On the Hawking effect

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Abstract

In terms of the Painlevé-Gullstrand-Lemaître coordinates a rather general scenario for the collapse of a star to a black hole is described by a manifestly $C^\infty$-metric. Without employing the geometric optics approximation the leading contributions to the Bogoliubov coefficients are calculated explicitly and the Hawking temperature is recovered. Depending on the particular dynamics of the collapse the final state represents either evaporation or anti-evaporation. In both cases the state after the collapse differs by an infinite amount of particles from the Unruh state.

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1 Introduction

One of the great challenges of theoretical physics is the quest for an underlying law that unifies quantum theory and general relativity. The investigation of quantum fields in curved space-times is expected to provide a chance of achieving some progress towards this aim. Fulling’s discovery [1] of the non-uniqueness of the particle interpretation in curved space-times may be regarded as a basis for various fundamental effects, see e.g. [1]–[20] and references therein. Perhaps the most prominent example is the Hawking effect [2] which predicts the evaporation of black holes. There are two alternatives for the investigation of this striking effect: Originally Hawking calculated the Bogoliubov coefficients via the geometric optics approximation (backwards ray-tracing) in Ref. [2]. In contrast to this dynamical treatment Unruh [4] imposed
boundary conditions on the state in the static regime in order to reproduce the main features of the Hawking effect. In particular, since the state defined in this way – the Unruh state – is completely stationary, it merely describes the late-time part of the radiation. Of course, in general there exists some amount of created particles that depends on the dynamics of the collapse. But according to Ref. [2] the number of these particles is finite with the result that they disperse after a finite period of time and thus do not affect the (divergent) late-time radiation. Ergo it appears quite natural to assume that the state after the collapse to a black hole coincides up to a finite number of particles with the Unruh state describing the black hole evaporation – independently of the particular dynamics of the collapse. The question of whether this assertion is strictly correct will by subject of the present article. For that purpose we shall calculate the number of created particles explicitly without employing the geometric optics approximation. It will turn out that the above statement is not justified for a rather general class of dynamics of the collapse.

This paper is organised as follows: In Section 2 we set up the basic properties of the quantum field under consideration. A brief introduction into the concept of Hadamard states is presented in Sec. 2.1. The number of created particles is calculated in Section 3. In Secs. 3.1 and 3.2 we deduce the eigenmodes in terms of the Schwarzschild and the Painlevé-Gullstrand-Lemaître coordinates, respectively. The Bogoliubov coefficients are derived explicitly in Sec. 3.3. We shall close with a summary, some conclusions, a discussion, and an outline.

Throughout this article natural units with \( G = \hbar = c = k_B = 1 \) will be used. Lowercase Greek indices such as \( \mu, \nu \) vary from 0 (time) to 3 (space) and describe space-time components (Einstein sum convention). Uppercase Roman indices \( I,J \) denote complete sets of quantum numbers.

## 2 General formalism

We consider a minimally coupled, massless and neutral (i.e. Hermitian) scalar (spin-zero) quantum field \( \hat{\Phi} \) propagating on a globally hyperbolic spacetime \( (M, g^{\mu\nu}) \). Global hyperbolicity demands strong causality and completeness, cf. [21]. (Without these requirements the time-evolution of the quantum system is not well-defined and unitary.) In the Heisenberg representation the kinematics of the field \( \hat{\Phi} \) is governed by the Klein-Fock-Gordon equation

\[
\Box \hat{\Phi} = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \hat{\Phi} \right) = 0.
\]

(1)

Strictly speaking, the quantum field is represented by an operator-valued distribution \( \hat{\Phi} \) and hence the above equation has to be understood in this sense: \( \Box F = 0 \rightarrow \hat{\Phi}[F] = 0 \). In a
globally hyperbolic space-time the wave equation (1) possesses unique advanced and retreated
Green functions $\Delta_{\text{adv}}(x,x')$ and $\Delta_{\text{ret}}(x,x')$, respectively. Employing these distributions one
may accomplish the canonical quantisation procedure via imposing the covariant commutation
relations

$$\left[ \hat{\Phi}(x), \hat{\Phi}(x') \right] = \Delta_{\text{ret}}(x,x') - \Delta_{\text{adv}}(x,x').$$ (2)

The solutions of the equation of motion (1) obey a symplectic structure induced by the inner
product

$$\langle F|G \rangle = i \int_{\Sigma} d\Sigma^\mu F^* \partial_\mu G,$$ (3)

with $F \partial_\mu G = F \partial_\mu G - G \partial_\mu F$. With the aid of Gauss’ law one can show that the inner
product (3) is independent of the particular Cauchy surface $\Sigma$ for any two solutions of the
Klein-Fock-Gordon equation $\Box F = \Box G = 0$, cf. [21]. It should be mentioned here that the
measure $d\Sigma^\mu$ used above already contains volume factors like $\sqrt{-g}\Sigma$ and is normalised according
to $d\Sigma_\mu dx^\mu = \sqrt{-g} d^4x$.

The canonical commutation relations (2) imply

$$\left[ \left( F|\hat{\Phi} \right), \left( \hat{\Phi}|G \right) \right] = \left( F|G \right).$$ (4)

As a result the inner product of the field $\hat{\Phi}$ with positive ($F_I$) and negative ($F^*_J$) frequency
solutions of the Klein-Fock-Gordon equation, respectively, with $(F_I|F_J) = -(F^*_I|F^*_J) = \delta(I,J)$
and $(F_I|F^*_J) = 0$ defines creation and annihilation operators, respectively. As it is well-known,
these operators and thus also the associated number operators depend on the particular choice
of the solutions $F_I$. This ambiguity represents the non-uniqueness of the particle interpretation
(see e.g. [1]) and may be regarded as the basis of the phenomenon of particle creation induced
by the gravitational field.

Averaging the operator-valued distributions $\hat{\Phi}(x)$ with $n$-point test functions $B_n \in C^\infty_0(M^n)$ via

$$\hat{\Phi}^n[B_n] = \int d^4x_1 \cdots \int d^4x_n \hat{\Phi}(x_1) \cdots \hat{\Phi}(x_n) B_n(x_1,\ldots,x_n)$$ (5)

we acquire well-defined operators $\hat{\Phi}^n[B_n]$. The complete set of all these operators (constructed
for all test functions) generates the $*$-algebra containing all possible observables of the quantum
system (with the unit element $1 = \hat{\Phi}^0[1]$).

