toward the black hole.

decayed to radial drift by the closed loops due to radial acceleration of their plasma.

If the accretion disk is strong enough, corotation would occur on the magnetic surface. Then, it is
consistent with the magnetic connection between the black hole and the disk breaks down.

ABSTRACT

driving inward and outward winds

black hole magnetospheres around thin disks

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1. INTRODUCTION

It is widely believed that a supermassive black hole surrounded by an accretion disk can work as a central engine of active galactic nuclei. If magnetic field exists around the black hole and the disk, it is important to understand the magnetospheric structure for explaining the highly energetic phenomena. The magnetohydrodynamical (MHD) approach in general relativity may be necessary to study various features of the black hole magnetospheres. Unfortunately, even if the magnetosphere is assumed to be stationary and axisymmetric, any global models describing both inward winds in the strong-gravity region near black hole and outward winds in the far distant region have not been constructed, in spite of a large number of works devoted to the MHD problems. This is mainly because it is very difficult to analyze the highly nonlinear Grad-Shafranov equation for the magnetic stream function which defines the poloidal field lines (see e.g., Beskin (1997) for review).

To avoid the mathematical difficulty in the full MHD system, one may consider vacuum solutions of the magnetic stream function as a practical approximation applicable to the magnetically dominated models. In particular, the well-known examples such as split-monopole, uniform and paraboloidal fields in Schwarzschild geometry (Michel 1973; Wald 1974; Blandford & Znajek 1977) have been used as the zeroth-order approximation for the perturbation method in slow-rotation and force-free fields, from which the first-order perturbation has been derived for giving a weak toroidal magnetic field and discussing the Blandford-Znajek process (Blandford & Znajek 1977). Another important MHD problem to be studied is fluid motion along poloidal field lines. For examples, the existence of accretion flows onto the black hole passing through some MHD critical points has been shown under a fixed shape of field lines (Takahashi et al. 1990). The vacuum solutions allowing a field configuration of astrophysical interest will become a useful tool in an attempt to discuss the trans-critical motion of outgoing and ingoing magnetized winds, except in the region where the effect of fluid inertia should dominate to change crucially the field structure.

Of course, even for the vacuum fields, some current distribution should be assumed to exist in a restricted region of the magnetosphere. In this paper we focus on the magnetic field generated by a toroidal current distribution in a thin disk around the Schwarzschild black hole. The source of the above-mentioned fields with split-monopole and paraboloidal structures can be also disk currents. However, such models contain only open field lines and fail to describe a magnetic connection of the disk with the black hole. This will be partially due to the absence of the disk inner edge separated from the event horizon. The magnetic connection by closed field lines is expected to exist and have astrophysically important effects (Punsly 1991a; Gruzinov 1999; Blandford 2000; Li 2000c; Krolik 2000). Hence, our purpose here is to present explicitly a global magnetospheric model with both closed and open field lines threading a thin disk and connecting magnetically the inner and outer parts, respectively, with the black hole and the far distant region, which is relevant to the problems of inward and outward disk-driven winds. Though the vacuum model would miss some important features of more realistic magnetospheres, our modeling could be a preliminary step for understanding the various MHD processes in further detail.
To obtain general models based on a disk current distribution, it may be convenient to consider linear superposition of the magnetic field generated by a single ring current located at a fixed radius on the equatorial plane which has been found in Kerr geometry (Petterson 1975). Assuming the current distribution from an inner edge to an outer one, Li (2000a) has derived a formal expression of the magnetic stream function in terms of the infinite sum of the multipole fields, without revealing the global shape of the field lines. Because the each multipole field of which the structure has been discussed by Ghosh (2000) still fails to give closed field lines regular at the event horizon, the infinite sum is essential to understanding the global shape.

In §2 we would like to develop a different approach to arrive at the infinite sum formula for the magnetic stream function. In §3 we succeed in reducing it to an analytical simple form by locating the disk inner edge at a radius much larger than the horizon radius and the outer edge at infinite radius. The global structure of closed and open field lines is shown, and it is found that only the very narrow region of the disk separated from the inner edge can be magnetically connected with the black hole, together with the result that the field lines become nearly cylindrical at the event horizon and conical at infinity. In §4 we also consider a modification of the global shape due to an addition of a uniform external field. Though collimated field lines appear at the far distant region, it is found that for a strong uniform field the magnetic connection between the black hole and the disk breaks down. Finally we discuss the MHD effect due to slow rotation of the magnetosphere (namely, angular momentum transfer by magnetically dominated inward winds from the disk to the black hole), which could lead to radial accretion of disk plasma and gradual disruption of the closed field lines (Punsly & Coroniti 1989). In the following we use the geometrical units with $G = c = 1$.

