Single State Supermultiplet in 1+1 Dimensions

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Abstract

We discuss the issue of multiplicity of the BPS states in (1+1)-dimensional models with minimal $\mathcal{N}=1$ supersymmetry. The previous argument that the irreducible representation of supersymmetry algebra is one-dimensional in this case is elaborated. Algebraic treatment is illustrated by explicit quasiclassical quantization in various weakly coupled models which present examples of one- and two-dimensional representations. For non-BPS multiplets, which are two-dimensional, one can introduce an operator of the fermion charge which differentiates between two distinct states in the multiplet. This operator is not defined for BPS states.

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1 Introduction

In many supersymmetric problems admitting solitons the supersymmetry (SUSY) algebra is centrally extended [1]. An example of this type which attracted much attention recently is the monopole solution in $\mathcal{N}=2$ SUSY Yang-Mills theory. The central charges in SUSY algebra imply the existence of special supermultiplets, usually called BPS (Bogomol'nyi-Prasad-Sommerfield) saturated. The BPS saturated solitons preserve a part of the original SUSY – usually 1/2 or 1/4. The corresponding supercharges annihilate the soliton state in question, while the remaining supergenerators act nontrivially.

Due to a lesser number of acting generators the BPS multiplets are shortened. For instance, in the monopole problem mentioned above the monopole multiplet is eight-dimensional while the minimal non-BPS multiplet is sixteen-dimensional. The multiplet shortening, if it occurs, leads to far reaching dynamical consequences. In particular, the mass of the state $M$ and the central charge $Z$ are rigidly related,

$$M - |Z| = 0.$$  \hspace{1cm} (1)

The shortening of the supermultiplet makes the relation (1) intact under small variations of parameters of the problem, and it is not corrected at the quantum level.

SUSY theories with central charges were thoroughly studied in 25 years of their existence. In certain instances the number of the supercharges realized nontrivially is odd. The supermultiplet structure in this case may be rather peculiar. The most spectacular example is provided by minimal $\mathcal{N}=1$ supersymmetric models in 1+1 dimensions. In this case one deals with two supercharges, only one of which is realized nontrivially on the BPS states. In Ref. [2] it was mentioned that in this case the irreducible representation of superalgebra consists of a single state, i.e. the representation is one-dimensional. This issue was not elaborated in detail in Ref. [2], although the assertion was in contradiction with the previous analysis of Ref. [3] where the authors had argued that the multiplet shortening did not occur in the minimal models\(^1\). A crucial point of their argumentation was realization of a global $Z_2$ symmetry.

The fact that the irreducible one-dimensional multiplet is realized without doubling was explicitly demonstrated in Ref. [4] where the consideration starts from the $\mathcal{N}=2$ extended version of the model. As was shown in [5], in such $\mathcal{N}=2$ model the solitonic multiplet is shortened (i.e. it is two- rather than four-dimensional). A soft breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ was then introduced in Ref. [4]. When the breaking parameter reaches some critical value, one of the two soliton states disappears from the physical spectrum\([4]\) via the phenomenon of delocalization. Delocalization means that fields are not localized near the soliton center. The BPS state that remains localized is single.

\(^1\)We have recently learned from P. van Nieuwenhuizen that the authors changed their opinion on the issue. In the forthcoming paper by A. Goldhaber, A. Litvintsev and P. van Nieuwenhuizen a new analysis will be presented.
In this paper we streamline the argument by demonstrating the same phenomenon of the one-dimensional BPS multiplet directly in the $\mathcal{N}=1$ model per se. A crucial point is the absence of $Z_2$ symmetry in the physical sector. Whenever such symmetry can be found, one deals in fact with non-shortened multiplets consisting of two states which are not protected against renormalization and are not expected to be BPS.

A nice illustration is provided by an early application of superalgebra with the odd number of fermionic generators – the Grassmannian description of nonrelativistic spin 1/2 discovered by Berezin and Marinov [6]. These authors introduced three Grassmann variables, $\xi_k$ ($k = 1, 2, 3$) which were quantized by anticommutators,

$$\{\xi_k, \xi_l\} = \delta_{kl}. \tag{2}$$

The irreducible representations of this algebra are two-dimensional. For instance, one can choose $\xi_k = \sigma_k/\sqrt{2}$. There exists a unitary nonequivalent choice, $\xi_k = -\sigma_k/\sqrt{2}$. The change of the sign of $\xi_k$ is a $Z_2$ symmetry which might be relevant to the problem. However, physically this $Z_2$ is not implemented. Indeed, observable are spin operators, $S_i = -(i/2) \epsilon_{ikl} \xi_k \xi_l$, which are bilinear in $\xi_k$. This is the reason why we deal here with a single two-dimensional representation rather than with two (degenerate) ones.

Although the $Z_2$ symmetry associated with the change of sign of all fermion fields is not implemented, for non-BPS states one can find another $Z_2$, which plays the role of the fermion number. (Note that there is no corresponding local current in the models with minimal $\mathcal{N}=1$ supersymmetry). This fermion number is not defined for the BPS states.

The paper is organized as follows. In Sec. 2 we discuss the minimal $\mathcal{N}=1$ Ginzburg-Landau model and the form of the $\mathcal{N}=1$ superalgebra. In Sec. 3 we reanalyze the algebraic aspects of the problem to show that in the minimal $\mathcal{N}=1$ model the irreducible representations are one-dimensional for BPS states. Similar to the spin example above, there is a sign ambiguity which is physically unobservable.