The states $\varrho$ of the quantum system can be introduced as linear $\varrho(\mu \hat{X} + \nu \hat{Y}) = \mu \varrho(\hat{X}) + \nu \varrho(\hat{Y})$
and non-negative $\varrho(\hat{Z}^\dagger \hat{Z}) \geq 0$ functionals over the $*$-algebra with unit norm $\varrho(1) = 1$. All these
states $\varrho$ build up a convex set, i.e., for any two states $\varrho_1$ and $\varrho_2$ also the convex combination

$$\varrho_\lambda = \lambda \varrho_1 + (1 - \lambda) \varrho_2$$

with $0 < \lambda < 1$ represents an allowed state. The extremal points of this convex set correspond to the pure states $\hat{\varrho} = \ket{\Psi} \bra{\Psi}$. Since every convex set is the convex hull of its extremal points all (mixed) states can be written as a (possibly infinite) linear combination of pure states.

In order to decide whether a state is pure or mixed in character one has to consider the complete algebra. Focusing on a sub-algebra a pure state may display properties that are usually connected with mixed states. This observation may be regarded as the basis of the thermo-field formalism, see e.g. [22] and [6].

It might be interesting to illustrate these points by some examples: If one describes the space-time of a black hole by the Schwarzschild metric the associated time coordinate represents a Killing vector. The ground state of the quantum field (with respect to that Killing vector) in the region outside the horizon is called the Boulware [3] state $\varrho_B$. It contains no particles – again with respect to the Killing vector measuring the time of an outside observer with a fixed spatial distance to the black hole. (A free-falling observer may well detect particles in that state.) Ergo the Boulware state is a pure state with respect to the algebra of the exterior region $\hat{\varrho}_B = \ket{\Psi_B} \bra{\Psi_B}$. The interior domain possesses no ground state at all, cf. [17]. (Again all assertions refer to the Killing field along the Schwarzschild time.) As further interesting states one may introduce the Kubo-Martin-Schwinger (KMS, [23]) states $\varrho_T$ describing thermal equilibrium at some given temperature $T$. Obviously these states are mixed in character – at least from the exterior point of view. One important KMS state is the Israel-Hartle-Hawking [5, 6] state $\varrho_{IHH}$ which corresponds to the Hawking temperature. It can be shown [11] that this state is indeed a pure state with respect to an enlarged algebra. The Israel-Hartle-Hawking state $\varrho_{IHH}$ contains the same number of ingoing and outgoing particles (thermal equilibrium). Hence the total energy flux vanishes. The phenomenon of the black hole evaporation can be described by the Unruh [4] state $\varrho_U$. This state is defined via two requirements: no ingoing/incoming particles/radiation at spatial infinity and thermal outgoing radiation near the horizon, see also [11]. If one considers a collapse of a star to a black hole and assumes the initial state to be pure in character (e.g. the vacuum) then the final state is – of course – also a pure state. The question of whether the initial state indeed transforms into the Unruh state will be subject of Section 3.

### 2.1 Hadamard states

In general, the complete convex set is too large and contains more states than physical reasonable. One way to restrict to physically well-behaving states is to impose the so-called Hadamard
condition. Hadamard states are states for which the symmetric part of the bi-distribution
\[ W^{(2)}(x,x') = \varrho \left( \hat{\Phi}(x)\hat{\Phi}(x') \right) = \text{Tr} \left\{ \hat{\varrho} \hat{\Phi}(x)\hat{\Phi}(x') \right\}, \tag{6} \]
the two-point Wightman [25] function, obeys the following singularity structure (in a 3+1 dimensional space-time)
\[ \frac{1}{2} \left( W^{(2)}(x,x') + W^{(2)}(x',x) \right) = -\frac{1}{(2\pi)^2} P \left( \frac{U(x,x')}{s^2} + V(x,x') \ln s^2 + W(x,x') \right), \tag{7} \]
where \( P \) symbolises the principal part. The antisymmetric part of \( W^{(2)} \) must be consistent with the commutation relation (2). \( s \) denotes the geodesic distance \( ds^2 = g_{\mu\nu}dx^\mu dx^\nu \) between the space-time points \( x \) and \( x' \) (which is at least in a neighbourhood of a regular point \( x \) unique). The functions \( U(x,x'), V(x,x') \) and \( W(x,x') \) are regular in the coincidence limit \( x \to x' \). Together with the normalisation \( U(x,x) = 1 \) the first two functions \( U(x,x') \) and \( V(x,x') \) are uniquely determined by the structure of space-time, e.g. \( V(x,x) = R^\mu_\mu/12 \) (with \( R^\mu_\mu \) being the Ricci tensor, see e.g. [24]). Hence all information about the state \( \varrho \) enters \( W(x,x') \) only. One important advantage of the Hadamard requirement may be illustrated by considering the regularisation of expectation values of two-field observables, for instance the energy-momentum tensor \( \hat{T}^\mu_\nu \). The Hadamard singularity structure ensures the validity of the point-splitting renormalisation technique, cf. [7]. It can be shown that for a globally hyperbolic \( C^\infty \) space-time \((M,g_{\mu\nu})\) the Hadamard condition is conserved, i.e., if the two-point function has the Hadamard singularity structure in an open neighbourhood of a Cauchy surface, then it does so everywhere [9].

If one considers the collapse of a star to a black hole which can be described by a \( C^\infty \)-metric the above theorem can be used to deduce the Hadamard condition for the final state. (The initial state is assumed to be a regular excitation over the ground state and thus satisfies the Hadamard requirement. The Minkowski vacuum of course meets the Hadamard structure with \( U = 1, V = 0 \) and \( W = 0 \).) On the other hand it can be shown that if the state of a field \( \Phi \) fulfils the Hadamard requirement (among other not as strict assumptions, cf. [12]) in the whole black hole space-time and especially at the horizon then the asymptotic expectation values correspond exactly to a thermal radiation with the Hawking temperature \( T = 1/(4\pi R) \) (see [12], [10] and [13]). Combining the two statements above we are able to deduce the Hawking effect for any collapse scenario that can be described by a \( C^\infty \)-metric.

It might be interesting to discuss the previous considerations by means of some examples. Applying the theorems above to the Boulware state, i.e. the ground state, it follows immediately that this state cannot satisfy the Hadamard requirement – at least at the horizon. Indeed this state is singular at the horizon – its (point-splitting) renormalised energy density diverges there
\[ \langle \Psi_B | \hat{T}_0^\dagger | \Psi_B \rangle_{\text{ren}} \downarrow -\infty \text{ for } r \downarrow R. \] It can be shown that the Boulware state as well as every KMS state (with an arbitrary temperature) fulfills the Hadamard requirement away from the horizon \( r > R \), see \[16\]. But only the KMS state corresponding to the Hawking temperature, i.e. the Israel-Hartle-Hawking meets the Hadamard structure at the horizon, see e.g. \[16, 13\]. However, the initial (approximately Minkowski) vacuum cannot transform into this state during a collapse of a star to a black hole, cf. \[17\]. In contrast to the Unruh state the Israel-Hartle-Hawking state represents thermal equilibrium also for \( r \uparrow \infty \) and the associated amount of particles and energy cannot be produced by a collapse.

