Gravitational Collapse in Constant Potential Bath

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Abstract

We analyse here the gravitational collapse of directed null radiation in a background with a constant potential such as one produced by a star system like galaxy in which the collapsing object is immersed. Both naked singularities and black holes are shown to be developing as the final outcome of the collapse. An interesting feature that emerges is that a part of the naked singularity spectrum in collapsing Vaidya region gets covered in the corresponding dual-Vaidya region, which corresponds to the Vaidya directed null radiation sitting in constant potential bath. The implications of such a result towards the issue of stability of naked singularities are discussed.

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1 Introduction

The spherical gravitational collapse in general relativity has been studied extensively in recent years by several authors (see e.g. [1], and references therein). The generic picture that emerges from these investigations is that both black holes and naked singularities develop as a result of collapse, depending on the nature of the initial data from which the collapse commences, for a rather general form of matter and equation of state. Also, various regularity conditions which are thought to be physically reasonable, are satisfied. While the strong gravity regions forming due to collapse get necessarily hidden behind an event horizon of gravity if a black hole forms, it is possible for radiations and energy to escape away from such regions to a faraway observer in the case of a naked singularity developing.

The key issue that remains open, and on which much debate is focused currently (see e.g. [2]), is the question of stability of such naked singularities. The point is, even if naked singularities develop, if they are not stable or generic in a suitable sense, they may not be physically realizable, and a physically realistic gravitational collapse will result in a black hole formation only. The problem however, in making any progress towards such a conclusion, is we have no well-defined notion of stability or genericity available presently in gravitation theory. It is in fact known that such a formulation may present formidable difficulties, and one may require new and sophisticated mathematical tools towards such a purpose. Under such a situation, a good way to make progress on this question is to subject the currently available scenarios of collapse to physically motivated perturbations, and then examine if the naked singularities of collapse persist or they disappear. Actually, such studies might even help us to generate a good definition of stability. There have been a few studies in this direction. For example, Joshi and Krolak [3] examined quasi-spherical collapse to consider non-spherical perturbations over the dust collapse models. Such studies imply that naked singularities do not necessarily go away as soon as we depart from spherical symmetry.

Our purpose here is to investigate a somewhat different type of perturbation over the usual scenarios considered, which is physically motivated as we explain below. In a typical scenario for collapse a cloud of matter distribution consisting of dust, radiation etc. collapses onto itself under its own gravity. The exterior field to the cloud is described by the asymptotically flat Schwarzschild solution. The assumption of asymptotic flatness essentially means that we are dealing with an isolated system with empty exterior. This is of course not a realistic situation, because a collapsing star would be a part of a star system like a galaxy. For a realistic description, it is required to break the asymptotic flatness of the Schwarzschild field. Since the latter is the unique solution of the Einstein vacuum equations, this would amount to introducing some energy distribution altering the vacuum character of the spacetime.

The question is, could this be achieved without disturbing the basic features of Schwarzschild field which have been observationally verified. The answer to this is in the affirmative. What we are seeking is the spacetime which is for all observationally tested properties equivalent to the Schwarzschild field, yet is not asymptotically flat. The Newtonian version of the situation would be that the collapsing system is sitting in a cavity of constant potential produced by the exterior universe. In the Newtonian theory constant potential has no dynamics, and hence has no effect on the dynamics of collapse. In contrast, this is not the case for the general theory of relativity. That is, the constant potential in the context of the situation under consideration is dynamically non-trivial. Note that the Schwarzschild solution written in
terms of the potential has the characteristic property that it does not allow the addition of a constant as was allowed by the Newtonian theory. General relativity in this sense determines the potential absolutely [4]. Without disturbing the Newtonian limit as well as the basic character of the field, we can add a constant to the potential which is the mark of the presence of exterior matter to the empty cavity embedding the collapsing body. It turns out that the two situations are electro-gravity dual of each other [8]. This duality is equivalent to the interchange of the Ricci and the Einstein tensors. This however does not make any difference for the Einstein vacuum equation but it does for the effective empty space equation for spherical symmetry given by $\rho = T_{ab} u^a u^b = 0$, $\rho_n = T_{ab} k^a k^b = 0$ where $u_a u^a = -1$, $k^a k_a = 0$. This admits the Schwarzschild solution as the general solution as the vacuum equations do. The dual set to these equations is $\rho_t = (T_{ab} - (1/2) T g_{ab}) u^a u^b = 0$, $\rho_n = 0$, which also admits the general solution in which the potential attains a constant non-zero value asymptotically [8]. It is thus asymptotically non-flat but is rather a spacetime of constant potential. Note that, if the effective equation is written in terms of the Ricci components, the dual equation would follow from it by replacing the Ricci by the Einstein component. It is an interchange between the Riemann curvature and the double left and right dual of the Riemann. Similar to the Schwarzschild solution, it is possible to construct solutions dual to other solutions [9], including the Vaidya solution.