2. THE MAGNETIC STREAM FUNCTION

Let us consider a stationary, axisymmetric magnetic field in Schwarzschild background of the metric

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) ,$$

where $M$ is the mass of the black hole. The poloidal components of the magnetic field may be given by

$$\vec{B}_p = \frac{\vec{\nabla} \Psi \times \vec{\phi}}{2\pi r \sin \theta} ,$$

where $\vec{\phi}$ is a unit vector in the azimuthal direction, and $\Psi(r, \theta)$ is the so-called magnetic stream function defining the poloidal field lines by $\Psi = \text{constant}$. If the vacuum field is assumed, the Maxwell equations reduce to the simple form

$$r^2 \frac{\partial}{\partial r} \left[ \left(1 - \frac{2M}{r}\right) \frac{\partial \Psi}{\partial r} \right] + \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right) = 0 ,$$

which allows the separable solutions

$$\Psi = R_p(r) \Theta_p(\theta) .$$
with the separation constant $\nu$

For the analysis of these functions it is convenient to introduce the variables

$$w = \frac{r}{M} - 1, \quad x = \cos \theta.$$  \hspace{1cm} (5)

Then, from the regularity at the polar axis $\theta = 0, \pi$ we obtain the angular part written by

$$\Theta_{\nu} = \int_{|x|}^{1} P_{\nu}(x') dx',$$  \hspace{1cm} (6)

where $P_{\nu}$ is the Legendre function of the first kind. Though in this paper we focus on the field with equatorial symmetry, the parameter $\nu$ is not limited to a positive odd integer for allowing a discontinuous change of the radial component of $\vec{B}_p$ on the upper and lower surfaces of the equatorial disk.

To determine the radial part $R_{\nu}$, the condition for regularity at the event horizon $r = 2M$ and at infinity $r \to \infty$ becomes important. We use the Legendre function $P_{\nu}(w)$ to give the solution regular at $w = 1$ of the form

$$R_{\nu} = (w + 1)P_{\nu}(w) - \int_{1}^{w} P_{\nu}(w')dw',$$  \hspace{1cm} (7)

where we assume to be $\nu = (-1/2) + ik$ with arbitrary positive $k$, because the oscillatory real function $P_{\nu}(w)$ has the amplitude decreasing in proportion to $1/\sqrt{w}$ at large $w$, and the linear superposition

$$\Psi(r, \theta) = \int_{0}^{\infty} f(k)R_{\nu}(r)\Theta_{\nu}(\theta)dk$$  \hspace{1cm} (8)

will be able to describe a global magnetic field satisfying the boundary conditions.

Now we calculate the coefficient $f(k)$ in the linear superposition corresponding to the case of a disk current with the inner edge at $w = w_0 > 1$. Recall that in previous works the solution was written by an infinite sum of the multipole fields with $\nu$ of natural numbers. Then, in the distant region $w > w_0$ the Legendre function $P_{\nu}(w)$ should be replaced by $Q_{\nu}(w)$ with a change of the coefficient due to the condition for continuity of $\Psi(r, \theta)$ at $w = w_0$. We can avoid these slightly complicated steps by using the integral form (8). Though the toroidal surface current in the thin disk is proportional to $r^{-2}(\partial \Psi/\partial \theta)$ at $\theta = \pi/2$, here we rather focus on the equality

$$\frac{\partial}{\partial r} \left( \frac{\partial \Psi}{\partial \theta} \right) = \frac{w + 1}{M} \frac{d}{dw} h(w),$$  \hspace{1cm} (9)

which is evaluated at the upper surface of the equatorial plane. The remarkable point is that the function $h(w)$ is given by

$$h(w) = \int_{0}^{\infty} g(k)P_{\nu}(w)dk,$$  \hspace{1cm} (10)

where we have

$$g(k) = P_{\nu}(0)f(k),$$  \hspace{1cm} (11)
and the inversion formula (Erdélyi et al. 1953) holds as follows,
\[ g(k) = k \tanh(\pi k) \int_1^\infty P_s(w) h(w) dw . \] (12)

Then, if the function \( h(w) \) is specified according to the current distribution, we can derive the coefficient \( f(k) \) giving the magnetic stream function \( \Psi(r, \theta) \) valid in the whole region.

Further, to obtain the explicit form of \( f(k) \) through the well-known integral formula involving two Legendre functions, let us assume in the disk region \( w > w_0 \) to be
\[
h(w) = A_0 Q_s(w) . \] (13)

Though \( A_0 \) may be an arbitrary real parameter, we limit \( s \) to be positive for assuring that \( \partial \Psi / \partial \theta \) does not diverge at large \( w \), where the Legendre function \( Q_s(w) \) of the second kind falls off as \( w^{-s-1} \).