### 3 Particle creation

Within the Heisenberg representation the time-evolution of the quantum system is governed by the operators while the states remain unaffected. Hence the investigation of the Hawking effect goes along with the question: How many (final) black hole particles contains the initial state? In general, this number depends on the particular initial state and the initial metric as well as the dynamics of the metric during the collapse. According to the considerations in the previous Section we assume a \( C^\infty \)-metric throughout. It can be shown that the Hawking effect (i.e. the late-time radiation) is independent of the (regular) initial space-time, see Sec. 3.3 below. Similarly any finite amount of particles being present initially does not alter the assertions concerning the Hawking effect (see the remarks at the end of Section 3.3 below). For that reason we assume the initial state to coincide with the (initial) vacuum. In this situation the number of final particles can be calculated via the Bogoliubov \( \beta \)-coefficients, see e.g. \[18\]

\[ N_J^{\text{out}} = \langle 0^{\text{in}} | \hat{N}_J^{\text{out}} | 0^{\text{in}} \rangle = \int_I | \beta_{IJ} |^2. \]  \[(8)\]

In order to calculate these coefficients we have to derive the structure of the initial modes \( F_J^{\text{in}} \) after the collapse and to compare them with the out-solutions \( F_J^{\text{out}} \) by means of the inner product in Eq. (3).

#### 3.1 Schwarzschild metric

The particle interpretation in quantum field theory is based on the selection of an appropriate time-like Killing vector. This choice refers to a certain class of associated observers whose time evolution is generated by the Killing field. For the flat space-time example, the Killing vector mediating the (Minkowski) time translation symmetry accords to a usual observer at
rest whereas special Lorentz boosts represent accelerated (Rindler) observers. Since, in general, different Killing vectors generate distinct particle definitions, the Rindler observer does not regard the Minkowski vacuum as empty with respect to (Rindler) particles. Instead, he experiences a thermal bath, a phenomenon which is called the Unruh-effect [4].

In analogy the time evolution of an observer at a fixed spatial distance to a black hole is generated by the Killing vector corresponding to the Schwarzschild time \( t \). The particles that are measured by such an observer can be described by positive frequency solutions – with respect to that time coordinate – of the Klein-Fock-Gordon equation. In contrast the evolution parameters of other coordinate representations of the black hole (e.g. the Kruskal metric) accord to different observers (e.g. the free-falling one) in general.

In terms of the Schwarzschild coordinates \( t, r, \vartheta, \varphi \) the space-time of an uncharged and non-rotating black hole assumes the well-known form

\[
 ds^2 = \left( 1 - \frac{R}{r} \right) dt^2 - \left( 1 - \frac{R}{r} \right)^{-1} dr^2 - r^2 d\vartheta^2 - r^2 \sin^2 \vartheta d\varphi^2 .
\]  

(9)

As it will become more evident later on, the most interesting region (with respect to the Hawking effect) is the vicinity of the horizon. In order to extract the features that are characteristic for this zone we introduce a dimensionless variable \( \chi \) via

\[
 \chi = \frac{r}{R} - 1 .
\]  

(10)

This quantity allows for a Taylor expansion in the vicinity of the horizon. As another useful tool we define the Regge-Wheeler tortoise coordinate

\[
 r_* = \int \frac{dr}{1 - R/r} = R \ln \chi + \mathcal{O}[\chi] .
\]  

(11)

This coordinate results in a conformally flat metric of the \((t, r)\)-sector

\[
 ds^2 = \left( 1 - \frac{R}{r} \right) \left( dt^2 - dr_*^2 \right) - r_*^2 d\vartheta^2 - r_*^2 \sin^2 \vartheta d\varphi^2 .
\]  

(12)

As a result the equation of motion simplifies to

\[
 \left( \frac{\partial^2}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial r_*} r_*^2 \frac{\partial}{\partial r_*} - \left( 1 - \frac{R}{r} \right) \nabla^2_{\vartheta \varphi} \right) \Phi = 0 .
\]  

(13)

After separating the angular dependence by spherical harmonics the centrifugal barrier and curvature scattering effects can be incorporated into an effective potential \( V_{\mathrm{eff}} \)

\[
 \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_{\mathrm{eff}}(r_*, \ell) \right) \phi_{\ell, m} = 0 .
\]  

(14)
\( \mathcal{V}_{\text{eff}} \) is strictly positive and approaches zero for \( r_* \uparrow +\infty \) and for \( r_* \downarrow -\infty \) with \( \mathcal{O}(1/r^2) = \mathcal{O}(1/r_*^2) \) and \( \mathcal{O}[\chi] = \mathcal{O}[\exp(r_*/R)] \), respectively. Unfortunately, no closed expression (in terms of well-known functions) for the eigenmodes is available. The asymptotic behaviour can be derived easily. For \( r_* \downarrow -\infty \) the positive frequency solutions behave as \( \exp(-i\omega t \pm i\omega r_*) \). These waves are purely ingoing or outgoing, respectively, for \( r_* \downarrow -\infty \). But every mode which is purely outgoing near the horizon contains for \( r_* \uparrow +\infty \) ingoing components as well owing to the scattering at the effective potential (inducing transmission and reflection coefficients) and vice versa. If we would divide the modes into purely ingoing/outgoing for \( r_* \uparrow +\infty \) they would be mixed at the horizon. In the following considerations we adopt the former choice where the \( \mathcal{O}[\chi] \)-approximated Schwarzschild eigenfunctions are given by

\[
F_{I}^{\text{out}}(x) = F_{\xi \omega \ell m}^{\text{out}}(t, \chi, \vartheta, \varphi) = N_{\omega \ell}^{\text{out}} e^{-i\omega t} \sqrt{\frac{\chi}{\omega \chi-i\xi \omega R}} Y_{\ell m}(\vartheta, \varphi) (1 + \mathcal{O}[\chi]) \tag{15}
\]

for \( r > R \) and vanish for \( r < R \) due to the horizon, cf. [17]. \( Y_{\ell m} \) denote the real-valued spherical harmonics, see e.g. [17]. The ingoing and outgoing modes are distinguished by \( \xi = \pm 1 \). \( N_{\omega \ell}^{\text{out}} \) symbolises a normalisation factor which may without any loss of generality chosen to be independent of \( \xi \). These eigenfunctions are rapidly oscillating near the horizon which again shows that there is the most interesting region.