We consider here a specific class of such models, the so called dual-Vaidya models, which represent the collapse of radiation shells when the exterior is no longer vacuum Schwarzschild in the sense described above. The paper is organised as follows. In Section 2 we discuss the the realistic setting with the dual-Vaidya models. In Section 3 we analyze the causal structure of the spacetime singularity. This is followed by the concluding Section 4.

2 The realistic setting

In the Newtonian theory the situation of spherical collapse would be envisioned as follows: a star or cloud of matter is collapsing in an empty cavity which is a part of a galaxy. Let the matter distribution exterior to the cavity be homogeneous, it would then produce a constant potential inside the cavity. The field inside the cavity would be described by the Laplace equation which will yield a free constant in its solution that could be matched with the constant potential of the exterior distribution. Since constant potential cannot affect the dynamics of collapse, it does not make any non-trivial contribution.

The situation is however different in general relativity. Consider, for example, the Schwarzschild case. It is asymptotically flat which means that the potential can vanish only at infinity and nowhere else. In general relativity a constant potential is thus not dynamically trivial. The Einstein vacuum equation for the situation under consideration ultimately reduces to the Laplace equation and its first integral. It is the latter that does not let the potential vanish anywhere else than infinity and consequently implying asymptotic flatness. The only alternative to break asymptotic flatness in the most harmless way is to relax the second equation and let the potential be given by the general solution of the Laplace equation with two constants of integration [4]. It would however amount to introduction of some energy distribution which conforms to a string dust distribution [5] or approximates at large distance to a global monopole [6]. It has been shown that this modification does not affect
the Schwarzschild field appreciably, the observational values of the physical parameters get
scaled by a very small factor [7].

It has been shown by one of us [8] that the effective empty space for the situation under
consideration could be defined by the equation

$$\rho = T^0_0 = 0, \rho_n = T^0_0 - T^1_1 = 0.$$  

It admits the Schwarzschild (as for the vacuum equation) as the unique solution. Then the
equation dual to it would read as [8]

$$\rho_t = T^0_0 - 1/2T = 0, \rho_n = 0$$

which follows from the effective equation by replacing the Einstein tensor by the Ricci tensor.
Note that under the duality transformation, $\rho \leftrightarrow \rho_t, \rho_n \rightarrow \rho_n$. Here by duality we mean
the interchange of the Ricci and the Einstein tensor in the equation. This is termed as
electro-gravity duality for it interchanges active and passive electric parts of the Riemann
curvature [9].

The remarkable thing is that the dual equation also admits the unique solution which
is nothing but reinstating the second constant of integration in the solution of the Laplace
equation. That is the dual solution to the Schwarzschild solution is not asymptotically
flat and its potential differs from that of the Schwarzschild only by a constant. Thus the
two solutions are absolutely equivalent at the Newtonian level. This is how asymptotically
flatness could be avoided in the most harmless manner allowing us to study the collapse in
the background of other distributions like galaxy or the rest of the Universe. This is however
not expected to alter the basic conclusions except scaling of the crucial parameter which
demarks black hole and naked singularity situations. In the same vein, the asymptotic limit
of the dual Schwarzschild solution is a spacetime of constant potential and it is dual flat.

We take a shell of radiation collapsing onto an empty cavity. The inner most cavity is
described by the constant potential (dual flat) spacetime, followed by the radiation shell
described by the dual Vaidya solution and finally it is the dual Schwarzschild solution in the
exterior. Essentially, all this simply amounts to putting the whole system into a constant
potential background. Since this is non-trivial in GR, it would be worthwhile to study its
effect on the process of gravitational collapse.

The spacetime under consideration can be divided into three regions I, II and III, corre-
sponding to the dual flat, the dual Vaidya and the dual Schwarzschild spacetime, respectively.

