined through:
\[
\psi \to B(B(\psi_1, \psi), \psi_2)
\]
The module product $\mathcal{H} \times \mathcal{H}_o \to \mathcal{H}_o$ is defined in the standard fashion, namely $\phi \psi := B(e(\phi), \psi)$. Note that complex multiplication in the boundary algebra is compatible with the extension map $e$ (since the later is linear). That is, the module structure of $(\mathcal{H}_o, B)$ over $(\mathcal{H}, C)$ determined by $e$ takes complex multiplication in $\mathcal{H}$ into complex multiplication in $\mathcal{H}_o$:

$$\lambda \psi = B(e(\lambda|0\rangle), \psi) \quad .$$

(111)

Boundary extensions correspond to open-closed topological field theories having a single boundary sector (i.e. when $\Lambda$ of Subsection 4.4. is a set with only one element). In Section 4, we expressed all sewing constraint in terms of the total boundary state space. In the language of the present section, what we did is to show that the axioms of open-closed topological field theory amount to the condition that the total boundary state space is an extension of the bulk algebra. To formulate this statement (and its converse) precisely, we need a few more mathematical definitions.

**Definition 5.4** A reduction of a boundary extension $(\mathcal{H}_o, B, \rho, e)$ is a (finite) direct sum decomposition $\mathcal{H}_o = \bigoplus_{a, b \in \Lambda} \mathcal{H}_{ab}$ into (nonzero) vector spaces, with the properties:

1. The decomposition is compatible with the grading, i.e. $\mathcal{H}_{ab} = (\mathcal{H}_{ab} \cap \mathcal{H}_o) \oplus (\mathcal{H}_{ab} \cap \mathcal{H}_o^0)$.

2. $\rho$ is zero except when restricted to subspaces of the form $\mathcal{H}_{ab} \times \mathcal{H}_{ba}$.

3. The product $B$ is zero except when restricted to subspaces of the form $\mathcal{H}_{ab} \times \mathcal{H}_{bc}$, and takes such a space into a subspace of $\mathcal{H}_{ac}$.

4. The map $e$ is ‘diagonal’, i.e. its image $e(\mathcal{H})$ is a subspace of $\bigoplus_{a \in \Lambda} \mathcal{H}_{aa}$.

Using condition (1), the requirement (2) can be reformulated as the statement that the triple correlator $\langle \psi_1, \psi_2, \psi_3 \rangle = \rho(\psi_1, B(\psi_2, \psi_3))$ equals zero except when restricted to subspaces of the form $\mathcal{H}_{ab} \times \mathcal{H}_{bc} \times \mathcal{H}_{ca}$. That is, the boundary topological metric and triple correlator must be ‘polygonal’ in the sense described in figure 17. An extension of the bulk algebra will be called reducible if it admits a reduction and irreducible otherwise. A reduction will be called trivial if the set $\Lambda$ has only one element (in this case, the reduction ‘does nothing’).

Figure 17. Graphical representation of the decomposition conditions for the boundary metric and triple correlator. This description is related to the category-theoretic interpretation of subsection 5.2.

---

20 This constraint is nontrivial, since it requires that any element $|\psi\rangle \in \mathcal{H}_{ab}$ be the sum of an even and an odd element of $\mathcal{H}_o$ both of which belong to $\mathcal{H}_{ab}$. 
The result of the sewing constraint analysis can now be formulated as follows:

**Proposition 5.1** An open-closed extension of a closed topological field theory is equivalent with an extension \((H_o, B, \rho, e)\) of the bulk algebra \((H, C, \eta)\), together with a (possibly trivial) reduction \(H_0 = \oplus_{a,b \in \Lambda} H_{ab}\).

The fact that an open-closed topological field theory defines such an extension and reduction was shown in Section 4. The converse (that an extension, together with a reduction, suffices to recover the data of Subsection 4.4) is rather obvious and I will not give all formal details here. The only slightly subtle point is that the map \(f : H_o \to H\) defined by the adjointness relation (109) has only ‘diagonal’ components, i.e. is of the form 

\[ f = \sum_{a \neq b} f(ab), \]

which must hold for all \(\phi\). Since \(\eta\) is non-degenerate, this requires \(\sum_{a \neq b} \psi_{ab} = 0\), so that \(f(\psi) = \sum_a f(aa)(\psi_{aa})\). Since \(\psi\) is arbitrary, it follows that the map \(f\) defined by the duality condition (109) is diagonal.