### 3.2 Painlevé-Gullstrand-Lemaître metric

The Schwarzschild metric is quite simple but exhibits a coordinate singularity at the horizon and is therefore not \( C^\infty \) there. Hence it is impossible to express a manifestly \( C^\infty \)-metric in terms of the Schwarzschild coordinates. For this purpose one has to employ other coordinates. As one possible candidate we consider the Painlevé-Gullstrand-Lemaître [26] coordinates \( t_{\text{PGL}}, r, \vartheta, \varphi \). These coordinates emerge from the Schwarzschild coordinates \( t_{\text{S}}, r, \vartheta, \varphi \) by means of the transformation

\[
dt_{\text{PGL}} = dt_{\text{S}} + \sigma \sqrt{\frac{R}{1-R/r}} dr.
\tag{16}
\]

There exist two branches of these coordinate set distinguished by \( \sigma = \pm 1 \). In the following we shall drop the index \( t = t_{\text{PGL}} \) for convenience. The metric transforms into

\[
ds^2 = \left(1 - \frac{R}{r}\right) dt^2 - 2\sigma \sqrt{\frac{R}{r}} dr dt - dr^2 - r^2 d\vartheta^2 - r^2 \sin^2 \vartheta d\varphi^2. \tag{17}
\]

In contrast to the Schwarzschild form the Painlevé-Gullstrand-Lemaître metric (and its inverse as well) is \( C^\infty \) except at the singularity at \( r = 0 \).
Since we shall calculate the inner product in terms of the new coordinates we have to transform the black hole eigenfunctions, i.e. the out-modes. This can be done by simply substituting the Schwarzschild time via

\[ t_{\text{PGL}} = t - \sigma R \ln \chi + \mathcal{O}[\chi] \]

\[ F_{\text{out}}(x) = \mathcal{N}_{\text{out}} e^{-i\omega t} \sqrt{\omega} \chi^{(\sigma-\xi)\omega R} \mathcal{Y}_{\ell m}(\partial, \varphi) (1 + \mathcal{O}[\chi]) . \]

(18)

One observes that the modes with \( \xi = \sigma \) are no longer singular (arbitrarily fast oscillating) at the horizon, only those with \( \xi = -\sigma \) still exhibit this property. As it will become evident later on, merely the singular modes with \( \xi = -\sigma \) will contribute to the Hawking effect.

Employing the Painlevé-Gullstrand-Lemaître coordinates it is possible to write down a manifestly \( C^\infty \)-metric modelling a collapse of a star to a black hole

\[ ds^2 = \left(1 - f^2(t,r)\right) dt^2 - 2\sigma f(t,r) dr dt - dr^2 - r^2 d\vartheta^2 - r^2 \sin^2 \vartheta d\varphi^2 , \]

(19)

with \( f \in C^\infty \). Initially the metric describes a star with a (relatively) dilute distribution of matter and can be approximated (locally) by the Minkowski metric \( f(t \downarrow -\infty, r) = f_{\text{in}}(r) \ll 1 \).

For reasons of simplicity we assume the horizon to be formed at \( t = 0 \), i.e. \( f(t \geq 0, r \geq R) = f_{\text{out}}(r) = \sqrt{R/r} \). (Note that we did not impose any conditions on the structure of \( f \) in the interior of the black hole, i.e. beyond the horizon.) Outside the (spherically symmetric) collapsing star the Birkhoff theorem demands a stationary metric \( f(t, r \gg R) = \sqrt{R/r} \).

The Jacobi determinant is simply given by \( \sqrt{-g} = r^2 \sin \vartheta \) and the metric as well as its inverse are smooth \( g_{\mu\nu}, g^{\mu\nu} \in C^\infty \). Of course, this assertion holds true only if we omit the formation of the singularity at \( r = 0 \). But the region beyond the horizon is causally separated from the outside domain and hence irrelevant for our purposes.

In order to calculate the Bogoliubov coefficients we have to deduce some informations about the in-modes. For that reason we adopt the eikonal \textit{ansatz} and divide the field into an amplitude and a phase

\[ F_{\text{in}}^{\xi, \omega, \ell, m}(t, r, \partial, \varphi) = \frac{1}{\sqrt{\omega}} A_{\xi, \omega}(t, r) \exp(-i\omega S_{\xi}(t, r)) \mathcal{Y}_{\ell m}(\partial, \varphi) \left(1 + \mathcal{O}\left[\frac{1}{\omega}\right]\right) . \]

(20)

This \textit{ansatz} will be justified for compact domains with smooth metrics and high (initial) frequencies. But as it will turn out later, this is exactly the limit that is relevant for the Hawking effect. Inserting the above expression into the Klein-Fock-Gordon equation the leading terms in \( \omega \) govern the kinematics of the phase function via

\[ (\partial_{\mu} S_{\xi}) g^{\mu\nu} (\partial_{\nu} S_{\xi}) = 0 \rightarrow (\partial_{r} S_{\xi} - \sigma f \partial_{r} S_{\xi})^2 = (\partial_{r} S_{\xi})^2 . \]

(21)

This non-linear equation has four separate branches of solutions – e.g. for \( f = 0 \) one may identify the positive and negative frequency solutions on the one hand and the ingoing and
outgoing components labelled by $\xi = \pm 1$ on the other hand

$$\partial_t S_\xi - \sigma f \partial_r S_\xi = \xi \partial_r S_\xi \nrightarrow \partial_t S_\xi = (\sigma f + \xi) \partial_r S_\xi. \quad (22)$$

If we assume a sufficiently well-behaving dynamics of $f$, e.g. if it transforms directly from $f_{in}$ to $f_{out}$ and does not oscillate or assume negative values, no bifurcation occurs and $\xi$ corresponds to the initial direction of propagation, see also the remarks in Section 6 below.

### 3.3 Bogoliubov coefficients

Now we are in the position to calculate the Bogoliubov coefficients and thereby the number of created particles explicitly. Unfortunately, it seems to be impossible to find a general solution for these overlap coefficients. Nevertheless, with an expansion into powers of the relative distance to the horizon $\chi$ and the inverse initial frequency $1/\omega$ it is possible to extract the leading contribution – the Hawking effect. (As it will turn out later, the sub-leading parts merely generate finite contributions and thus do not affect the late-time radiation.) Per definition the Hawking radiation is exactly that part of the radiation which persists at arbitrarily late times (if we neglect the back-reaction). Hence the number of created particles accounting for the Hawking effect has to diverge. Any finite amount of particles would disperse after a finite period of time and cannot generate late-time radiation. (This is a consequence of the spectral properties of the wave equation. It possesses a purely continuous spectrum and thus does only allow for scattering states but no bound states, see e.g. [17] and [19].) As demonstrated in Ref. [17], the divergent number of particles is necessary for thermal behaviour in an infinite volume. In order to isolate the divergent part of the number of created particles we have to consider the Bogoliubov $\beta$-coefficients (see e.g. [18])

$$\beta_{IJ} = i \int d\Sigma^\mu \overrightarrow{F}_{I}^{\text{in}} \partial_{\mu} \overleftarrow{F}_{J}^{\text{out}}. \quad (23)$$