(If any different model of the current distribution becomes necessary, the linear superposition of \( Q_s \) with various \( s \) in the range \( 0 < s < \infty \) is a possible procedure to construct the solution \( \Psi \). However, we would like to emphasize that this simple example for the current distribution can be useful for revealing interesting features of the magnetospheric structure, as will be shown in the following.) Because in the inner region \( 1 \leq w < w_0 \), where the field lines should vertically thread the equatorial plane, we have
\[
h(w) = 0 , \] (14)
it is easy to arrive at the result
\[ g(k) = \frac{A_0 k \tanh(\pi k)}{k^2 + [s + (1/2)]^2} F_\nu , \] (15)
where
\[ F_\nu = (w_0^2 - 1) \left[ P_s(w_0) \frac{dQ_s(w_0)}{dw_0} - Q_s(w_0) \frac{dP_s(w_0)}{dw_0} \right] . \] (16)

Strictly speaking, even if the condition (14) is assumed, equation (9) only implies that \( \partial \Psi / \partial \theta \) is constant at the upper or lower surface of the equatorial plane in the inner range \( 1 \leq w < w_0 \). Hence, we must confirm the validity of equation (15), by calculating the value on the event horizon \( w = 1 \), which is given by
\[ \frac{\partial \Psi}{\partial \theta} = \int_0^\infty 2g(k) dk , \] (17)
at the upper surface of the equatorial plane. In fact, the right-hand side of equation (17) is shown to vanish by virtue of the equality
\[ Q_s(w_0) = \int_0^\infty \frac{k \tanh(\pi k)}{k^2 + [s + (1/2)]^2} P_s(w_0) dk \] (18)
which is a result of the inversion formula (12).

We have obtained the magnetic stream function of the integral form (8) in which the coefficient \( f(k) = g(k)/P_s(0) \) is given by equation (15). To calculate the integral with respect to \( k \) with the
application of the residue theorem, it will be useful to consider the integration contour along the real axis \(( -\infty < k < \infty )\) and the large semi-circle in the complex \(k\)-plane, based on the relation between two Legendre functions written by

\[
P_\nu(z) = \frac{i}{\pi} \cot(\pi k) [Q_\nu(z) - Q_{-\nu-1}(z)] .
\]  

(19)

If \( w < w_0 \) we replace \( P_\nu(w_0) \) in the integrand according to equation (19), and the large semi-circle is chosen to be in the lower (or upper) half of the complex \(k\)-plane for the integral of the term involving \( Q_\nu \) (or \( Q_{-\nu-1} \)). Then, by taking account of the contribution only from the poles present at \( k = \pm i[s + (1/2)] \) and at \( k = \pm i[2n + (3/2)] \) where \( P_\nu(0) \) vanishes for \( n = 0, 1, 2, \cdots \), we obtain the result valid in the inner region \( 1 \leq w < w_0 \) as follows,

\[
\Psi = \sum_{n=0}^{\infty} \alpha_n G_{2n+1} R_{2n+1}(r) \Theta_{2n+1}(\theta) ,
\]  

(20)

where

\[
\alpha_n = \frac{A_0 (-1)^n (4n + 3)}{(2n + s + 2)(2n + 1 - s)} \frac{(2n + 1)!!}{(2n)!!}
\]

and

\[
G_{2n+1} = (w_0^2 - 1) \left[ Q_{2n+1}(w_0) \frac{dQ_n(w_0)}{dw_0} - Q_n(w_0) \frac{dQ_{2n+1}(w_0)}{dw_0} \right] .
\]  

(22)

In the outer region \( w > w_0 \), however, we must use equation (19) for \( P_\nu'(w) \) [instead of \( P_\nu(w_0) \)] with the same choice of the large semi-circle, and the contribution from the poles is divided into the two parts

\[
\Psi = \Psi_1 + \Psi_2
\]  

(23)

with the forms written by

\[
\Psi_1 = \sum_{n=0}^{\infty} \alpha_n \Theta_{2n+1}(\theta) \left[ F_{2n+1} R_{2n+1}(r) - G_{2n+1} \int_1^{w_0} P_{2n+1}(w') dw' \right] ,
\]  

(24)

and

\[
\Psi_2 = \frac{A_0}{\sqrt{\pi}} \Gamma \left( \frac{1 - s}{2} \right) \Gamma \left( 1 + \frac{s}{2} \right) \Theta_s(\theta) \tilde{R}_s(r)
\]

(25)

where the function \( \tilde{R}_\nu(r) \) for any \( \nu \) is defined by

\[
\tilde{R}_\nu(r) = (w + 1) Q_\nu(w) - \int_{w_0}^{w} Q_\nu(w') dw'
\]

(26)

and \( \Gamma(z) \) is the gamma function.