We supplement this algebraic consideration by explicit construction of BPS states in a simple weakly coupled model. To provide an infrared regularization we put the system into a finite spatial box of the size $L$ with certain boundary conditions which preserve the residual supersymmetry. The model can be treated quasiclassically – we introduce the mode decomposition and quantize the corresponding coefficients. The soliton supermultiplet is determined by the zero modes. In the model at hand we have one bosonic (the soliton center) and one fermionic zero mode. This is in one-to-one correspondence with the algebraic construction mentioned above which leads to the one-dimensional supermultiplet (Sec. 4).

Section 5 is devoted to modifications of the model such that the number of the fermion zero modes on the kink is two. This is achieved by making the spatial dimension compact, or, alternatively, by introducing extra fields in the minimal model of Sec. 2 (the field content we consider in Sec. 5.2 is precisely that of the $\mathcal{N}=2$ model). Then, there are two classical BPS soliton states. The multiplet
shortening provides no protection. In fact, these states pair up (typically, at one-loop level in the supercharge) and form a two-dimensional non-BPS multiplet. We show how this happens; a crucial element of our demonstration is the anomaly. General statements which follow from our analysis of particular models are presented in Sec. 6. Our conclusions are summarized in Sec. 7.

2 Ginzburg-Landau models with minimal supersymmetry

In (1+1)-dimensional space, \( x^\mu = (t, z) \), the \( \mathcal{N}=1 \) superfield contains a real boson field \( \phi \) and a two-component Majorana spinor \( \psi_\alpha (\alpha = 1, 2) \). The Lagrangian of the \( \mathcal{N}=1 \) Ginzburg-Landau model with \( n \) superfields \( \{ \phi_i, \psi_{i\alpha} \} \) in the component form is

\[
\mathcal{L} = \frac{1}{2} \left\{ \partial_\mu \phi_i \partial^\mu \phi_i + i \tilde{\psi}_i \gamma_\mu \partial_\mu \psi_i - \frac{\partial W}{\partial \phi_i} \frac{\partial W}{\partial \phi_i} - \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \tilde{\psi}_i \psi_j \right\}, \tag{3}
\]

where summation over \( i \) is implied. The Dirac conjugation for the real spinor \( \psi \) is defined as \( \tilde{\psi} = \psi^T \gamma^0 \), and the Majorana basis for the two-dimensional \( \gamma \) matrices is

\[
\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_3, \quad \gamma^5 = \gamma^0 \gamma^1 = -\sigma_1. \tag{4}
\]

The superpotential \( W(\phi_i) \) is an arbitrary function of the fields \( \phi_i \). Although minimal supersymmetry does not protect the superpotential against radiative corrections the model is super-renormalizable. Some of such models are known to be exactly integrable \([7, 5]\). However, following Refs. \([2, 4]\) we will limit our consideration to the quasiclassical regime assuming that the expansion parameter is small.

If equations \( \partial W/\partial \phi_i = 0 \) have several solutions then the theory has several classical vacua and admits solitons interpolating between these vacua. For example, the model with one superfield \( \{ \phi, \psi \} \) (i.e. \( n = 1 \)) and the superpotential \( W \)

\[
W(\phi) = \frac{m^2}{4\lambda} \phi - \frac{\lambda}{3} \phi^3 \tag{5}
\]

(the so-called polynomial model) has two vacua, \( \phi = \pm m/\lambda \), and admits the BPS soliton which in the quasiclassical approximation, \( \lambda/m \ll 1 \), has the form

\[
\phi_0 = \frac{m}{2\lambda} \tanh \frac{mz}{2}. \tag{6}
\]

Two supercharges \( Q_\alpha \) of the model are defined as

\[
Q_\alpha = \int dz j_\alpha^0, \quad j^\mu = (\partial_\nu \phi_i) \gamma^\nu \gamma^\mu \psi_i + i \frac{\partial W}{\partial \phi_i} \gamma^\mu \psi_i. \tag{7}
\]
They generate the following superalgebra:

\[ \{Q_\alpha, \bar{Q}_\beta\} = 2(\gamma^\mu)_{\alpha\beta}P_\mu + 2i(\gamma^5)_{\alpha\beta}Z, \]  

(8)

where \( \bar{Q}_\beta = Q_\alpha(\gamma^0)_{\alpha\beta} \) and \( Z \) is the central charge,

\[ Z = \int dz \partial_z \phi_i \frac{\partial W}{\partial \phi_i} = W(z \to \infty) - W(z \to -\infty). \]  

(9)

Note also that the superalgebra (8), although it was derived in a particular model, is in fact the most general \( \mathcal{N}=1 \) superalgebra.

As was shown in [2] the algebra (8) is preserved at the quantum level. The quantum corrections lead to the anomaly in the central charge \( Z \), namely,

\[ W \to W + \frac{1}{4\pi} \frac{\partial^2 W}{\partial \phi_i \partial \phi_i}. \]  

(10)

This replacement must be also done in the supercurrent \( J^\mu \) and in the energy-momentum tensor \( \theta^{\mu\nu} \).