Since the Painlevé-Gullstrand-Lemaître coordinates are completely regular the measure $d\Sigma^\mu$ does not contain any singularities. As we have observed in the previous Sections, the modes $\overrightarrow{F}_{I}^{\text{in}}$ and $\overleftarrow{F}_{J}^{\text{out}}$ are bounded. In addition, the Birkhoff theorem implies that the modes at very large spatial distances to the collapsing star are not affected by the collapse. Consequently this region does not contribute to the $\beta$-coefficients and generates a $\delta(\omega - \omega')$-term for the $\alpha$, see also [2]. In summary we arrive at the conclusion that all (single) Bogoliubov $\beta$-coefficients are finite. As a result the only left way to generate a divergence is given by the summation/integration
over the initial quantum numbers \( I = (\xi, \omega, \ell, m) \)

\[
N_{J}^{\text{out}} = \sum_{I} |\beta_{IJ}|^2 .
\] (24)

For a spherically symmetric collapse the summation over the angular quantum numbers \( \ell \) and \( m \) breaks down. Consequently the divergence of \( N_{J}^{\text{out}} \) must be traced back to the integration over the initial frequencies \( \omega \). There are two possibilities for a singularity, the IR- and the UV-divergence. In the limit of small frequencies \( \omega \) the modes become space- and time-independent and approach a constant – unaffected by the Klein-Fock-Gordon equation. Ergo in the limiting case \( \omega \downarrow 0 \) the in- and out-modes coincide and thus possess a vanishing overlap with all other modes corresponding to finite frequencies. As a consequence the \( \omega \)-integration of the (absolute values squared of the) Bogoliubov coefficients is IR-save. In summary the infinite amount of particles has to be caused by the UV-divergence of the integration over the initial frequencies in consistency with Ref. [2]. (The Hawking effect is dominated by large (initial) frequencies only if one considers a fundamental quantum field theory without any kind of dispersion. Introducing a cut-off, see e.g. [27], as an effective description of some underlying theory the calculations are different.)

Recalling the structure of the initial eigenfunctions in Eq. (20) we arrive at the conclusion that only singularities of the out-modes may induce a UV-divergence. The convolution of regular expressions with the for \( \omega \uparrow \infty \) arbitrarily fast oscillating in-modes yields results of order \( 1/\omega \). Ergo the subsequent \( \omega \)-integration would be UV-save. Indeed, the out-modes are not regular at the horizon – the region that is naturally relevant for the Hawking effect. Thus it is sufficient to consider the vicinity of the horizon and the high (initial) frequency limit in order to extract the Hawking effect. As it will become more evident later, exactly the leading contributions in \( \chi \) and \( 1/\omega \) allow for the derivation of the thermal radiation.

If we choose the Cauchy surface according to \( \Sigma = \{ t = 0 \} \) the surface element assumes the form \( d\Sigma_\mu = (d^3 r, 0) \) and the \( \beta \)-coefficients transforms into

\[
\beta_{IJ} = i \int d^3 r \ F_{I}^{\text{in}} \left( \vec{\partial}_t - \sigma f \vec{\partial}_r \right) F_{J}^{\text{out}} ,
\] (25)

with the quantum numbers \( I = (\xi, \omega, \ell, m) \) and \( J = (\xi', \omega', \ell', m') \). The integration over the angular coordinates involves the spherical harmonics and simply yields \( \delta_{\ell \ell'} \delta_{mm'} \). Inserting the result of the previous Section \( \partial_\xi S_\xi - \sigma f \partial_r S_\xi = \xi \partial_r S_\xi \) we arrive at

\[
\beta_{IJ} = \int d^3 r A_{\xi, \ell} \exp (-i \omega S_\xi) \frac{i \omega \xi \partial_r S_\xi - i \omega' (1 + \sigma f [\sigma - \xi']/\chi)}{\sqrt{\omega \omega'}} \chi^{i(\sigma - \xi')/\omega' R} \times \mathcal{N} (1 + \mathcal{O}[\chi]) \left( 1 + \mathcal{O} \left[ \frac{1}{\omega} \right] \right) \delta_{\ell \ell'} \delta_{mm'} .
\] (26)
In view of the dominance of the vicinity of the horizon we may Taylor expand the amplitude $A_{\xi,\omega,\ell}(t = 0, r) = A_{\xi,\omega,\ell}(t = 0, r = R)(1 + \mathcal{O}[\chi])$. The zeroth-order term can be absorbed into the overall normalisation factor $\mathcal{N}$ and the higher order terms are omitted. (Here and in the following we do not change the symbol $\mathcal{N}$ for the normalisation factor and use the same letter also for the modified pre-factors.) A similar procedure can be performed with the phase function $S_{\xi}$. But owing to the pre-factor $\omega$ it is necessary to expand it up to first order $S_{\xi}(t = 0, r) = S_{\xi}(t = 0, r = R) + \partial_r S_{\xi}(t = 0, r = R)R\chi + \mathcal{O}[\chi^2]$. Again the zeroth-order term $S_{\xi}(t = 0, r = R)$ may be absorbed by a redefinition of $\mathcal{N}$. Since we have to integrate over the initial frequency $\omega$ the remaining unknown first-order term $\partial_r S_{\xi}(t = 0, r = R)$ can be eliminated by a re-scaling of the initial frequency via $\omega \rightarrow \omega \xi \partial_r S_{\xi}(t = 0, r = R)$. The Jacobi factor arising from the change of the integral measure modifies the normalisation $\mathcal{N}$ only. Assuming a very abrupt change of the metric (sudden approximation) the final phase function coincides with its initial form. For the Minkowski example it is simply determined by $\partial_r S_{\xi}(t = 0, r = R) = \xi$ and no redefinition is necessary. For other initial metrics the redefinition of the frequency exactly corresponds to the fact that the Hawking effect is independent of the initial (regular and stationary) space-time. The undetermined normalisation factor will be fixed later by virtue of the completeness relation in Eq. (32) below. After an analogous Taylor expansion of the volume element $r^2 dr = R^3 d\chi(1 + \mathcal{O}[\chi])$ and the function $f(t = 0, r) = 1 + \mathcal{O}[\chi]$ we find

$$
\beta_{IJ} = \int_0^\infty d\chi \exp(-i\xi\omega R\chi) \frac{\omega - \omega'\sigma[\sigma - \xi']/\chi}{\sqrt{\omega\omega'}} \chi^{i(\sigma - \xi')\omega'R} 
\times \mathcal{N}(1 + \mathcal{O}[\chi]) \left(1 + \mathcal{O}\left[\frac{1}{\omega}\right]\right) \delta_{\ell\ell'} \delta_{mm'}.
$$

(27)

As expected from the previous consideration, the Bogoliubov $\beta$-coefficients contribute only for $\sigma = -\xi'$ and vanish (within the used approximations) for $\sigma = \xi'$. In that case the out-modes are not singular (at the horizon) – only for $\sigma = -\xi'$ they display the arbitrarily fast oscillating behaviour. Hence – depending on the sign $\sigma$ – either only ingoing or only outgoing particles are produced (in an infinite amount). It should be repeated here that the classification above refers to the behaviour at the horizon and not at spatial infinity.