It is easy to check that the continuity of these expressions (20) and (23) for \( \Psi \) at the boundary \( w = w_0 \) holds, because through the application of the expansion in terms of the Legendre polynomials to the function \( u(x) \) defined by \( u = P_\nu(x) \) for \( x > 0 \) and \( u = -P_\nu(|x|) \) for \( x < 0 \) we
\begin{equation}
\sum_{n=0}^{\infty} \frac{(-1)^n(4n+3)}{(2n+s+2)(2n+1-s)} \frac{(2n+1)!!}{(2n)!!} P_{2n+1}(x) = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1-s}{2} \right) \Gamma \left( \frac{1+s}{2} \right) u(x) .
\end{equation}

Further, equation (27) is useful to show that for a positive \( s \) the asymptotic behavior of \( \Psi \) at \( w \to \infty \) becomes

\[ \Psi \simeq -A_0 \int_{v_0}^{\infty} Q_4(w') dw' \times v(x) , \]

where the dependence on \( x \) is given by

\[ v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(4n+3)}{(2n+1)(2n+2)} \frac{(2n+1)!!}{(2n)!!} \int_{x=1}^{x} P_{2n+1}(x') dx' . \]

By substituting the value of \( s = 0 \) into the equality (27), we can calculate the summation to be \( v = 1 - |x| \). Hence, we conclude that this black hole magnetosphere based on a disk current model has the asymptotically conical shape of magnetic field lines. This asymptotic structure is a consequence of the disk currents extending to infinite distance. Though such a current distribution is unphysical, the existence of open field lines will be useful to discuss the problem of outgoing winds from the disk to infinity in our modeling.

3. THE GLOBAL STRUCTURE OF FIELD LINES

Though the asymptotic behavior of \( \Psi \) at infinity is clear, the expression written by the infinite sum of the multipole fields is still complicated for understanding the features of the field line structure near the inner edge and the event horizon. Hence, let us assume that the inner edge is far distant from the event horizon. For example, the inner edge of the disk around the Schwarzschild black hole may be located at the innermost stable circular orbit \( r = 6M \) on the equatorial plane. Then, we have \( w_0 = 5 \), for which the approximation \( w_0 \gg 1 \) will be roughly allowed.

In the following we focus on equation (20), because the multipole sum should arrive at the same result even if equation (23) is used. By virtue of the approximation \( w_0 \gg 1 \), we obtain the dominant behavior of the Legendre function such that \( Q_4(w_0) \sim w_0^{-3/2} \). Except in the region near the event horizon where \( w \) is of order unity, we can calculate equation (20) under the assumption \( 1 \ll w \leq w_0 \), which leads to the approximate relation

\[ P_{2n+1}(w)Q_{2n+1}(w_0) \simeq \frac{2n+1}{(2n+2)(4n+3)} \left( \frac{w}{w_0} \right)^{2n+2} . \]

Of course, for the angular part involving the integral of \( P_{2n+1}(x) \) we cannot use such an approximation. Hence, our key step is to rewrite the Legendre polynomial through the integral formula

\[ P_{2n+1}(\cos \theta) = \frac{1}{\pi} \int_{0}^{\pi} (\cos \theta + i \sin \theta \cos \varphi)^{2n+1} d\varphi , \]
which allows us to obtain

$$\Psi \simeq A_0 Q_s(w_0) w \int_{|x|}^1 dx' \left[ \frac{1}{\pi} \int_0^\pi d\varphi K(y) \right]. \quad (32)$$

The variable $y$ and the function $K$ are defined by

$$y = \frac{w}{w_0} \left[ x' + i \cos \varphi \sqrt{1 - (x')^2} \right], \quad (33)$$

and

$$K = \sum_{n=0}^\infty \frac{(-1)^n(2n+1)(2n+1)!}{2n+2+s} \frac{(s+1)y^{n+1}}{(2n+2)^{n+1}}, \quad (34)$$

for which the infinite sum becomes possible to give

$$K = \frac{1}{y} \left( 1 - \frac{1}{\sqrt{1+y}} \right) - (s+1)y^{s-1} \int_0^y (y')^{s-1} \left[ 1 - \frac{1}{\sqrt{1+(y')^2}} \right] dy'. \quad (35)$$

This expression of $K(y)$ is also valid in the outer region $w > w_0$, and it is easy to verify that at infinity where $y \gg 1$ the magnetic stream function given by equation (32) can keep the same asymptotic form as equation (28).