The dual vacuum spacetime (region I) can be conveniently written in the $(V,R,\theta,\phi)$
coordinates as

$$ds^2_I = -(1 + k)dV^2 + 2dVdR + R^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

here null coordinate $V$ denotes the advanced time and constant negative $k$ characterizes
the strength of the constant field resulting from the dual vacuum in the limit of vanishing
mass. The range of $k$ is restricted to $-1 < k \leq 0$, in order to preserve the signature of the
metric, as otherwise the metric becomes degenerate. The only nonvanishing component of
the stress-energy tensor is

$$T^v_v = -\frac{k}{R^2}.$$
The weak as well as strong energy conditions are satisfied in the spacetime. We note that there is a mild singularity at the center $r = 0$ in the dual Minkowski spacetime above. However, as will be clear, it is not a genuine physical feature in any sense because $\sigma^2 R_{ij} K^i K^j$ goes to zero in the limit, $\sigma \to 0$, of approach to the singularity (here $K^i$ is the tangent vector to nonspacelike geodesics, and $\sigma$ is the affine parameter along the same). Hence no radial test particles (ingoing or outgoing) feel any gravitational tidal forces. The initial data is regular because there are no trapped surfaces on any spatial slice below the genuine curvature singularity at $v = 0$, $r = 0$, the first point of the dual Vaidya Spacetime. In the Newtonian limit the spacetime corresponds to a constant potential which understandably does not give rise to any physical pathology.

Region II consists of a dual Vaidya spacetime, by which we mean an imploding radiation field over a background constant potential field generated by $k$. It can be given in general by a metric of the form

$$ds^2_{II} = -(1 - \frac{2g}{r})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

(2)

Here $g(v, r)$ is interpreted as the mass function, which can be defined in general in spherically symmetric spacetimes, as

$$g(v, r) \equiv \frac{r}{2} R^{32}_{232} = \frac{r}{2} (1 - g^{\mu\nu} \partial_\mu r \partial_\nu r),$$

(3)

where $r$ is the areal radius and $R^{32}_{232}$ is the component of the Riemann-Chritoffel tensor. It is a positive definite quantity due to the positivity of matter density.

As $r$ is a coordinate here we have

$$g(v, r) = \frac{r}{2} (1 - g^{rr}).$$

The null coordinate $v$ is defined for range $v \geq 0$. Regions I and II are matched across $v = 0$ null hypersurface where radiation starts to collapse. For a realistic evolution it is necessary that there is no flux of radiation across the $v = 0$ null line, and that the mass function attains the same value at it when approached from either side. From equation (3) the above stated matching implies following condition on the mass function,

$$g(v, r) \big|_{v=0} = -\frac{kr}{2}.\tag{4}$$

For simplicity we work throughout with a source of radiation given by a linear function such that

$$2g(v, r) = \lambda v - kr.\tag{5}$$

(Note that the choice above ensures matching of mass function along $v = 0$ null hypersurface). The only nonvanishing components of the stress-energy tensor are

$$T^v_v = -\frac{k}{r^2}, T^r_v = \frac{\lambda}{r^2}.\tag{5}$$
The weak as well as strong energy conditions are satisfied in the spacetime because we have \( \lambda > 0 \) and \( k < 0 \).

The source of radiation is switched off at \( v = T \) and the spacetime (Region II) is here joined with a dual Schwarzschild spacetime (Region III)

\[
ds_{III}^2 = -(1 - \frac{\lambda T}{r} + k)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)
\]

with mass \( M = (\lambda T - kr)/2 \) (see Fig. 1).

\[\begin{array}{ll}
0 < v < 0 & \text{for } v < 0 \\
0 < v < T & \text{for } 0 < v < T \\
\lambda T - kr & \text{for } v > T \\
\end{array}\]

**Figure 1** Gravitational collapse of radiation shells

### 3 Nature of the final singularity

Earlier investigations of collapse models with a linear mass function were done by Kuroda [10] and Papapetrou [11] where they showed the existence of naked singularity for weak enough radiation collapsing along \( v = \text{constant} \) lines. Radial null geodesics were analysed by Papapetrou clarifying the null structure of spacetime. The cases with more general mass function were analysed by Rajagopal and Lake [12] and others [2]. Also it is well-known that singularities arising in these spacetimes are strong curvature singularities.

For our purpose here we have restricted the discussion to the choice of a linear mass function (5). This has the advantage of a clearer physical interpretation in that it relates the outcome of collapse as black hole or a naked singularity in terms of rate of collapse or mass implosion at the center. The key idea of the analysis here is that the nature of monopole field maintains the self-similar nature of the spacetime metric and we could use the analysis developed earlier (see, e.g., chapter 6 [2]). The dual-Vaidya region admits a homothetic...
killing vector 

\[ \xi = v \left( \frac{\partial}{\partial v} \right) + r \left( \frac{\partial}{\partial r} \right) \]

which is given by the Lie derivative 

\[ \mathcal{L}_\xi g_{\mu \nu} = 2g_{\mu \nu} \] (7)