The most simple \( \mathcal{N}=1 \) Ginzburg-Landau model operates with one superfield, we will consider it in Sec. 4. In this case there is no conserved fermion number current — no currents can be constructed from one Majorana field \( \psi \). With two superfields \( \{\phi_1, \psi_1\}, \{\phi_2, \psi_2\} \) the model (3) can possess a larger \( \mathcal{N}=2 \) supersymmetry. It happens provided that

\[ \frac{\partial^2 W}{\partial \phi_1 \partial \phi_1} + \frac{\partial^2 W}{\partial \phi_2 \partial \phi_2} = 0, \]  

(11)

i.e. when \( W(\phi_1, \phi_2) \) is a harmonic function. It means that \( W \) is a real part of a holomorphic function of \( \phi_1 + i\phi_2 \). The \( \mathcal{N}=2 \) model can be viewed as dimensional reduction of the four-dimensional Wess-Zumino model.

## 3 Representations of superalgebra

To construct representations of the algebra (8) let us pass to the rest frame where \( P_\mu = (M, 0) \) and the algebra takes the form

\[ Q_1^2 = M + Z, \quad Q_2^2 = M - Z, \quad \{Q_1, Q_2\} = 0, \]  

(12)

where \( M \) and \( Z \) can be treated as \( c \)-numbers. Positive definiteness leads to

\[ M^2 \geq Z^2. \]

If \( M^2 \neq Z^2 \) it is the Clifford algebra with two generators and its irreducible representation is two-dimensional. For instance, one can choose

\[ Q_1 = \sigma_1 \sqrt{M + Z}, \quad Q_2 = \sigma_2 \sqrt{M - Z}. \]  

(13)
When $M^2 = Z^2$ we deal with a special case of the BPS multiplets. By definition we choose the topological charge $Z$ to be positive for soliton (negative for antisoliton). Then for the BPS soliton $M = Z$, and the supercharge $Q_2$ is trivial, $Q_2 = 0$. Thus, we are left with a single supercharge $Q_1$ realized nontrivially, and the algebra reduces to a single relation

$$Q_1^2 = 2Z.$$  \hfill (14)

The irreducible representations of this algebra are one-dimensional \cite{2}, i.e.

$$Q_1 |\text{sol} \rangle = \sqrt{2Z} |\text{sol} \rangle.$$  \hfill (15)

A natural question to be addressed is the uniqueness of this representation. Indeed, it is clear that there is a sign ambiguity in the representation of $Q_1$,

$$Q_1 = \pm \sqrt{2Z}.$$  \hfill (16)

It is clear that these two representations are unitary nonequivalent. Is this sign ambiguity physically observable? In particular, do we deal here with two distinct soliton species, or with one? Let us argue, that this sign ambiguity is physically unobservable and we deal in fact with one species.

The change of sign of $Q_1$ can be related to the change of sign of all fermion fields,

$$\psi \rightarrow - \psi,$$  \hfill (17)

which is an invariance of the theory. The $Z_2$ symmetry generated by Eq. (17) is evidently present in any theory with fermions. It is just this $Z_2$ symmetry that led people to conclude that the total number of the BPS states is two, i.e. the same as for the non-BPS multiplets.

All possible observables are represented by operators even in $\psi$ fields. This is equivalent to the superselection rule known from the time of introduction of fermions, it could be traced back to the double-valuedness of the spinor representation of the rotational group in three dimensions. Certainly, in one spatial dimension there is no rotational group and the superselection rule for physical operators should be imposed externally. Once it is imposed, two unitary nonequivalent representation of $Q_1$ in Eq. (16) become equivalent physically.

This remark concludes our proof that in the case at hand the BPS supermultiplet is one-dimensional. It is instructive to examine how this $Z_2$ symmetry is implemented in the non-BPS (two-dimensional) representation (13). In this case the change of signs of $Q_1$ and $Q_2$ leads to the unitary equivalent representations, so there is no need in the superselection rule.

The unitary rotations (i.e. the unitary equivalence transformations) which have been just mentioned in the previous paragraph are generated by supersymmetry itself. Namely, let us consider the bosonic generator

$$R = \frac{i}{2} \sqrt{M^2 - Z^2} = \frac{1}{2} \sqrt{P_\mu P^\mu - Z^2}.$$  \hfill (18)
The expression after the second equality sign is not bound to the rest frame. This generator has the following features:

\[ R^2 = 1, \quad \{ R, Q_\alpha \} = 0, \]

\[ [R, Q_\alpha] = -\frac{2}{\sqrt{P_\mu P^\mu - Z^2}} (\gamma^\mu P_\mu + i\gamma^5 Z)_{\alpha\beta} Q_\beta \]

\[ [R, P_\mu] = 0, \quad [R, Z] = 0. \quad (19) \]

For a finite SO(2) rotation generated by \( R \) we get

\[ e^{i\alpha R/2} Q e^{-i\alpha R/2} = \left[ \cos \alpha - i \sin \alpha \frac{\gamma^\mu P_\mu + i\gamma^5 Z}{\sqrt{P_\mu P^\mu - Z^2}} \right] Q. \quad (20) \]

Note an analogy between the introduction of the operator \( R \) above, in constructing representations of the superalgebra, and the introduction of the Pauli-Lubanski spin operator \( S^\mu = \epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_\sigma / (2\sqrt{P_\sigma P^\sigma}) \) for the Poincaré group. In this case there is no local current too, and the Pauli-Lubanski operator is not defined for massless particles, \( P_\sigma P^\sigma = 0 \).

The eigenvalues of the operator \( R \) are \( \pm 1 \) due to the fact that \( R^2 = 1 \). Shifting \( R \) by a constant one can introduce a kind of fermionic charge \( F \)

\[ F = \frac{1 + R}{2}, \quad (21) \]

which has eigenvalues 0 and 1. Indeed, the operator \( F \) acts as a projection operator, \( F^2 = F \). It measures the number of fermions modulo two. E.g., the elementary excitations of the fields \( \phi \) and \( \psi \) in the Lagrangian (3) have \( F = 0 \) and 1, respectively.