Next, let us give the magnetic stream function near the event horizon. Because $w$ is of order unity, the term with $n = 0$ dominates in the multipole sum (20) under the approximation $w_0 \gg 1$, and we have

$$\Psi \simeq \frac{A_0 Q_s(w_0)}{4(s+2)w_0} (1-x^2)(w+1)^2, \quad (36)$$

which shows the cylindrical magnetic field written by $\Psi = \pi B_0 r^2 \sin^2 \theta$ with the strength

$$B_0 = \frac{A_0 Q_s(w_0)}{4\pi M^2(s+2)w_0}. \quad (37)$$

This is in accordance with the result claimed for general multipole expansions by King, Lasota, & Kundt (1975). However, if $w_0$ is of order unity, the dominance of the cylindrical field in the sum (20) apparently breaks down even near the event horizon, and a more complicated structure of field lines will become possible there.

We hereafter fix $B_0$ to be positive in equation (37). Then, for $s > 0$, the magnetic stream function becomes negative at infinity, where we have

$$\Psi \simeq - \left( 1 + \frac{2}{s} \right) 4\pi B_0(Mw_0)^2 (1 - |x|). \quad (38)$$

Because this change of $\Psi$ occurs for all $\theta$, a closed field line of $\Psi = 0$ connecting the equatorial plane with the polar axis should exist in the magnetosphere. This critical field line can thread the black hole along the polar axis $|x| = 1$ and divide the magnetosphere into the inner region with
closed field lines and the outer region with open field lines. Note that if equation (32) is written by the sum of the multipole fields we obtain the \( r \)-dependence at \( x = 0 \) as follows,

\[
\Psi \approx B_0(Mw)^2 \sum_{n=0}^{\infty} \frac{2(s+2)}{2n+2+s} \frac{(2n+1)!n!}{(2n+2)!} \left( \frac{w}{w_0} \right)^{2n},
\]

which claims \( \Psi \) to be positive everywhere on the inner equatorial plane \( w < w_0 \) between the disk and the black hole. Hence, the critical field line of \( \Psi = 0 \) should thread the disk at some point which we denote by \( w = w_c \) (or \( r = r_c \)). For example, if we consider the model of \( s = 2 \) with \( K \) of the form

\[
K = \frac{3}{y^3} \left( \sqrt{1 + y^2} - 1 \right) - \frac{1}{y} \left( \frac{1}{2} + \frac{1}{\sqrt{1 + y^2}} \right),
\]

the numerical estimation of equation (32) gives \( w_c \approx 1.9w_0 \). (The critical field line also reaches to the polar axis at \( w \approx 1.8w_0 \), along which \( w \) is approximately constant.)

To give the magnetic stream function approximately valid in the whole range from \( w \approx 1 \) to \( w \approx w_0 \), we modify equation (32) which reduces to \( \Psi \to \pi B_0(Mw)^2(1 - x^2) \) in the limit \( w \ll w_0 \).

We note that \( \Psi \) given by equation (32) is smoothly matched to the cylindrical field (36) which is valid near the event horizon, only by multiplying the factor \( (w + 1)^2/w^2 \) which becomes equal to unity in the region \( w \gg 1 \). Hence, the modified form of the magnetic stream function should be

\[
\Psi = 4(s+2)B_0(w+1)^2\frac{w_0}{w}M^2 \int_{|z|}^{1} dx' \int_0^{\pi} d\varphi K(y)
\]

with \( K \) given by equation (35).

We have arrived at the main result (41) available as a model for describing the global structure of the black hole magnetosphere. The numerical example of magnetic field lines for \( s = 2 \) and \( w_0 = 10 \) is drawn in Figures 1 and 2, which are useful to see the magnetic connection between the disk and the black hole in the inner region and the conical shape at large \( r \), respectively. The magnetic spot in the disk connected with the black hole is found to be a very narrow region. This is because \( \Psi \approx B_0M^2 \) near the event horizon, while \( \Psi \approx B_0(Mw_0)^2 \) on the disk unless \( w \approx w_c \). If the width of the magnetic spot is denoted by \( \Delta w_c \) (or \( \Delta r_c \)), at \( w = w_c \) we obtain

\[
B_0M^2 \approx \left( \frac{\partial \Psi}{\partial w} \right) \Delta w_c,
\]

