DYSON-SCHWINGER EQUATIONS
– ASPECTS OF THE PION

M.B. HECHT, C.D. ROBERTS and S.M. SCHMIDT
Physics Division, Bldg 203, Argonne National Laboratory
Argonne, IL 60439-4843, USA

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The contemporary use of Dyson-Schwinger equations in hadronic physics is exemplified via applications to the calculation of pseudoscalar meson masses, and inclusive deep inelastic scattering with a determination of the pion’s valence-quark distribution function.

Keywords: Dyson-Schwinger equations, Goldstone bosons, Heavy-Quarks, Valence-quark distribution functions.

The Dyson-Schwinger equations (DSEs)\(^1\) provide an approach well-suited to the calculation of pion observables. Since a chiral symmetry preserving truncation scheme exists,\(^2\) they provide a framework in which the dichotomous bound-state/Goldstone-mode character of the pion is easily captured.\(^3,4\) Furthermore, because perturbation theory is recovered in the weak coupling limit, they combine; e.g., a description of low-energy \(\pi-\pi\) scattering\(^5\) with a calculation of the electromagnetic pion form factor, \(F_\pi(q^2)\), that yields\(^6\) the \(1/q^2\)-behaviour expected from perturbative analyses at large spacelike\(q^2\) and a calculated evolution to the \(\rho\)-meson pole in the timelike region.\(^7\) These and other contemporary applications are reviewed in Refs. [8,9].

As an illustration, using the inhomogeneous Bethe-Salpeter equations for the axial-vector and pseudovector vertices; the dressed-quark DSE; and the fact that a nonperturbative Ward-Takahashi identity preserving truncation of the DSEs is possible, it was shown in Ref. [3] that for flavour nonsinglet pseudoscalar mesons

\[
f_H m_H^2 = \mathcal{M}_H^{\zeta},
\]

with \(\mathcal{M}_H^{\zeta} := \text{tr}_{\text{flavour}}[M_\zeta\{T^H, (T^H)^t\}]\), where \(M_\zeta = \text{diag}(m_\zeta^u, m_\zeta^d, m_\zeta^s, \ldots)\), with \(\zeta\) the renormalisation point, and \((\cdot)^t\) indicates matrix transpose, so that \(\mathcal{M}_H^{\zeta}\) is the sum of the constituents’ current-quark masses. This model-independent identity is valid for all current-quark masses, irrespective of their magnitude, and therefore provides a single formula that unifies the light- and heavy-quark regimes.

In Eq. (1), \(f_H\) is the leptonic decay constant,

\[
f_H P_\mu = Z_2 \int_0^\Lambda \frac{d^4 q}{(2\pi)^4} \frac{1}{2} \text{tr} [\left(T^H\right)^t \gamma_5 \gamma_\mu \chi_H(q; P)],
\]

1
where $Z_2 = Z_2(\zeta, \Lambda)$ is the dressed-quark wave function renormalisation constant, with $\Lambda$ the regularisation mass-scale. (The advantages of employing a translationally invariant regularisation scheme when Ward-Takahashi identities are involved should be obvious.) The r.h.s. is gauge-invariant and independent of $\zeta$ and $\Lambda$. The other factor is

$$\nu^\zeta_H = Z_4 \int_q^\Lambda \frac{1}{2} \text{Tr} \left[ (T^H)^4 \gamma_5 \chi_H(q; P) \right],$$

where $Z_4 = Z_4(\zeta, \Lambda)$ is the dressed-quark mass renormalisation constant. Here the r.h.s. is gauge invariant and cutoff-independent, and $Z_4$’s $\zeta$-dependence ensures that the product on the r.h.s. of Eq. (1) is independent of the renormalisation point. Using these formulae it can be established that Eq. (1) reproduces the so-called Gell-Mann–Oakes-Renner relation in the limit of small current-quark masses and also predicts that heavy-meson masses increase linearly with the mass of their heaviest constituent.$^{10}$ However, the latter “heavy-quark limit” provides a poor approximation to the pseudoscalar meson mass trajectory for current-quark masses less than that of the $b$-quark.$^{11}$

Another more recent application is a calculation of the pion’s valence-quark distribution,$^{12}$ of which a very accurate measurement is possible given a high-luminosity electron-proton collider.$^{13}$ Using an algebraic DSE-model,$^{14}$ employed successfully in studies of a wide range of hadronic observables; e.g., Refs. [10,11], it is straightforward to calculate the “handbag” contributions to the virtual photon-pion forward Compton scattering amplitude. These are the only impulse approximation diagrams that survive when calculating the pion’s structure function in the deep inelastic Bjorken limit and yield the results in Fig. 1.