We mentioned earlier, that there is no local current associated with this symmetry in the framework of the minimal \( \mathcal{N}=1 \) model with one superfield. Such current does exists in the case of extended \( \mathcal{N}=2 \) supersymmetry, where fermion number has a local definition. The two fermion charges are different, generally speaking. It is known that the fermion charge defined by the local current is noninteger for solitons [5] (the fractional fermion charge of the soliton was discovered by Jackiw and Rebbi [8]). At the same time the fermion charge (21) is always integer. In the topologically trivial one-particle sector of \( \mathcal{N}=2 \) theories both fermion charges coincide.

Let us emphasize once more that \( R \) is not defined for the BPS states which is clearly visible from the singularity \( 1/\sqrt{P_\mu P^\mu - Z^2} \) in its definition (18).

4 One-superfield model, quantization in the box

In this section we consider the \( \mathcal{N}=1 \) Ginzburg-Landau model (3) with one superfield \( \{ \phi, \psi \} \), following the treatment developed in Ref. [2]. Although the superpotential
$\mathcal{W}(\phi)$ can be arbitrary, for illustrative purposes we will sometimes present explicit forms in the polynomial model (5).

The system is placed in a large spatial box, i.e. the boundary conditions at $z = \pm L/2$ are imposed. The conditions we choose are

$$[\partial_z \phi - \mathcal{W}'(\phi)]|_{z=\pm L/2} = 0, \quad \psi_1|_{z=\pm L/2} = 0,$$

$$[\partial_z - \mathcal{W}''(\phi)] \psi_2|_{z=\pm L/2} = 0,$$  \hspace{1cm} (22)

where $\psi_{1,2}$ denote the components of the spinor $\psi_\alpha$. The first line is nothing but a supergeneralization of the BPS equation, $D_1 \Phi(t, z = \pm L/2, \theta) = 0$ at the boundary. The second line is the consequence of the Dirac equation of motion, if $\psi$ satisfies the Dirac equation there is essentially no boundary conditions for $\psi_2$. Therefore, it is not an independent boundary condition in the solution of the classical equations of motion. We will use these boundary conditions later for the construction of modes in the differential operators of the second order.

The above choice is particularly convenient because it is compatible with the residual supersymmetry in the presence of the BPS soliton. The boundary conditions (22) are consistent with the classical solutions, both for the flat vacuum and for the kink. In particular, the soliton solution $\phi_0$ of Eq. (6) satisfies $\partial_z \phi - \mathcal{W}' = 0$ everywhere.

The next step is to introduce the expansion in modes for deviations from the soliton solution (6). For the mode expansion we use the second order Hermitean
differential operators $L_2$ and $\tilde{L}_2$,

$$L_2 = P\dagger P, \quad \tilde{L}_2 = PP\dagger,$$  \hspace{1cm} (23)

where

$$P = \partial_z - \mathcal{W}'|_{\phi=\phi_0(z)}, \quad P\dagger = -\partial_z - \mathcal{W}'|_{\phi=\phi_0(z)}.$$  \hspace{1cm} (24)

The operator $L_2$ defines the modes of $\chi \equiv \phi - \phi_0$, and those of the fermion field $\psi_2$, while $\tilde{L}_2$ does this job for $\psi_1$. The boundary conditions for $\psi_{1,2}$ are given in Eq. (22), for $\phi - \phi_0$ they follow from the expansion of the first condition in Eq. (22),

$$[\partial_z - \mathcal{W}''(\phi_0(z))] \chi|_{z=\pm L/2} = 0.$$  \hspace{1cm} (25)

It is easy to verify that there is only one zero mode $\chi_0(z)$ for the operator $L_2$ which has the form,

$$\chi_0 \propto \frac{d\phi_0}{dz} \propto \mathcal{W}'|_{\phi=\phi_0(z)} \propto \frac{1}{\cosh^2(mz/2)}.$$  \hspace{1cm} (26)

This is the zero mode for the boson field $\chi$ (translational mode) and for fermion $\psi_2$ (supersymmetric mode).
The operator $\tilde{L}_2$ has no zero modes at all. Let us emphasize that the absence of the zero modes for $\tilde{L}_2$ is not because the solution of $\tilde{L}_2 \tilde{\chi} = 0$ is non-normalizable (we keep the size of the box finite) but because of the boundary conditions $\tilde{\chi}(z = \pm L/2) = 0$.

The translational and supersymmetric zero modes discussed above imply that the soliton is described by two collective coordinates: its center $z_0$ and a “fermionic” center $\eta$,

$$\phi = \phi_0(z - z_0) + \text{nonzero modes}, \quad \psi_2 = \eta \chi_0 + \text{nonzero modes},$$

where $\chi_0$ is the normalized mode given by Eq. (26). The nonzero modes are those of the operator $L_2$. As for $\psi_1$ it is given by the sum over the nonzero modes of the operator $\tilde{L}_2$.