This result reveals a certain ambiguity in the construction of boundary sectors. In fact, one can have two realizations of the structure in Subsection 4.4, such that they both define the same total boundary extension \((H_o, B, \rho, e)\). Such realizations are indistinguishable from the physical point of view, and thus they should be identified.

Consider the implications of this observation for the meaning of ‘topological D-branes’. The relevant problem is to give a precise formulation of what is means to determine ‘all’ admissible boundary sectors \(a\). Since any such sector \(a\) defines an extension \((H_{aa}, B(aaa), \rho(aa), e(a))\) of the bulk algebra, it is clear that each admissible label \(a\) in Section 2 must correspond to an object of the type described in Definition 5.3. However, such an extension need not be irreducible, and a reduction leads to a refinement of the admissible set of boundary labels. It is clear that the meaningful procedure is to look for all irreducible boundary extensions. This allows us to give the following

**Definition 5.5** A topological D-brane (or ‘abstract boundary condition’) is an irreducible boundary extension of the bulk algebra \((H, C, \eta)\).

We can now give a precise formulation of the boundary extension problem for a given bulk topological field theory. Let \(S\) be the set of (isomorphism classes of) irreducible boundary extensions of a given bulk algebra (this set may be infinite). Assuming \(S\) is known, a boundary extension of the bulk theory is constructed as follows. First, pick a (finite) subset \(S_0\) of \(S\), enumerated by a set of labels \(\Lambda\), and let \((H^{(a)}_o, B^{(a)}, \rho^{(a)}, e^{(a)}) \in S_0\) be the boundary extension associated to \(a \in \Lambda\). Then boundary extensions of the given bulk theory, determined by this set of topological D-branes, correspond to structures of the type described in Subsection 4.4., based on the set of labels \(\Lambda\) and having the property that \((H_{aa}, B(aaa), \rho(aa), e(a)) = (H^{(a)}_o, B^{(a)}, \rho^{(a)}, e^{(a)})\) for all \(a\)
in $\Lambda$. This is a well-defined mathematical problem, lying at the intersection between algebra and category theory (see the next subsection). To my knowledge, this program has not been carried out even for a single nontrivial example.

5.1 Category-theoretic interpretation

The presence of various boundary sectors gives a category-theoretic flavor to the boundary data of Subsection 4.4. Since this interpretation is quite obvious and not particularly deep, I will keep the following remarks short. The categorical formulation arises by viewing the labels $a$ as objects and the open string states in $H_{ba}$ as morphisms from $a$ to $b$: $\text{Hom}(a, b) = H_{ba}$. Then one defines composition of morphisms $\text{Hom}(a, b) \times \text{Hom}(b, c) \to \text{Hom}(a, c)$ to be given by the boundary product $B(ca)$. This gives a category due to associativity of the boundary product $B$. Note that we have $\mathbb{Z}_2$-gradings on the spaces $H_{ba}$ and the composition $b$ is a degree zero map. Thus, we have a $\mathbb{Z}_2$-graded category. The boundary topological metric gives non-degenerate bilinear pairings $\rho(ba)$ between $\text{Hom}(a, b)$ and $\text{Hom}(b, a)$ which obey the cyclicity constraint (40). Restricting to degree zero morphisms $H^0_{ba}$ gives a ‘Grassmann-even’ sub-category. These are the fundamental objects behind the ‘categories of D-branes’ recently considered in the literature [7] from a space-time perspective. For the reader familiar with [7], we mention that morphism compositions in our approach arise naturally from the physical boundary product, or equivalently from triple boundary correlators. This gives the physical origin of morphism compositions, which are not restricted to the even subspaces $H^0_{ba}$. The application of our framework to open-closed topological sigma models is discussed in [34].