which leads to \( \Delta w_c \approx 1/w_0 \ll 1 \) (i.e., \( \Delta r_c \ll M \)). Because the area \( 2\pi r_c\Delta r_c \) of the magnetic spot is just of order of the horizon area \( 16\pi M^2 \), no significant amplification of the magnetic field occurs near the event horizon. Steady inflows of the disk plasma to the black hole may occur along the magnetic field lines, starting from the magnetic spot of the disk which is apart from the inner edge. Then, the inflows reach to the polar region at a distance roughly equal to \( Mw_0 \) and finally fall to the black hole along the cylindrical field lines.
Plasma outflows from the outer part of the disk may also emanate along the open field lines of $\Psi < 0$ extending to infinity. It should be remarked, however, that the co-existence of two topologically distinct zones in the magnetosphere is not due to a general relativistic effect of the black hole, because such a global structure is also possible for an appropriate superposition of a dipole and a uniform field, giving closed and open field lines, respectively. The role of the black hole is rather to prohibit the presence of a dipole field at the event horizon. Hence, the closed field lines should be sustained by disk currents only, which is the point essential to our modeling.

In astrophysical magnetospheres various MHD processes are expected to become important for producing highly energetic phenomena. For example, in the region very close to the event horizon, strong gravity forces the plasma to accrete in a radial direction and to bend the cylindrical field lines (Punsly 1991b; Hirota et al. 1992). The magnetic field (41) generated by disk currents provides a starting point to analyze such MHD effects of ambient and disk plasma on the magnetospheric structure, which will be discussed in the next section.

4. DISCUSSION

If the plasma inertia is taken into account even under the condition of magnetic domination, the asymptotic collimation from the conical shape of field lines should occur in logarithmic scales of the cylindrical radius (Chiu et al. 1991; Tomimatsu 1994). Apparently such an inertial back-reaction of the plasma on the poloidal field does not work in the vacuum fields. Then, we present a model with collimated structure, by adding the uniform field (which is a solution of the vacuum Maxwell equations)

$$\Psi_u = \pi B_u r^2 \sin^2 \theta$$

(43)
to $\Psi$ given by equation (41). Let us denote the new magnetic stream function by $\Psi' = \Psi + \Psi_u$. Because $\Psi$ is negative in the far distant region, $B_u$ should be chosen to be negative for allowing a smooth collimation of the conical field lines. If the uniform field is weak, namely $B_0 + B_u > 0$, the value of $\Psi'$ remains positive at the event horizon to keep the magnetic connection between the disk and the black hole (Fig. 3). However, if $B_0 + B_u$ becomes negative, the field lines threading the black hole can reach to infinity only (Fig. 4).

Of course, this addition of a uniform field is just a mathematical procedure, and the strong collimation shown in Figures 3 and 4 is not a consequence of the physical confinement mechanisms by self-generated toroidal fields and/or by external pressure with a boundary at finite cylindrical radius. It remains quite uncertain whether the details of the field-line shape shown in these figures become approximately valid in magnetically dominated wind regions or not. Nevertheless, it is interesting to note that the vacuum models can successfully describe the transition of the structure of field lines threading the black hole (i.e., from magnetic connection with the disk to that with the remote load), which may occur in more realistic magnetospheres as an observable change of the astrophysical activity.
To discuss the magnetic activity by using the vacuum field $\Psi'$ (or $\Psi$) as a background field perturbed by magnetically dominated winds from the disk, let us consider the magnetosphere slowly rotating with the angular velocity $\Omega_F$ of a magnetic field line and the hole’s one $\Omega_H$ (i.e., $r \sin \theta \Omega_F \ll 1$ and $M \Omega_H \approx a/4M \ll 1$). If a magnetic field line threading the black hole is closed in the disk at $r = r_d$ (see Figs. 1 and 3), the angular velocity $\Omega_F$ is nearly equal to the local Keplerian one, i.e.,

$$\Omega_F \simeq \Omega_K \equiv (M/r_d^3)^{1/2}$$

and we have $r_d \approx M\chi w_0$, where $\chi$ is a numerical factor in the range $1 < \chi \leq 1.9$ for the $s = 2$ model. Now the approximation of slow rotation is assured by virtue of the estimation such that $r_d \Omega_F \approx 1/\sqrt{\beta w_0} \ll 1$. However, the inner edge in realistic accretion disks cannot be too distant from the event horizon (namely, $w_0$ cannot be too large), and from the relation

$$\frac{\Omega_H}{\Omega_F} \simeq \frac{M \Omega_H (\beta w_0)^{3/2}},$$

we expect to be $\Omega_H < \Omega_F$ as a typical case for slowly rotating black holes. Hence, the inward winds from the magnetic spot carry angular momentum from the disk to the black hole, and the disk plasma which remains in the magnetic spot without flowing into the magnetic tube will be torqued and fall inward. This means that the disk currents located in the inner region and sustaining the loop field lines become unstable for radial accretion, and the closed loop field disconnected from the disk must disappear owing to the infinite redshift effect at the event horizon (Punsly & Coroniti 1989).