Substituting the mode expansion in the supercharges (7) we arrive at

$$Q_1 = 2\sqrt{Z} \eta + \text{nonzero modes}, \quad Q_2 = \sqrt{Z} \dot{z}_0 \eta + \text{nonzero modes}.$$  

Now we can proceed to the quasiclassical quantization. Projecting the canonic equal-time commutation relations for the fields $\phi$ and $\psi$ on the zero modes we get

$$[p, z_0] = -i, \quad \eta^2 = \frac{1}{2},$$

where $p = Z \dot{z}_0$ is the canonical momentum conjugated to $z_0$. It means that in quantum dynamics of the soliton moduli $z_0$ and $\eta$ the operators $p$ and $\eta$ can be realized as

$$p = -i \frac{d}{d z_0}, \quad \eta = \frac{1}{\sqrt{2}}.$$  

It is clear that we could have chosen $\eta = -1/\sqrt{2}$. This is the same unobservable ambiguity that was discussed in Sec. 3, the supercharge $Q_1$ is linear in $\eta$.

Thus, the supercharges depend only on the canonic momentum $p$,

$$Q_1 = \sqrt{2Z}, \quad Q_2 = \frac{p}{\sqrt{2Z}}.$$  

In the rest frame in which we perform our consideration $\{Q_1, Q_2\} = 0$, and the only value of $p$ consistent with it is $p = 0$. Thus, for the soliton $Q_1 = \sqrt{2Z}, \quad Q_2 = 0$ in full agreement with the general construction discussed in Sec. 3.

Note that the representation (31) can be used at nonzero $p$ as well. It reproduces the superalgebra (8) in the nonrelativistic limit, with $p$ having the meaning of the total spatial momentum $P_1$.

In passing from Eq. (28) to (31) we have omitted the nonzero modes. For each given nonzero eigenvalue there is one bosonic eigenfunction (in the operator $L_2$), the same eigenfunction in $\psi_2$ and one eigenfunction in $\psi_1$ (of the operator $\tilde{L}_2$). The quantization of the nonzero modes is quite standard. The corresponding additional terms in $Q_{1,2}$ can be easily written in term of the creation and annihilation operators.
They describe excitations of the BPS solitons. These excitations form long (two-dimensional) multiplets. Both supercharges do not vanish and one can introduce the fermion number (18), (21).

The multiplet shortening guarantees that the equality $M = Z$ is not corrected. For the exactly solvable $\mathcal{N}=1$ models [7, 5], such as that with the superpotential $\mathcal{W} = mv^2 \sin(\phi/v)$, the soliton mass is known exactly. In Ref. [2] it was explicitly checked that $M$ is equal to the matrix element of $Z$ (see Eq. (9) with the account for the anomaly (10)) up to two loops. Moreover, it was seen that the coupling constant expansion has a finite radius of convergence (no essential singularity at small coupling).

5 Models with two soliton states

In this section we consider modifications such that the number of the fermion zero modes is even, which leads to two rather than one soliton states. In the leading approximation we deal with two degenerate one-dimensional supermultiplets. A special feature of these cases is that there is no obstacle to forming a long multiplet with the mass larger than the central charge $Z$. That’s exactly what happens. In typical cases this can be seen already at one loop in the supercharge (two loops in the soliton mass). In one example to be considered below $Q_2$ remains zero at one loop, and we conjecture that $M - Z > 0$ may occur at the nonperturbative level. The analog of such phenomenon was demonstrated previously in an $\mathcal{N}=2$ model [9].

5.1 Minimal model on the circle

In the model with Lagrangian (3) with one superfield let us assume that the field $\phi$ lives on the circle of circumference $2\pi v$. This implies that $\mathcal{W}'(\phi)$ is periodic, with the period $2\pi v$. Moreover, we assume that the spatial coordinate $z$ is also compact and defined on a circle, i.e. the points $z$ and $z + L$ are identified. As was shown in [10], the BPS saturated solitons are possible provided the superpotential $\mathcal{W}$ is a multivalued function such that $\mathcal{W}'$ is single-valued. Let us take, for instance

$$\mathcal{W}(\phi) = c\phi + w(\phi), \quad \mathcal{W}'(\phi) = c + w'(\phi),$$

(32)

where $w(\phi)$ is a $2\pi v$ periodic function and $c$ is an appropriately chosen numerical coefficient. The central charge will be equal to $2\pi vc$. As an example one can have in mind $w = mv^2 \sin(\phi/v)$.

The BPS equation

$$\frac{d\phi}{dz} = \mathcal{W}'(\phi)$$

(33)

has an implicit solution

$$\int_{\phi(0)}^{\phi(z)} \frac{d\phi}{\mathcal{W}'(\phi)} = z.$$  

(34)
The function $W'(\phi)$ must be positive everywhere on the target space circle. We choose the value of $\phi(0)$ to have $W'(\phi(0)) = \text{Max}\{W'\}$, it puts the center of the soliton at $z = 0$.

The condition of periodicity

$$\int_0^{2\pi v} \frac{d\phi}{W'(\phi)} = L, \quad (35)$$

fixes the value of $c$, assuming that Eq. (35) has a solution, which is a generic situation. We denote the solution $\phi_0(z)$.

The mode expansion of $\phi - \phi_0$ and $\psi_{1,2}$ is performed in the eigenmodes of differential operators $L_2$ and $\tilde{L}_2$, in the same way it was done in the previous section. The only difference is in the boundary conditions. Now, instead of Eq. (22), we require periodicity. In noncompact space the operator $L_2$ had a zero mode while $\tilde{L}_2$ had no zero mode. Now, in the compact space, both have zero modes, we denote them as $\chi_0$ for $L_2$ and $\tilde{\chi}_0$ for $\tilde{L}_2$,

$$\chi_0 \propto \exp\left\{ \int_0^z W''(\phi_0(z)) dz \right\} \propto \frac{d\phi_0}{dz} \propto W'(\phi_0),$$

$$\tilde{\chi}_0 \propto \exp\left\{ - \int_0^z W''(\phi_0(z)) dz \right\} \propto \frac{1}{W'(\phi_0)}. \quad (36)$$

Note that while the zero mode $\chi_0$ (in $\phi$ and $\psi_2$ fields) is localized on the kink, the mode $\tilde{\chi}_0$, i.e. that of $\psi_1$ is localized off the kink. The zero mode balance is the same as for nonzero modes: we have one bosonic mode and two fermionic.