In this calculation, dressed-quarks with a valence-quark mass of $\bar{\cal M} = 301$ MeV carry 71% of the pion’s momentum at a resolving scale $q_0 = 0.54$ GeV = 1/(0.37 fm). The remainder is carried by the gluons that effect the binding of the pion bound state. The second and third moments of the distribution are $\langle x^2 \rangle_{q_0} = 0.18$, $\langle x^3 \rangle_{q_0} = 0.10$. To determine the resolving scale, $q_0$, we employed the 3-flavour ($N_f = 3$) QCD renormalisation group (evolution) equations to evolve the distribution in Fig. 1 up to $q = 2$ GeV, and required agreement between the first and second moments of our evolved distribution and those calculated from the phenomenological fits in Ref. [15]. $q_0 = 0.54$ GeV gives

<table>
<thead>
<tr>
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<th>$\langle x \rangle_q$</th>
<th>$\langle x^2 \rangle_q$</th>
<th>$\langle x^3 \rangle_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calc.$^{12}$</td>
<td>0.24</td>
<td>0.098</td>
<td>0.049</td>
</tr>
<tr>
<td>Fit$^{15}$</td>
<td>0.24 ± 0.01</td>
<td>0.10 ± 0.01</td>
<td>0.058 ± 0.004</td>
</tr>
<tr>
<td>Latt.$^{18}$</td>
<td>0.27 ± 0.01</td>
<td>0.11 ± 0.3</td>
<td>0.048 ± 0.020</td>
</tr>
</tbody>
</table>

with the valence-quarks now carrying a momentum-fraction of 0.49.

The evolved distribution is also shown in Fig. 1. The evident accentuation via evolution of the convex-up behaviour of the distribution near $x = 1$ is characteristic of the renormalisation group equations, which populate the sea-quark distribution

$^*$Herein we have corrected a minor numerical error discovered in the calculations of Ref. [12].
Figure 1: Dashed line: calculated form of $xu_v(x; q_0)$; Solid line: evolved distribution, $xu_v(x; q = 2\, \text{GeV})$; Dotted line: $xu_v(x; q = 4.05\, \text{GeV})$, evolved from $xu_v(x; q = 2\, \text{GeV})$ with $\Lambda_{QCD} = 0.204\, \text{GeV}$; and Dot-dashed line: phenomenological fit in Ref. [15], which takes the form $xu_v(x) \propto x^\alpha (1 - x)^\beta$. The data are obtained with an invariant $\mu^+\mu^-$-mass $> 4.05\, \text{GeV}$ and inferred from the differential pion-nucleon Drell-Yan cross section using simple distribution parametrisations of the type just indicated, yielding $\alpha = 0.64 \pm 0.03$, $\beta = 1.15 \pm 0.02$. This data was part of the set employed in the fit of Ref. [15]. Fits to our calculation using the parametrisation in Eq. (5) are indistinguishable from our result on the scale of this figure.

at small-$x$ at the expense of large-$x$ valence-quarks. The simple parametrisation: 

$$xu_v(x; q) \propto x^\alpha (1 - x)^\beta$$

is not flexible enough to provide a good pointwise fit to our calculated distribution. However, the modernised fitting form

$$xu_v(x) \propto x^{\eta_1} (1 - x)^{\eta_2} (1 - \epsilon_u \sqrt{x} + \gamma_u x),$$

with parameters: $\eta_1, \eta_2, \epsilon_u, \gamma_u$, can describe our calculated result very well.

The importance of this is apparent when one appreciates that the functional form

$$xu_v(x; q) \propto x^\alpha (1 - x)^\beta, \alpha = 0.67, \beta = 1.13,$$

can be obtained via the evolution from $q_0 = 0.35\, \text{GeV}$ of $u_v(x) = \theta(x)\theta(1 - x)$, which latter distribution corresponds to the valence-quark carrying each and every fraction of the pion’s momentum with equal probability, and to a momentum-independent pion Bethe-Salpeter amplitude; i.e., it is equivalent to representing the pion as a point particle. The convex-up character of our result is a characteristic feature of calculations in which the pion is described by a finite-size Bethe-Salpeter amplitude (see, e.g., Refs. [20]). Hence the convexity of the valence-quark distribution can be interpreted as a signature of binding in a nonpointlike pion.

While the moments of our calculated distribution are indistinguishable from
those of that fitted to data, there is a pointwise discrepancy between our result and the data. Currently we judge that this discrepancy can be attributed to the restricted function space used thus far in parametrising pion data, especially as the modernised fitting form, Eq. (5), can describe our calculation. We would be much interested in a reanalysis of existing data using the updated parametrisations and, indeed, in new experiments; e.g., Ref. [13], with small errors on the valence-quark domain, which might expose what we have described as the signature of quark-antiquark binding in QCD’s Goldstone mode.

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References