In the magnetically-dominated wind region the toroidal magnetic field $B_T$ is approximately given by

$$B_T \simeq -4\pi \eta L/\alpha \varpi,$$

where $\alpha$ and $\varpi$ are the lapse function and the cylindrical radius, respectively, in Kerr geometry. By virtue of the approximation of slow rotation we obtain $\alpha = \sqrt{1 - (2M/r)}$ and $\varpi = r \sin \theta$. Along a poloidal field line the particle flux $\eta$ per unit flux tube and the total angular momentum $L$ of magnetized fluid are conserved, and the angular momentum flux per unit area carried by the winds towards the black hole is given by $-\eta LB_p$, which is a consequence of exerting the magnetic torque on the disk plasma. (Now $L$ is positive, while $\eta$ is negative for ingoing winds.) Then, the characteristic timescale $t_1$ to extract the plasma angular momentum in the disk would be

$$t_1 \sim \frac{h \rho_d r_d^2 \Omega_K}{(-\eta LB_p)} \sim \frac{(r_d/h)}{\beta e \Omega_K},$$

where $\rho_d$ is the mass density in the disk with the thickness $h$, and $\beta e$ is the ratio of the external magnetic stress $B_T B_p$ at the disk surface to the local pressure in the disk. One may also consider the internal torque transporting angular momentum radially outward through the disk, for which the accretion timescale $t_2$ is estimated to be

$$t_2 \sim \frac{(r_d/h)^2}{\beta e \Omega_K},$$
and the ratio $\beta_i$ of the internal stress to the local pressure is usually supposed to be less than unity (e.g., $\beta_i \approx 0.1$). In our modeling the external torque responsible for carrying angular momentum into the winds should be dominant if compared with the internal torque, and we require the condition $\beta_e > \beta_i h/r_d$, which becomes valid for $h \ll r_d$, unless the turbulent internal disk field is supposed to be much stronger than any ordered vertical field at the disk surface. Hence, for $\beta_e \approx \beta_i$, the duration of the closed loop structure could be quite longer than the dynamical timescale $\Omega_K^{-1}$, particularly in a thin disk.

In significant long term accretion of plasma beyond the timescale $t_1$ one must consider the change of the closed field-line topology, unless dynamo action in the inner disk region can efficiently work to recover the closed loops. The process of disruption of the closed loops would be quasi-stationary, and our models could give a qualitative picture of the structure change as follows: Initially strong disk currents can sustain nearly circular loops of field lines shown in Figure 1 (i.e., $B_0 \gg |B_d|$). As the disk currents become weaker (i.e., as the ratio $B_0/|B_d|$ decreases) by virtue of the radial accretion, the loops begin to collapse towards the black hole (especially in the vertical direction), as was shown in Figure 3. Finally, only the loops disconnected from the horizon remains near the inner edge (see Fig. 4 for $B_0 \ll |B_d|$). The loop field completely annihilates as a set of O-points, if the radial accretion from the inner edge continues still more (Punsly & Coroniti 1990b). During this process, a part of electromagnetic energy stored in loop field lines could be outward transferred as a Poynting flux to infinity. We expect that the magnetic field (41) to become a convenient initial state for analytical and numerical MHD calculations to study the problem of non-steady energy release in plasma accretion toward the black hole (see, e.g., Koide et al. (2000) for numerical investigations, in which the initial magnetic field is chosen to be the Wald solution in Kerr background).

In the slowly rotating magnetospheres the closed field line plays the role of a path for the energy and angular momentum transfer from the disk to the black hole. [ Li (2000c) claimed that the transfer from the black hole to the disk occurs only for $a/M > 0.36$.] However, the spin-up of the black hole should stop when the structure shown in Figure 4 has been established. Then, the energy extraction from the rotating black hole becomes possible as a mechanism of energy release. The angular velocity $\Omega_F$ of the open field lines connecting the black hole with the remote load may satisfy the condition $\Omega_F \approx \Omega_H/2$, which allows the twisted field with the negative toroidal component $B_T$ (i.e., $\eta < 0$ and $L < 0$) to bring negative angular momentum and energy to the black hole. The efficiency of this Blandford-Znajek mechanism is still controversial, because the validity of the condition $\Omega_F \approx \Omega_H/2$ is based on the assumption that a causal contact between the black hole and the remote plasma holds. In plasma accretion onto the black hole the poloidal inward velocity exceeds the fast magnetosonic speed near the horizon, and no information could propagate outward across some critical surface (Punsly & Coroniti 1990a). In fact, the black hole-driven winds satisfying $\Omega_F \ll \Omega_H$ have been found to exist (Punsly 1998). For such winds the main mechanism of energy extraction from the black hole could be the screw instability, which makes the twisted magnetic tubes unwind and causes a sudden transport of the magnetic free-energy into
the plasma's kinetic energy. For the cylindrical magnetic field shown in Figure 4 we can use the instability condition known as the Kruskal-Shafranov criterion (Kadomtsev 1966) of the form