Retaining only the zero modes we have the following expansion for the bosonic and fermionic fields:

$$\phi(z) = \phi_0(z - z_0), \quad \psi_1 = \xi \tilde{\chi}_0, \quad \psi_2 = \eta \chi_0, \quad (37)$$

where $\xi$ and $\eta$ are the fermion collective coordinates. This leads to exactly the same supercharges as in Eq. (28). The difference lies in the quantization relations,

$$[p, z_0] = -i, \quad \eta^2 = \xi^2 = \frac{1}{2}, \quad \{\eta, \xi\} = 0. \quad (38)$$

Due to $\{\eta, \xi\} = 0$ the representation now is two-dimensional.

In the leading approximation above both soliton states are BPS since $Q_2 = 0$. However, shortly we will show that already at the one-loop level the supercharge $Q_2$ does not vanish. Thus, the long (two-dimensional) multiplet is formed. The states are non-BPS, their mass exceeds the central charge by a two-loop correction.

The easiest way to demonstrate the phenomenon is the explicit calculation of $Q_2$ with account of the anomaly (10),

$$Q_2 = \xi \int dz \left[ \partial_z \phi_0 - W'(\phi_0) - \frac{W'''(\phi_0)}{4\pi} \right] \tilde{\chi}_0(z), \quad (39)$$
where we substituted the classical soliton solution and the zero mode for \( \psi_2 \) in the definition (7). The zero mode of \( \psi_1 \) drops out from \( Q_2 \) at \( p = Z \dot{z}_0 = 0 \). The term \( W'''(\phi_0)/4\pi \) is due to the anomaly. On the classical solution the first two terms in the square brackets cancel each other, only the anomalous term survives. Thus, we see that \( Q_2 \neq 0 \),

\[
Q_2 = -\frac{1}{4\pi} \xi \left[ \int \frac{d\phi}{(W')^3} \right]^{-1/2} \int d\phi \frac{W'''}{(W')^2}, \tag{40}
\]

where we used expression (36) for the zero mode \( \tilde{\chi}_0 \). It means that the excess of the soliton mass over the central charge is

\[
M - Z = Q_2^2 = \frac{1}{32\pi^2} \left[ \int \frac{d\phi}{(W')^3} \right]^{-1} \left[ \int d\phi \frac{W'''}{(W')^2} \right]^2. \tag{41}
\]

Note that taking account of the anomaly in the model of Sect. 4 (in the box) does not lead to nonvanishing \( Q_2 \) because of the absence of the zero mode in \( \psi_2 \). Its effect on \( Q_1 \),

\[
\Delta Q_1 = \frac{1}{\sqrt{2}} \int dz \left[ \frac{W'''(\phi_0)}{4\pi} \right] \chi_0(z) = \frac{1}{\sqrt{2Z}} \frac{1}{4\pi} \left[ W'(z \to \infty) - W'(z \to -\infty) \right], \tag{42}
\]

amounts to the shift in the classical value of \( Z \) (see Eq. (9)) caused by the anomaly by virtue of the substitution (10).

### 5.2 Two-superfield model

Let us return to noncompact space and design an \( \mathcal{N} = 1 \) model similar to that of Sec. 5.1 – two fermion zero modes on the kink, rather than one. To this end, we start from extended supersymmetry, \( \mathcal{N} = 2 \), and then break it down to \( \mathcal{N} = 1 \).

As it was mentioned in Sec. 2 the Lagrangian (3) with two real superfields \( \{ \phi_i, \psi_i \} \) \((i = 1, 2)\) has \( \mathcal{N} = 2 \) supersymmetry if the superpotential \( \mathcal{W}(\phi_1, \phi_2) \) is a harmonic function,

\[
\Delta_\phi \mathcal{W} = \frac{\partial^2 \mathcal{W}}{\partial \phi_i \partial \phi_i} = 0 \quad \text{for } \mathcal{N} = 2. \tag{43}
\]

It means, in particular, the absence of the anomaly in the central charge – the superpotential is not changed by radiative corrections. The \( \mathcal{N} = 2 \) supersymmetry makes the model finite, while in \( \mathcal{N} = 1 \) it was superrenormalizable. A polynomial example of a harmonic superpotential is

\[
\mathcal{W}(\phi_1, \phi_2) = \frac{m^2}{4\lambda} \phi_1 - \frac{\lambda}{3} \phi_1^3 + \lambda \phi_1 \phi_2^2 \tag{44}
\]

How one can introduce breaking of \( \mathcal{N} = 2 \)? To this end, consider a more general case of nonharmonic \( \mathcal{W}(\phi_1, \phi_2) \),

\[
\mathcal{W}(\phi_1, \phi_2) = \frac{m^2}{4\lambda} \phi_1 - \frac{\lambda}{3} \phi_1^3 + \lambda \phi_1 \phi_2^2 + \frac{pm}{2} \phi_2^2 + \frac{q\lambda}{3} \phi_2^3, \tag{45}
\]
where $p$ and $q$ are dimensionless parameters. For $p, q \neq 0$, the extended $\mathcal{N} = 2$ supersymmetry is explicitly broken down to $\mathcal{N} = 1$. The parameter $p$ introduces soft breaking of $\mathcal{N} = 2$ which preserves finiteness of the theory and the absence of the anomaly. The nonvanishing $q$ breaks the finiteness (the theory stays superrenormalizable, however) and introduces the anomaly, $\Delta_\phi \mathcal{W} = 2q \lambda \phi_2$.