Define now \( K^\mu = dx^\mu / ds \) as tangent to non-spacelike geodesics, where \( s \) is an affine parameter. Then,

\[ g_{\mu \nu} K^\mu K^\nu = B \] (8)

where \( B = 0 \) for null and \(+1(−1)\) for spacelike (timelike) vectors. Along non-spacelike geodesics we have

\[ \frac{d}{ds}(\xi^\mu K_\mu) = \xi_{\mu \nu} K^\mu K^\nu \] (9)

which can be integrated, after using eqns (7) and (8) to reduce the expression on the right hand side of equation above to \( B \), as

\[ \xi^\mu K_\mu = Bs + C \] (10)

where we have used \( K^\mu K^\nu = 0 \) and \( C \) is the integration constant. From the expression for Killing vector we get from (10)

\[ rK_r + vK_v = Bs + C. \] (11)

Now defining \( P(v, r) \) as

\[ K^v = \frac{P(v, r)}{r} = \frac{dv}{ds} \] (12)

and using eqn. (8) we get

\[ K^r = \left( 1 - \frac{2g}{r} \right) \frac{P}{2r} + r \left( B - \frac{l^2}{r^2} \right). \] (13)

Writing (8) explicitly and using eqn (11) for \( K_v \) we have

\[ r^2[X + kX - \lambda X^2 - 2](K_r)^2 + 2r(Bs + C)K_r + X(l^2 - Br^2) = 0, \] (14)

where we have defined the similarity variable \( X = v/r \). Eqn (14) can be solved for \( K_r \) as

\[ rK_r = \frac{(Bs + C) \pm \sqrt{(Bs + C)^2 + X(l^2 - Br^2)(2 + \lambda X^2 - X - kX)}}{2 + \lambda X^2 - X - kX}. \] (15)
and $K_v$ can be obtained using (11). Since $K^v = g^{\nu\rho}K_{\nu\rho}$ we get the expression for $P$, defined in (12), using (15) as

$$P(v, r) = \frac{C(1 + As) + C\sqrt{(1 + As)^2 + X[L^2 - (Ar^2/C)][2 + \lambda X^2 - X - kX]}}{2 + \lambda X^2 - X - kX}$$

(16)

where $A = B/C$ and $L = l/C$.

The point $v = 0, r = 0$ is the first point on the singularity curve. To analyse nature of this point one can try to analyse the outgoing singular geodesics terminating at the singularity in the past. We consider a parameterization ($s$) of these outgoing singular geodesics in such a way that in the limit $s \to 0$ we approach singular point $(0, 0)$. Let

$$\lim_{s \to 0} v = \lim_{s \to 0} r = X_0,$$

(17)

along a singular geodesic. Using eqns. (12), (13) and (16) in eqn. (17) above we get

$$X_0 = \lim_{s \to 0} \frac{dv/ds}{dr/ds} = \frac{2[1 + Q(X_0)]^2}{(1 - \lambda X_0 + k)[1 + Q(X_0)]^2 - L^2(2 + \lambda X_0 - kX_0 - X_0)^2}$$

(18)

where $Q(X) = [(1 + As)^2 + X[L^2 - (Ar^2/C)][2 + \lambda X^2 - X - kX])^{1/2}$ and $r = r(X)$.

Since we want to analyze the effect of constant potential term $k$ on the final state of collapse, it is sufficient for our purpose to analyse the simple case of radial null geodesics, i.e., $L = 0$ and $B = 0$. Eqn. (18) simplifies to

$$X_0 = \frac{2[1 + Q(X_0)]^2}{(1 - \lambda X_0 + k)[1 + Q(X_0)]^2}$$

(19)

and hence, on further simplification,

$$X_0 = a_\pm = \frac{(1 + k) \pm \sqrt{(1 + k)^2 - 8\lambda}}{2\lambda}.$$  

(20)

Therefore, real values of $a_\pm$ are possible only if $\lambda \leq (1 + k)^2/8$ (See Fig. 2).
The singularity arising in the Vaidya-Papapetrou model was shown to be a strong curvature singularity, and also a scalar polynomial singularity [2]. In fact, all known cases of strong curvature singularities in Einstein equations are also scalar curvature singularities. In the model here also the singularity continues to be a strong curvature singularity, as expected, with the Kretschmann scalar

$$K = R^{ijkl}R_{ijkl} = 4\frac{k^2}{r^4} - 8\frac{k\lambda X}{r^4} + 12\frac{\lambda^2 X^2}{r^4}$$

which always diverges at the singularity. It is interesting to note that the curvature scalar $R \equiv g^{ij}R_{ij}$ and the Ricci scalar $R^{ij}R_{ij}$ also diverge in the limit of approach to the singularity,

$$R = -2\frac{k}{r^2}, R^{ij}R_{ij} = 2\frac{k^2}{r^4}$$

which were vanishing in the Vaidya-Papapetrou case.