$$|B_T|/2\pi \delta > |B_p|/S,$$

(49)

where $\delta$ and $S$ are the width and the length of the cylindrical magnetic tube. This has been proved by Gruzinov (1999) in the force-free magnetosphere, but it should be noted that the analysis has been limited to the case such that $\Omega_F = 0$ and $S \gg M$ (i.e., the approximation of no rotation and no gravity). Then, a naive application (Li 2000b) of the Kruskal-Shafranov criterion to the magnetosphere where $B_T$ is roughly estimated to be $\sin \theta \Omega_F B_p$ seems to be very dubious. Here, the toroidal field is estimated by equation (46), which remains valid even for $\Omega_F = 0$, and considering the region well away from the event horizon (i.e., $\alpha \approx 1$), we rewrite the instability condition into the form

$$S > |B_p| \delta^2/(2|\eta L|),$$

(50)

of which the right-hand side is constant if the magnetic flux $|B_p|\delta^2$ in the cylinder is conserved. This means that if a larger angular momentum flux $|\eta LB_\eta|\delta^2$ is carried away from the black hole by magnetically dominated winds in the cylinder with the width $\delta \sim M$, the twisted field lines become screw-unstable in a shorter scale $S$. After the disruption of the screw-unstable structure of magnetic field, the rotating black hole would begin to twist again the magnetic field lines, and the energy release from the black hole could quasi-periodically occur.

As was first proposed by Gruzinov (1999), the closed field lines connecting the black hole with the disk may be also screw-unstable. However, for the field lines closed in the disk the toroidal component will be mainly generated by disk rotation with the angular velocity $\Omega_K \approx \Omega_F$. Then, we must discuss the instability condition by including the effect of $\Omega_F$, which is a problem studied in future works (Matsuoka, Tomimatsu, & Takahashi 2001).

Finally it should be emphasized that the closed field-line topology presented in this paper is a very transient structure in rapidly rotating plasma-filled magnetospheres. The angular velocity $\Omega_F \approx \Omega_K$ of closed loops connecting the black hole with the disk will be smaller than the hole’s angular velocity $\Omega_H$, and the energy and angular momentum transfer from the black hole to the disk could spin up the disk plasma (Li 2000c). Then, as a consequence of this energy extraction from the black hole, outward flows of plasma from the disk would be generated, and owing to the frozen-in condition the field lines threading the black hole are eventually disconnected from the disk. The closed field lines anchored in plasma orbiting in the ergosphere would also open up to infinity by the strong dynamo action, as was conjectured to occur in field-aligned pulsars (Punsly & Coroniti 1990b). Further, the timescale $t_2$ for radial accretion due to internal torque may become shorter than the timescales to open up the loops closed in the disk. Then, the closed loops accrete toward the black hole and annihilate owing to the infinite redshift effect (Punsly & Coroniti 1990b). These are astrophysically interesting processes of rapid energy release, and we could observe transient flaring states of black hole magnetospheres, if the repeated formation of closed magnetic loops due to strong disk currents in the inner region occurs.
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Fig. 1.— Poloidal magnetic field lines for the axisymmetric model of $s = 2$ and $w_0 = 10$. The value of $\Psi/4\pi M^2 B_0$ is denoted on each field line, and the structure is assumed to have the equatorial symmetry. Further, the central black hole with the radius $r = 2M$ and the surrounding disk with the inner edge at $r = 11M$ are displayed. We can see the magnetic connection of the black hole with a very narrow region in the disk.
Fig. 2.— Same as Fig. 1 for large scale of $r$, to show the asymptotic shape of field lines.
Fig. 3.— Poloidal magnetic field lines for the model $\Psi'$ involving a weak uniform field, i.e., $B_u = -0.9 B_0$. The magnetic field generated by the disk current is same as Fig. 1. The magnetic connection between the black hole and the disk still remains, and the collimated field lines appear in large $r$. 
Fig. 4.— Same as Fig. 3, except for the existence of a strong uniform field, i.e., $B_x = -1.1B_0$. Only the open field lines can thread the black hole without any connection with the disk, though the loop field lines still exist near the inner edge.