The integral in Eq. (27) involves generalised eigenfunctions which do not belong to the Hilbert space $L_2$ but are distributions, cf. [17]. Hence it cannot be interpreted as a well-defined Riemann integral. But – as demonstrated in Ref. [17] – it is possible to approximate (locally) the generalised eigenfunctions with well-defined wave-packets. One way to simulate such an approximation is to introduce a convergence factor $\varepsilon$ via $\chi^\varepsilon \exp(-\varepsilon \chi)$. For $\sigma = -\xi'$ the above integral can be solved in terms of $\Gamma$-functions. After insertion of the
convergence factor we can make use of the formula [28]

\[ \int_0^\infty dx \, e^{-xy} x^{z-1} = y^{-z} \Gamma(z) , \]  

(28)

which holds for \( \Re(y) > 0 \) and \( \Re(z) > 0 \), and we arrive at (remember \( \Gamma(z+1) = \Gamma(z) \))

\[ \beta_{IJ} = N \delta_{\sigma-\ell'} \delta_{\xi \ell'} \delta_{mm'} \sqrt{\omega / \omega'} \Gamma(2i\sigma \omega R) (i\xi \omega R + \varepsilon)^{2i\xi' \omega' R} \left( 1 + \mathcal{O} \left[ \frac{1}{\omega} \right] \right) . \]  

(29)

In view of Eq. (28) the higher order terms in \( \chi \) cause increasing arguments \( z \). Ergo these terms result in higher orders in \( 1/y \) – i.e. \( 1/\omega \) – consistently with our approximation and the arguments at the beginning of this Section. In order to evaluate the absolute value squared of the \( \beta \)-coefficient we may utilise the identity [28]

\[ \Gamma(z) \Gamma(-z) = -\frac{\pi}{z \sin \pi z} \]  

(30)

to obtain the final result

\[ |\beta_{IJ}|^2 = \frac{N}{\omega} \delta_{\sigma-\ell'} \delta_{\xi \ell'} \delta_{mm'} \exp(4\pi \omega' R) - 1 \left( 1 + \mathcal{O} \left[ \frac{1}{\omega} \right] \right) . \]  

(31)

This expression confirms the argumentation at the beginning of this Section. The remaining \( \omega \)-integration is indeed UV-divergent. In addition we observe that the terms of higher order in \( 1/y \) (and thus \( \chi \)) that we have neglected in our calculations are not UV-divergent and hence do not contribute to the Hawking effect. This observation provides an a posteriori justification of our expansion into powers of \( 1/\omega \) and \( \chi \) and the neglect of the sub-leading contributions. The UV-divergence can be interpreted with the aid of the well-known completeness relation

\[ \sum_I \int \alpha_{IJ}^* \alpha_{IK} - \beta_{IJ} \beta_{IK}^* = \delta(J, K) , \]  

(32)

where \( I \) symbolises the initial quantum number. This equality reflects the completeness of the initial modes. Special care is required concerning the derivation of an analogue expression involving the out-modes since these solutions are restricted to the region outside the horizon and thereby they are not complete in the full space-time. In order to apply this relation we have to deduce the \( \alpha \)-coefficients as well. For that purpose we define slightly modified Bogoliubov coefficients via

\[ \tilde{\beta}(\omega, \omega') = \sqrt{\omega \omega'} \beta_{\omega \omega'} , \]  

(33)
and in analogy the $\alpha$-coefficient. The modified Bogoliubov coefficients can be analytically continued into the complex $\omega'$-plane where the relations $\tilde{F}_{\text{out}}^* (\omega') = F_{\text{out}} (-\omega')$ and hence

$$\tilde{\alpha}(\omega, \omega') = \tilde{\beta}(\omega, -\omega')$$

(34)

hold. This enables us to derive the Bogoliubov $\alpha$-coefficient for large initial frequencies $\omega$. Substituting $\omega' \to -\omega'$ in Eq. (29) together with the complex conjugation the only difference between $|\alpha|$ and $|\beta|$ is the sign in front of the term $i \xi \omega R$. Dividing the absolute values of the two coefficients all other terms cancel and the convergence factor $\varepsilon$ determines the side of the branch cut of the logarithm in the complex plane. Hence we find for large frequencies $\omega$

$$|\beta_{IJ}| = \exp(-2\pi \omega'R |\alpha_{IJ}|) \left(1 + O \left[ \frac{1}{\omega} \right] \right) .$$

(35)

Inserting Eq. (35) into the completeness relation (32) and considering the singular coincidence $J = K$ it follows

$$N_J = \langle 0_{\text{in}} | N_J^{\text{out}} | 0_{\text{in}} \rangle = \int \int \int |\beta_{IJ}|^2$$

$$= \delta_{\sigma-\xi'} \frac{\delta_-(I, I)}{\exp(4\pi \omega'R) - 1} + \text{finite}$$

$$= \delta_{\sigma-\xi'} \frac{N_V^- V}{\exp(4\pi \omega'R) - 1} + \text{finite} .$$

(36)

According to the results of Ref. [17] the UV-divergence of the $\omega$-integration of the absolute values squared of the $\beta$-coefficients in Eq. (31) exactly corresponds to the singular quantity $\delta_-(I, I) = \delta_-(\omega, \omega)$ and thus represents the near-horizon ($r \downarrow R$, i.e. $r_* \downarrow -\infty$) part $N_V^- V$ of the infinite volume divergence $N_V V = N_V^- V + N_V^+ V$ of the continuum normalisation. As explained in Ref. [17], the infinitely large amount of particles is necessary for (quasi) thermal behaviour in an unbounded volume.

It is also possible to calculate the Bogoliubov coefficients for regular modes (wave packets instead of plane waves), cf. [17]. In this case no divergences occur and all quantities are finite. However, in this situation it is rather difficult to distinguish between the (late-time) Hawking effect and the (collapse-dependent or initially present) finite amount of particles.

As mentioned before, an initial state $\varrho_{\text{in}}$ with a finite number of particles does not change the final results concerning the Hawking effect. Inserting the Bogoliubov transformation the expectation value counting the number of black hole particles equals the Hawking term plus
For a state \( \varrho_{\text{in}} \) that contains a finite number of initial particles the above expectation values vanish in the high (initial) frequency limit \( \omega_I, \omega_K \uparrow \infty \). As a result the \( I \) and \( K \) summations/integrations are not UV-divergent. Hence the additional contributions are finite and do not affect the (divergent) Hawking effect. E.g., if we assume the collapsing star to be enclosed by a (arbitrarily large but finite) box with Dirichlet boundary conditions we may describe an initial thermal equilibrium state via the canonical ensemble. In view of the previous arguments we arrive at the conclusion that any initial temperature does also not affect the final (Hawking) temperature.