The classical solution for the kink is the same as in the $\mathcal{N} = 1$ model with one superfield considered in Sec. 2. The second field $\phi_2$ vanishes,

$$\langle \phi_1 \rangle_{\text{sol}} = \phi_0(z) = \frac{m}{2\lambda} \tanh \frac{mz}{2}, \quad \langle \phi_2 \rangle_{\text{sol}} = 0. \quad (46)$$

The mode expansion is again based on operators $L_2 = P^i P$, $\tilde{L}_2 = P P^i$ where operator

$$P_{ij} = \partial_z \delta_{ij} - \frac{\partial^2 \mathcal{W}(\phi = \phi_{\text{sol}})}{\partial \phi_i \partial \phi_j} \quad (47)$$

now has a matrix form. The matrix is diagonal in our case,

$$P_{ij} = \begin{pmatrix} \partial_z - 2\lambda \phi_0(z) & 0 \\ 0 & \partial_z + 2\lambda \phi_0(z) + pm \end{pmatrix}. \quad (48)$$

The zero modes for the fields $\phi_1, \psi_1$ are the same as in Sec. 4. A new zero mode appears in the field $\psi_2$,

$$\langle \psi_2 \rangle_{\text{zero mode}} = \xi N \frac{\exp(-pmz)}{\cosh^2(mz/2)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (49)$$

where $\xi$ is the operator coefficient, $N$ is the normalization factor. At $p = 0$ it has the same functional form as the old fermionic mode in $\psi_1$. This is not surprising because of $\mathcal{N} = 2$ supersymmetry at $p = 0$. What is crucial is that the zero mode (49) is not lifted even at nonvanishing $p$. This feature is due to the Jackiw-Rebbi theorem [8].

One boson and two fermion zero modes mean that we have two soliton states which are the BPS states in the leading approximation. In this approximation $Q_2 = 0$. Let us show that the one-loop anomaly makes $Q_2 \neq 0$. The anomalous part in $Q_2$ is

$$Q_2 = -\frac{1}{4\pi} \int dz \left[ \frac{\partial}{\partial \phi_2} \Delta_\phi \mathcal{W} \right] \langle \psi_2 \rangle_{\text{zero mode}}$$

$$= -\xi \frac{q\lambda N}{2\pi} \int dz \frac{\exp(-pmz)}{\cosh^2(mz/2)} = -\xi \frac{q\lambda}{\pi m} \sqrt{\frac{3mp}{2(1-4p^2) \tan \pi p}}, \quad (50)$$

\[2\] The term $pm \phi_2^2/2$ leads to a constant in $\Delta_\phi \mathcal{W}$ which shifts the superpotential by an unobservable constant.
Correspondingly, the shift of the soliton mass from $\mathcal{Z}$ is

$$M - \mathcal{Z} = Q_2^2 = \frac{3q^2\lambda^3}{4\pi^2m} \frac{p\pi}{(1 - 4p^2) \tan p\pi}. \quad (51)$$

Let us summarize the situation. At $p, q = 0$ we deal with the two-dimensional $\mathcal{N}=2$ multiplet (short for $\mathcal{N}=2$). Introducing $q \neq 0$ one does not lift the degeneracy but breaks the BPS saturation. Now one deals with a non-BPS multiplet (long for $\mathcal{N}=1$). As for the parameter $p$, its value in Eq. (51) can be arbitrary in the interval $|p| < 1$. Note the absence of singularity at $p = 1/2$. The singularity at $|p| = 1$ reflects a phase transition: the mode (49) becomes non-normalizable and one of two soliton states disappears from the physical spectrum, the soliton multiplet becomes one-dimensional, i.e. BPS. This transition was discussed in detail in [4].

Now let us discuss a special case $q = 0$, $0 < |p| < 1$. In this case there is no anomaly and $Q_2$ remains zero at one loop. We did not analyze higher loops, our suspicion is that a nonvanishing $Q_2$ is generated by nonperturbative effects. If it is the case $M - \mathcal{Z} \propto \exp(-c/\lambda)$.

6 General statements

Having considered a representative set of examples we now are in position to formulate some general assertions referring to the construction of BPS multiplets in 1+1 with minimal ($\mathcal{N}=1$) supersymmetry.

(i) Assume that in the theory under consideration a physical $R$ symmetry exists (at the quantum level) such that $R^2 = 1$, $[R, H] = 0$, and $\{R, Q_\alpha\} = 0$ ($R$ may be a part of a larger invariance of the theory). Then all representations are even-dimensional. One-dimensional representations do exist only provided that such $R$ symmetry is not implemented.