One can further measure the rate of curvature growth along the singular geodesics which come out, in the limit of approach to the singularity. The result is,

$$\lim_{s \to 0} s^2 R_{ij}V^i V^j = \frac{4\lambda}{(1 + k)}$$

where $s$ is the affine parameter, and we restrict to outgoing radial ($l = 0$) null ($B = 0$) geodesics. We work in the parameterization $s = 0$ corresponding to the first point on the singularity curve. It is thus seen that the singularity continues to be a strong curvature singularity. Again, setting $k = 0$ gives us the earlier results in the Vaidya models.

Figure 2 Final singularity in gravitational collapse

\[\begin{array}{c}
\text{Space-time Singularity} \\
\text{Event Horizon}
\end{array}\]

Dual-Schwarzschild
Dual-Flat space-time
Dual-Vaidya

\[\begin{array}{c}
\text{Space-time Singularity} \\
\text{Event Horizon}
\end{array}\]

Dual-Schwarzschild
Dual-Flat space-time
Dual-Vaidya
4 Concluding remarks

We studied here the Vaidya collapse with a monopole field and examined the formation of black holes and naked singularities. Our purpose was to examine how the perturbation induced by the external matter fields affect the formation or otherwise of the naked singularity. The relevant question is whether the effect of such external fields could remove the occurrence of the same. We find that a part of the range of the parameter space gets covered once the external potential is taken into account.

This scenario is motivated by the fact that in the real Universe a collapsing object is always sitting in a cavity which is enclosed by the other matter distribution like a galaxy, cluster of galaxies, rather the rest of the Universe. Strictly speaking, asymptotic flatness character of the Schwarzschild solution does not permit the existence of any other matter in the Universe. To make the setting accord with the non-empty rest of the Universe and Mach’s principle, we need to break the asymptotic flatness of the Schwarzschild solution [13]. The constant $k$ is the measure of the uniform potential produced by the rest of the stars in the galaxy in the empty cavity enclosing the collapsing system. That is why it would be negative and its measure would be of the order of $O(10^{-6})$ for the star velocity of $200$ km/sec. It would therefore not disturb any observations materially.

It is worth noting however, that since the condition $|k| < 1$ is always respected, there would always (even for large $k$) exist a nonzero measure range of parameter space available for which the naked singularity definitely develops. In that sense, the potential generated by the exterior universe does not remove the naked singularity, which thus displays stability with respect to this particular mode of perturbation. Since there is no general definition available to check the stability of naked singularities as indicated earlier, the only way available is to check the stability with respect to classes of perturbations which are essentially physically motivated (see e.g. [14]). It would be reasonable to subject various classes of collapse models resulting into naked singularities to such physical perturbations to check for the stability (see e.g. [15] for a detailed discussion on dust models and how initial data is related to the formation of naked singularities and black holes). This may be perhaps the only possible way available towards a suitable formulation of the cosmic censorship hypothesis.

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Appendix

The geodesic equations, for the spacetime metric (2) in region II, are written as

$$\frac{d}{ds}[r^2 \sin \theta K^\phi] = 0 \quad (A1)$$
\[ \frac{dK^\theta}{ds} + \frac{2}{r} K^\theta K^\theta - \sin \theta \cos \theta (K^\phi)^2 = 0 \quad (A2) \]

\[ \frac{dK^r}{ds} + 2 \left( \frac{g}{r^2} - \frac{g_r}{r} \right) K^r K^v + \left[ \frac{g_v}{r} - \left( \frac{g}{r^2} - \frac{g_r}{r} \right) \left( 1 - \frac{2g}{r} \right) \right] (K^v)^2 \]
\[ - \left( 1 - \frac{2g}{r} \right) r[(K^\theta)^2 + \sin^2 \theta (K^\phi)^2] = 0 \quad (A3) \]

\[ \frac{dK^v}{ds} + \left( \frac{g}{r^2} - \frac{g_r}{r} \right) (K^v)^2 - r[(K^\theta)^2 + \sin^2 \theta (K^\phi)^2] = 0. \quad (A4) \]

Here superscripts denote corresponding partial derivatives. Integrating eqns. (A1) and (A2) gives solution of the form

\[ K^\theta = \frac{l \sin \beta \cos \phi}{r^2}, \quad K^\phi = \frac{l \cos \beta}{r^2 \sin^2 \theta} \quad (A5) \]

where \( l \) is impact parameter and \( \beta \) is an isotropy parameter satisfying \( \sin \phi \tan \beta = \cot \theta \).

**References**