With the aid of similar arguments one can show that the Hawking effect – i.e. the late-time radiation – is also independent of the initial metric (as long as it is regular). The number of particles created during the transition from one to another regular metric is finite. These particles disperse after some finite period of time and do not affect the (divergent) late-time part of the radiation in accordance with the arguments in the previous paragraph. In terms of the Bogoliubov coefficients this degree of freedom exactly corresponds to the redefinition of the initial frequency \( \omega \). (We did not need to specify the initial metric \( f_{\text{in}}(r) \) in Sec. 3.2.)

4 Summary

In terms of the Painlevé-Gullstrand-Lemaître coordinates it is possible to model a collapse of a star to a black hole with a manifestly \( C^\infty \)-metric. This set of coordinates possesses two separate branches (labelled by \( \sigma = \pm 1 \)). Depending on the particular branch (i.e. the sign of \( \sigma \)) either only ingoing or only outgoing particles are created in an infinite amount. Since the classification above refers to the behaviour of the modes in the vicinity of the horizon particles that are purely ingoing at spatial infinity are produced in any case (in an infinite amount).
5 Conclusions

The theorems presented in Section 2.1 imply that during every collapse that can be described by a $C^\infty$-metric an infinite number of particles with a thermal spectrum corresponding to the Hawking temperature is created. This assertion was verified for a rather general ansatz for a $C^\infty$-metric in Eq. (19). For that purpose it was neither necessary to impose any conditions on the metric beyond the horizon nor to specify the explicit dynamics of $f(t,r)$ during the collapse (as long as it is regular, i.e. $C^\infty$).

However, the properties of the produced particles crucially depend on the branch of the Painlevé-Gullstrand-Lemaître metric under consideration. Adopting the Schrödinger representation the two distinct branches generate completely different final states $\rho_\sigma$. Only one state represents the phenomenon of black hole evaporation while the other state corresponds to anti-evaporation.

In addition, even if one assumes that only the branch causing evaporation is physically reasonable, the initial state differs – after a collapse according to Eq. (19) – by an infinite amount of particles from the Unruh state. One basic requirement in the definition of that state is the condition 'no ingoing/incoming particles/radiation at spatial infinity'. Ref. [4] states explicitly: Note that we have not defined the vacuum by minimizing some positive-definite-operator expectation value (e.g. the Hamiltonian), but we have defined the vacuum as the state with no incoming particles. It was shown in Ref. [17] that this requirement cannot be used to define the initial vacuum uniquely and consistently. In order to decide whether it may apply to the final state one may consider the Bogoliubov $\beta$-coefficients corresponding to a mode which is purely ingoing at spatial infinity $r_\sigma \uparrow \infty$. Owing to the barrier penetration effect governed by the effective potential $V_{\text{eff}}$ such a mode contains outgoing components as well in the vicinity of the horizon $r_\pm \downarrow -\infty$. Hence the $\omega$-integration of the absolute values squared of the $\beta$-coefficients is UV-divergent indicating an infinite number of created particles.

6 Discussion

Perhaps the most striking outcome of the presented calculation is the fact that – depending on the particular branch $\sigma$ of the dynamics during the collapse – not necessarily an evaporating but also an anti-evaporating black hole may emerge. The phenomenon of anti-evaporation has already been discussed in the literature, see e.g. [15], but in a different context (Schwarzschild-de Sitter black holes). In contrast the calculation in the present article applies to asymptotically flat space-times.

Although the particular asymptotic metrics with $\sigma = \pm 1$ are related to each other via a simple
change of the coordinates the distinction between the different ways of collapse for \( \sigma = +1 \) and 
\( \sigma = -1 \), respectively, cannot be removed by any transformation. (It is not possible to find a 
globally integrating factor for the differential form.) Within the picture of the maximally extended 
Kruskal manifold the two distinct branches of the Painlevé-Gullstrand-Lemaître metric in Eq. (17) contain merely 
the black hole or the white hole singularity, respectively. Nevertheless, the Hawking effect is not the result of a space-time 
singularity but a consequence of the formation of a horizon. For the derivation of the Bogoliubov coefficients no assertions about the 
metric inside the black hole \( f(t \geq 0, r < R) \) are necessary at all. Moreover, an observer at a finite 
spatial distance to the collapsing star cannot distinguish a priori between the two branches 
\( \sigma = \pm 1 \). The different results for \( \sigma = +1 \) and \( \sigma = -1 \) concerning the late-time radiation can 
be traced back to the Planck scale vicinity of the horizon. Adopting the widely accepted point 
of view that general relativity represents the low-energy effective theory of some (still to be 
found) underlying theory assertions concerning the region beyond the Planck scale are a very 
delicate matter. Ergo, without any knowledge about the value of \( \sigma \) during the collapse the most natural 
_ansatz_ for the state governing the measurements of an outside observer is given by (remember the convexity of the states discussed in Sec. 2)

\[
\varrho_0 = \frac{\varrho_+ + \varrho_-}{2}.
\]

(Again we adopt the Schrödinger representation with \( \varrho_\pm = \varrho_\sigma \).) This state describes some kind 
of quasi-thermal equilibrium – it contains the same (infinite) number of ingoing and outgoing 
particles with a thermal spectrum corresponding to the Hawking temperature.

It should be mentioned here that this quasi-thermal equilibrium state does not coincide with 
the Israel-Hartle-Hawking state, which describes (at least with respect to the algebra of observables 
outside the black hole) real thermal equilibrium. The expectation value of the number of particles in the Israel-Hartle-Hawking state \( \varrho_{\text{IHH}} \) exhibits the complete infinite volume divergence, i.e. the near-horizon part \( r_s \downarrow -\infty \) as well as the usual spatial infinity \( r_s \uparrow \infty \), cf. [17].

In contrast the analogue expectation value in the state \( \varrho_0 \) contains the near-horizon part only, 
see Sec. 3.3. As a consequence the renormalised expectation value of the energy density in the 
state \( \varrho_0 \) decreases for large distances \( r \) whereas the same quantity approaches a constant value 
(in view of the Stefan-Boltzmann law proportional to \( T^4 \)) in the Israel-Hartle-Hawking state \( \varrho_{\text{IHH}} \).

The fact that the state after a (rather general) collapse according to Eq. (19) does not coincide 
with the Unruh state may be regarded as the second new result of this article. This discrepancy 
can probably traced back to the different approaches. In Ref. [4] the requirement 'no 
ingoing/incoming particles/radiation' was postulated in order to describe the state representing 
the black hole evaporation via imposing boundary conditions in the static regime, see also
However, as demonstrated in the present article, the initial state does not satisfy these boundary condition after a collapse described by Eq. (19).