(ii) The existence/nonexistence of such symmetry can be traced back to the index $\nu$ of the Dirac operator in the soliton background, i.e. to the index of

$$P_{ij} = \partial_z \delta_{ij} - \frac{\partial^2 \mathcal{W}(\phi = \phi_{\text{sol}})}{\partial \phi_i \partial \phi_j}.$$ 

The index $\nu$ is defined as the difference between the number of normalizable zero modes of the operators $L_2 = P^\dagger P$ and $\bar{L}_2 = PP^\dagger$. If the index vanishes (modulo 2) then the $R$ symmetry with the above properties is operative, and short (one-dimensional) representations are absent. In particular, this index is always zero in the case of the compact spatial dimension (i.e. the theory on circle), for the arbitrary choice of the superpotential $\mathcal{W}$. This follows from the Atiyah-Singer theorem. Correspondingly, $\mathcal{N}=1$ theories in 1+1 with the compact spatial dimension have no one-dimensional representations. The BPS-saturated solitons are not expected to survive at the quantum level, even though they may exist at the classical level.
Note that the index \( \nu = 0 \) in the topologically trivial sector, only in the soliton sector \( \nu \) can be nonvanishing.

(iii) For the noncompact spatial dimension (cf. Sec. 4) the index of the operator \( P \) in the soliton background was found by Witten [11], in a general form, for the arbitrary number of fields \( \phi_k \) involved in the problem. Witten’s construction can be summarized as follows. At first one must identify the initial \((i)\) and final \((f)\) vacua between which the soliton in question interpolates (the vacua are the solutions of the equations \( \partial W / \partial \phi_k = 0 \) for all \( k = 1, \ldots, n \)). Then one calculates the matrices of the second derivatives (the fermion mass matrix)

\[
\frac{\partial^2 W}{\partial \phi_k \partial \phi_l}
\]

at both critical points and diagonalizes them. The index \( \nu \) is the difference between the numbers of the negative eigenvalues in the initial and final vacua,

\[ \nu = n_i - n_f. \]

This result immediately shows that \( \nu = 0 \) in the \( \mathcal{N}=2 \) case and its small deformations. Indeed, due to the harmonicity of \( W \) in this case, \( \Delta \phi W = 0 \), which leads to one negative eigenvalue in the matrix of the second derivatives in each vacua (per pair of fields related by \( \mathcal{N}=2 \)).

If the index \( \nu = 0 \) modulo 2 the \( R \) symmetry is operative, all representations are even-dimensional. This is the case in the model of Sec. 5.2 at \( |p| < 1 \). The model considered in Sec. 4 has \( \nu = 1 \); correspondingly, there is no physical \( R \) symmetry, and in this model the irreducible representation is one-dimensional.

Note that at \( |p| = 1 \) one of the eigenvalues of the fermion mass matrix vanishes in the model Sec. 5.2 [2], while at \( |p| > 1 \) two new vacua appear and, simultaneously, the above index becomes unity: \( \nu \) jumps from 0 to 1 at \( |p| = 1 \).

This example shows a subtlety in the definition of the index. It refers to normalizable modes and can jump in the case of the phase transition (at \( |p| = 1 \)). On the other hand, we can define an index in the finite box, using our boundary conditions. Then normalizibility of the zero modes does not play a role. With such a definition, the index is preserved even in the case of the phase transition. However, after the phase transition, some zero modes are localized at the boundaries of the box rather than on the soliton. Then, the transition from localization to delocalization occurring at \( |p| = 1 \) replaces the jump in the traditionally defined index.

7 Conclusions

We analyzed models with minimal \( \mathcal{N}=1 \) supersymmetry and central charges in \((1+1)\) dimensions. For the BPS states only one out of two supercharges is realized nontrivially which leads to one-dimensional irreducible representations of the superalgebra. The non-BPS supermultiplets are two-dimensional. Our main topic was
the soliton multiplet structure in various models at weak coupling. We presented
models of two types: in some of these models at the classical level we deal with one
BPS state, in others with two. At the quantum level the fate of these solitons is
different. In the first case they remain BPS, the multiplet shortening prevents them
from leaving the BPS bound. In the second case, generally speaking, the quantum
corrections pair the two states making them non-BPS. We have explicitly demon-
strated this phenomenon in certain models by calculating the classically vanishing
supercharge $Q_2$ at one loop and observing that at one loop $Q_2 \neq 0$. This leads to
$M = Z \neq 0$ at two loops.

The demonstration was based on the central charge anomaly. In special cases,
where this anomaly is absent, $Q_2$ is still zero at one loop, and we conjecture that it
may be generated nonperturbatively.

We observe that although the theories under consideration have no conserved
fermion current, supersymmetry itself allows us to introduce a conserved fermion
charge which distinguishes between two states in the two-dimensional multiplet (i.e.
fermion number modulo 2). It is related to an $R$ symmetry which is operative in
this case. This fermion charge is not defined for the short multiplets.

Our results naturally “blend in” into a general picture. The BPS saturation was
studied in detail in the $\mathcal{N}=2$ theories in (1+1) dimensions [12] and in the $\mathcal{N}=2$,
$\mathcal{N}=4$ theories in (3+1)-dimensions. In all these cases, when dealing with a single
short multiplet, its BPS nature is protected by shortening. When at the classical
level the theory contains enough short multiplets to form a long one, generically, it
does happen at the quantum level.

In a broader context, we found another example of a remarkable phenomenon
first discussed by Witten [13] – supersymmetry without the full fermion-boson de-
genereacy. In the vacuum, supersymmetry is operative ensuring that the vacuum
energy density vanishes. At the same time, for some one-particle states there is no
doubling – the state with the given mass is unique. If such theories could be found
in four dimensions, this would be “a dream came true.” In Witten’s example the
action of supercharges on certain one-particle states was ill-defined. In our case the
fermion charge for certain one-particle states is ill-defined.

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