Finiteness of a spinfoam model for euclidean quantum general relativity

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We prove that a certain spinfoam model for euclidean quantum general relativity, recently defined, is finite: all its all Feynman diagrams converge. The model is a variant of the Barrett-Crane model, and is defined in terms of a field theory over $SO(4) \times SO(4) \times SO(4) \times SO(4)$.

I. INTRODUCTION

In reference [1], Barrett and Crane have introduced a model for quantum general relativity (GR). The model is based on the topological quantum $SO(4)$ BF theory, and is obtained by adding a quantum implementation of the constraint that reduces classical BF theory to euclidean GR [2,3]. To make use of the Barrett-Crane construction in quantum gravity, two issues need to be addressed. First, in order to control the divergences in the sum defining the model, the Barrett-Crane model is defined in terms of the $q$-deformation of $SO(4)$. In a realistic model for quantum Euclidean GR, one would like the limit $q \to 1$ to be well defined.

Second, the Barrett-Crane model is defined over a fixed triangulation. This is appropriate for a topological field theory such as BF theory, which does not have local excitations. But the implementation of the BF-to-GR constraint breaks topological invariance and frees local degrees of freedom. The restriction of the model to a fixed discretization of the manifold can therefore be seen only as an approximation. In order to capture all the degrees of freedom of classical GR, and restore general covariance, an appropriate notion of sum over triangulations should be implemented (see for instance [4]). A novel proposal to tackle this problem is provided by the field theory formulation of spin foam models [5,6]. In this formulation, a full sum over arbitrary spin foams (and thus, in particular, over arbitrary triangulations) is naturally generated as the Feynman diagrams expansion of a scalar field theory over a group. The sum over spinfoams admits a compelling interpretation as a sum over 4-geometries. The approach represents also a powerful tool for formal manipulations and for model building: examples of this are Ooguri’s proof of topological invariance of the amplitudes of quantum BF theory in [7] and the definition of a spinfoam model for Lorentzian GR in [8].

Using such framework of field theories over a group, a spinfoam model for Euclidean quantum GR was defined in [9]. This model modifies the Barrett-Crane model in two respects. First, it is not restricted to a fixed triangulation, but it naturally includes the full sum over arbitrary spinfoams. Second, the natural implementation of the BF-to-GR constraint in the field theory context fixes the prescription for assigning amplitudes to lower dimensional simplices, an issue not completely addressed in the original Barrett-Crane model. This same prescription for lower dimensional simplices amplitudes (but in the context of a fixed triangulation) was later re-derived by Oriti and Williams in [12], without using the field theory.

The model introduced in [9] appeared to be naturally regulated by those lower dimensional amplitudes. In particular, certain potentially divergent amplitudes were shown to be bounded in [9]. These results motivated the conjecture that the model could be finite. That is, that all Feynman diagrams might converge. In this letter we prove this conjecture.

This paper is not self-contained: familiarity with the formalism defined in [5,6] is assumed. The definition of the model is summarized in the section II; for a detailed description of the model we refer to [9]. In section III, a series of auxiliary results is derived. The proof of finiteness is given in section IV.

II. THE MODEL

Consider the fundamental representation of $SO(4)$, defined on $\mathbb{R}^4$, and pick a fixed direction $\hat{t}$ in $\mathbb{R}^4$. Let $H$ be the $SO(3)$ subgroup of $SO(4)$ that leaves $\hat{t}$ invariant. The model is defined in terms of a field $\phi(g_1, g_2, g_3, g_4) = \phi(g_i), i = *Andrew Mellon Predoctoral Fellow.
1...4 over $SO(4) \times SO(4) \times SO(4) \times SO(4)$, invariant under arbitrary permutations of its arguments. We define the projectors $P_h$ and $P_g$ as

$$
P_h \phi(g_i) := \int dh_i \phi(g_i h_i), \quad P_g \phi(g_i) := \int dg \phi(g_i g),$$

where $h_i \in H$, and $g \in SO(4)$. The model introduced in [9] is defined by the action

$$
S[\phi] = \int dg_i \left[ P_g \phi(g_i) \right]^2 + \frac{1}{5!} \int dg_i \left[ P_g P_h \phi(g_i) \right]^5,
$$

where $g_i \in SO(4)$, and the fifth power in the interaction term is notation for

$$
\left[ \phi(g_i) \right]^5 := \phi(g_1, g_2, g_3, g_4) \phi(g_5, g_6, g_7) \phi(g_8, g_9, g_{10}) \phi(g_{11}, g_{12}, g_{13}, g_{14}).
$$

$P_g$ projects the field into the space of gauge invariant fields, namely, those such that $\phi(g_1, g_2, g_3, g_4) = \phi(g_5, g_6, g_7, g_8, g_9)$ for all $g \in SO(4)$. The projector $P_h$ projects the field over the linear subspace of the fields that are constants on the orbits of $H$ in $SO(4)$. When expanding the field in modes, that is, on the irreducible representations of $SO(4)$, this projection is equivalent to restricting the expansion to the representations in which there is a vector invariant under the subgroup $H$ (because the projection projects on such invariant vectors). The representations in which such invariant vectors exist are the “simple”, or “balanced”, representations namely the SO representations of $H$ that are constants on the orbits of $H$. When expanding the field in modes, that is, on the irreducible representations of $SO(4)$, this projection is equivalent to restricting the expansion to the representations in which there is a vector invariant under the subgroup $H$ (because the projection projects on such invariant vectors). The representations in which such invariant vectors exist are the “simple”, or “balanced”, representations namely the ones in which the spin of the self dual sector is equal to the spin of the antiselfdual sector. In turn, the simple representations are the ones whose generators have equal selfdual and antiselfdual components, and this equality, under identification of the $SO(4)$ generator with the $B$ field of BF theory is precisely the constraint that reduces BF theory to GR. Alternatively, this constraint allows one to identify the generators as bivectors defining elementary surfaces in 4d, and thus to interpret the coloring of a two-simplex as the choice of a (discretized) 4d geometry $[10,11,1,4]$. Using the Peter-Weyl theorem one can write the partition function of the theory