In Ref. [2] the dynamical period during the collapse was treated within the geometric optics approximation, i.e. the phase function was obtained via backwards ray tracing. The applicability of this approximation requires some additional conditions on the dynamics of the space-time. It should be mentioned here that the calculation in the present article is not based on the geometric optics approximation but only on the eikonal ansatz which merely represents a consequent expansion into inverse powers of the initial frequency $1/\omega$.

In addition, it turns out that only the region near the horizon generates contributions that are relevant with respect to the Hawking effect. Exactly the leading terms in $1/\omega$ and $\chi$ give rise to the UV-divergence accounting for the Hawking effect. The notion of the vicinity of the horizon as the region that is essential for the Hawking effect may be illustrated via the following gedanken experiment: Let us imagine a very thin shell of matter with slowly decreasing radius. As long as the radius of the shell is larger than the associated Schwarzschild radius the number of created and radiated particles remains finite as a consequence of the regularity of the metric. If the shell were to stop shrinking before it reached its Schwarzschild radius, no Hawking effect would be observed. Accordingly, the creation of particles accounting for the Hawking effect occurs exactly in the space-time region of the formation of the horizon.

Strictly speaking, there exist several definitions of a horizon, for example the event, the apparent, and the putative horizon, cf. [21] and [20]. The notion of the event horizon refers to the global structure of the space-time (asymptotical reachability) whereas the apparent horizon can be defined by strictly local considerations (trapped surfaces). Together with some additional requirements (e.g. asymptotical flatness, cf. [20]) also the putative horizon represents a local condition: ‘time slows to a stop’. Within our investigations we always refer to a locally defined horizon – such as the apparent horizon.

As demonstrated in Ref. [17], a regular spherically symmetric space-time without horizon does not allow for the definition of ingoing and/or outgoing particles. The eigenmodes are standing waves, i.e. linear combinations of ingoing and outgoing components with equal weights. In view of this observation one might wonder whether the separation of the different branches for the solution of the phase function in Eq. (22) is justified. Indeed, the bouncing-off effect at $r = 0$ mixes the ingoing and outgoing components during static as well as during the dynamical period. In an effectively 1+1 dimensional consideration this ”reflection” may be simulated by an effective boundary condition at $r = 0$. Selecting appropriate coordinates the point $r = 0$ becomes time-dependent. E.g., in terms of length and time scales associated to an outside observer the centre of the collapsing star goes to infinity (asymptotically at a null line) owing to the formation of the horizon. In terms of these particular coordinates the origin $r = 0$
corresponds to an accelerated mirror. Ref. [8] presents a derivation of the Hawking effect based on the moving mirror analogue. However, in terms of the Painlevé-Gullstrand-Lemaître coordinates the origin \( r = 0 \) obeys no time-dependence at all. Ergo the ingoing and outgoing components are asymptotically unaffected in the limit of high frequencies. In addition, we may consider a conceptual clear scenario – where the effective boundary condition at \( r = 0 \) does not contribute at all – described in the following *gedanken* experiment: At first we suppose a small amount of highly charged matter to collapse at the centre of the star forming a tiny extreme Reissner black hole. The surface gravity of such an object vanishes with the result that there is no Hawking radiation (at this stage). After the formation of the small black hole the point \( r = 0 \) is hidden by the corresponding horizon. Consequently, there is no ”reflection” at the origin \( r = 0 \) in this case. (It is possible to define ingoing and outgoing particles separately, cf. [17].) If we now suppose the star (enclosing the tiny black hole) to collapse only the dynamics of the metric might generate a mixing of the different branches (e.g. ingoing and outgoing). Combining the above statements we arrive at the conclusion that the different branches in Eq. (22) are indeed effectively independent – at least if we assume the metric to change fast enough: Imposing the same requirement as already used in Ref. [4] we consider a rapidly collapsing star where no light ray which is ingoing at the beginning of the collapse can escape. In such a scenario no information about a possible ”reflection” at \( r = 0 \) can attain the relevant region (accessible to an outside observer).

7 Outline

The present article considers the most simple example of a quantum field theory, i.e. the neutral, massless, and minimally coupled scalar field \( \Phi \). Further investigations should be devoted to fields obeying more complicated equations of motion. For the spin-zero field example one may incorporate potential terms including masses \( m^2 \Phi^2 \) or conformal couplings \( R \mu^2 \Phi^2 / 6 \) and consider charged (i.e. non-Hermitian) fields. Furthermore it would be interesting to extend the examination to fields with higher spin, e.g. the electromagnetic field. Nevertheless, there is no obvious reason why the main conclusions of this article should not persist. The evaluation of the Hawking effect for interacting fields with non-linear equations of motion seems to be rather challenging.

Similarly the space-time under consideration describes the most simple example of a black hole. The Schwarzschild solution represents an uncharged and non-rotating black hole where the Einstein tensor and thereby also the energy-momentum tensor vanish for \( r > 0 \). The extension of the results presented in this article to more general static (i.e. non-rotating) black-
holes – e.g. the Reissner solution – seems to be straight-forward, see also [17]. In contrast the investigation of rotating (i.e. stationary, but not static) black-hole space-times – e.g. the Kerr solution – holds more difficulties.

Apart from the Painlevé-Gullstrand-Lemaître coordinates there are several other coordinate sets that describe a black hole space-time by a manifestly $C^\infty$-metric, e.g. the Eddington-Finkelstein coordinates. It might be interesting to consider a collapse model in terms of these coordinates in analogy to Eq. (19) and to compare the results.

As it became evident in Section 3.3 the properties of the particles created during the collapse depend on its dynamics, in particular on $\sigma$. It could be interesting to investigate the properties of the different time-dependent metrics $g_{\mu\nu}(t, r, \sigma)$, for instance via calculating the associated Ricci tensor $R_{\mu\nu}(t, r, \sigma)$. By virtue of the Einstein equations this quantity reveals the corresponding energy-momentum tensor $T_{\mu\nu}(t, r, \sigma)$ which could be compared with an appropriate model of the collapsing star. In addition it could be used to test the energy conditions.

However, one should be aware that all of the previous considerations neglect the back-reaction of the quantum field onto the metric. So far the quantum field is treated as a test field propagating on a given (externally prescribed) space-time. If one attempts to leave this formalism several problems arise: The concept of Hadamard states as described in Eq. (7) is restricted to free fields obeying linear equations of motion. The two-point function of interacting fields possesses additional singularities in general. Consequently – if one regards the treatment of quantum fields in classical (general relativistic) space-times as a low-energy effective theory of some underlying theory – the imposition of the Hadamard condition is not obviously justified. Similarly the requirement of a smooth $C^\infty$-metric may be questioned from this point of view. Accordingly, it might be interesting to examine the consequences of collapse dynamics that are not $C^\infty$ regarding the Hawking effect.

An exhaustive clarification of these problems probably requires the knowledge of an underlying law that unifies quantum field theory and general relativity.

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