If we view the boundary data in this fashion, then the bulk-boundary product acts as an exterior multiplication on the morphism spaces $\text{Hom}(a, b)$ and endows them with a module structure over the Frobenius algebra $(H, C)$. The topological Cardy condition (77) gives a constraint between this exterior multiplication and the boundary product $B$.

\[21\] The authors of [7] consider the Calabi-Yau B-model [3, 4] in the presence of D-branes, some of which can be described as holomorphic bundles over the target space. For such D-branes, they consider the category of holomorphic bundles and bundle morphisms, and then proceed to restrict to the BPS saturated case, for which they propose a modified stability condition inspired by mirror symmetry arguments. The physical origin of this category-theoretic structure is not discussed in [7]. As we show in [34], the category of holomorphic bundles arises by considering the realization of our structure for the open-closed B-model. For those topological D-branes which can be described through holomorphic bundles $V_a$, the (physically relevant) morphism spaces are $H_{ba} = \text{Ext}^*(V_a, V_b) = \oplus_j \text{Ext}^j(V_a, V_b)$, as proposed by Kontsevich in [15] from mathematical considerations. The boundary product $B$ is the Yoneda product. The grading on the total extension spaces corresponds to a $\mathbb{Z}$-grading induced by the global $U(1)$ worldsheet symmetry of the model. The category of holomorphic bundles considered in [7] arises from this by restricting to degree zero morphisms (with respect to this grading), i.e. to the spaces $\text{Ext}^0(V_a, V_b) = \text{Hom}(V_a, V_b)$, in which case the boundary product reduces to composition of morphisms.
6 Conclusions and directions for further research

We gave a systematic derivation of the ‘on-shell’ structure of two-dimensional topological field theories on oriented open-closed Riemann surfaces. The analysis of sewing constraints allowed us to encode all information contained in such a theory in the well-defined algebraic structure summarized in Subsection 4.4. This mathematical object can be used as a definition of open-closed ‘on-shell’ topological field theories, and can be taken as the starting point in the study of boundary extensions, as well as of the ‘on-shell’ deformation problem. We drew attention to the central role of the bulk-boundary and boundary-bulk maps and gave a detailed analysis of the topological version (69) of the (generalized) Cardy constraint.

Consideration of an arbitrary number of ‘open string sectors’ led us to the problem of decompositions of boundary extensions, a mathematical formulation of which was given in Subsection 5.1. We also extracted a category structure from the boundary products and proposed a precise definition of topological D-branes as irreducible boundary extensions.

It is natural to ask about the off-shell counterpart of this framework in open-closed cohomological field theories. A correct treatment of this problem requires consideration of topological open-closed string field theory along the lines of [13], generalized to the case of an arbitrary number of boundary sectors. In fact, open-closed string field theory holds the key to a physical understanding of recent mathematical work on homological mirror symmetry, as we explain in more detail somewhere else. The tree-level off-shell structure was recently discussed in [25], under restrictive conditions on the boundary data

Understanding homological mirror symmetry [15] from a string field theoretic point of view requires certain generalizations of the topological open-closed string theories of [2]. In particular, one has to consider such systems in the presence of an arbitrary number of D-branes, which leads to boundary-condition-changing sectors. This problem is discussed somewhere else [34].

Acknowledgments The author thanks Jae-Suk Park for many stimulating conversations and Martin Rocek for his sustained interest in his work. He also wishes to thank Sorin Popescu for some helpful mathematical observations.

---

22 The authors of [25] give a partial off-shell description of the tree-level algebraic structure in a boundary sector containing a single D-brane. In fact, they mainly restrict to the case where the boundary product is associative off-shell, since they are interested in giving a physical interpretation to recent results of Kontsevich and Soibelman which gave a proof of Deligne’s conjecture [26]. Since they work at open string tree level, they do not consider the (off-shell) version of the topological Cardy constraint, which appears from a bulk-boundary sewing condition on the cylinder (however, this conditions does constrain tree level data). The work of [25] focuses on the boundary deformation problem, which they relate to the mathematical results just cited. The sewing bulk-boundary constraints considered in [25] can be viewed as a particular case of the more general string vertex equations of [13], as they apply to their situation.
References