$$
Z := \int D\phi \ e^{-S[\phi]}
$$

as a perturbative sum over the amplitudes $A(J)$ of Feynman diagrams $J$. This computation is performed in great detail in [9], yielding

$$
Z = \sum_J A(J) = \sum_J \sum_N \prod_{f \in J} \Delta_{N_f} \prod_{e \in J} A_e(N_e) \prod_{v \in J} A_v(N_v).
$$

The first summation is over pentavalent 2-complexes $J$, defined combinatorially as a set of faces $f$, edges $e$ and vertices $v$, and their boundary relations. The second sum is over simple $SO(4)$ representations $N$ coloring the faces of $J$. $\Delta_N$ is the dimension of the simple representation $N$. The amplitude $A_e(N)$ is a function of the four colors that label the corresponding faces bounded by the edge. It is explicitly given by

$$
A_e = \frac{\Delta_{N_1} \ldots \Delta_{N_4}}{(\Delta_{N_1} \ldots \Delta_{N_4})^2},
$$

where $\Delta_{N_1} \ldots N_4$ is the dimension of the space of the intertwiners between the four representations $N_1, \ldots, N_4$ [9]. The vertex amplitude $A_v$ is the Barrett-Crane vertex amplitude, which is a function of the ten colors of the faces adjacent to the 5-valent vertex of $J$. The Barrett-Crane vertex amplitude can be written as a combination of $15j$ symbols.

$^1$Representations of $SO(4)$ can be labeled by two integers $(n_1, n_2)$. In terms of those integers, the dimension of the representation is given by $\Delta_{(n_1, n_2)} = (n_1 + 1)(n_2 + 1)$. Simple representations are those for which $n_1 = n_2 = N$.

$^2$The Feynman diagrams of the theory are obtained by connecting the five-valent vertices with propagators. At the open ends of propagators and vertices there are the four group variables corresponding to the arguments of the field. A 2-complex is given by a certain vertices-propagators topology plus a fixed choice of a permutation on each propagator (see [9]). Strictly speaking, Feynman diagrams of the field theory are given by the five valent graphs. On this 1-skeleton, a 2-complex is defined by each one of the permutations. However, following [9], we will call here “Feynman diagram” each one of such 2-complexes.
However, as it was shown by Barrett in [14], it can also be express as an integral over five copies of the 3-sphere—a representation with a nice geometrical interpretation. This representation is better suited for our purposes so we give it here explicitly

\[
A_v(N_1, \ldots, N_{10}) = \int_{S^3} dy_1 \ldots dy_5 \ K_{N_1}(y_1, y_5) K_{N_2}(y_1, y_4) K_{N_3}(y_3, y_3) K_{N_4}(y_1, y_2) \\
K_{N_5}(y_2, y_5) K_{N_6}(y_2, y_4) K_{N_7}(y_2, y_3) K_{N_8}(y_3, y_4) K_{N_9}(y_4, y_5),
\]

where \( dy \) denotes the invariant normalized measure on the sphere. If we represent the points in the 3-sphere as unit-norm vectors \( y^i \) in \( \mathbb{R}^4 \), and we define the angle \( \Theta_{ij} \) by \( \cos(\Theta_{ij}) = y_i^a y_j^a \delta_{\mu\nu} \), then the kernel \( K_N \) is given by

\[
K_N(y_i, y_j) = \frac{\sin((N + 1) \Theta_{ij})}{\sin(\Theta_{ij})}.
\]

This is a smooth bounded functions on \( S^3 \times S^3 \), with maximum value \( N + 1 \).

### III. Bounds

Our task is to prove convergence of the Feynman integrals of the theory. In the mode expansion, potential divergences appear in the sum over representations \( N \). Therefore we need to analyze the behavior of vertex and edge amplitudes for large values of \( N \).

An arbitrary point \( y \in S^3 \) can be written in spherical coordinates as

\[
y = (\cos(\psi), \sin(\psi)\sin(\theta)\cos(\phi), \sin(\psi)\sin(\theta)\sin(\phi), \sin(\psi)\cos(\theta)),
\]

where \( 0 \leq \psi \leq \pi, 0 \leq \theta \leq \pi, \) and \( 0 \leq \phi \leq 2\pi \). The invariant normalized measure in this coordinates is

\[
dy = \frac{1}{2\pi^2} \sin^2(\psi) \sin(\theta) \ d\psi \ d\theta \ d\phi.
\]

Using the gauge invariance of the vertex, the invariance of the three sphere under the action of \( SO(4) \) and the normalization of the invariant measure, we can compute (7) by dropping one of the integrals and fixing one point on \( S^3 \), say \( y_1 = (1, 0, 0, 0) \). Thus equation (7) becomes

\[
A_v(N_1, \ldots, N_{10}) = \frac{1}{16} \int d\psi_2 \ldots d\psi_5 \ d\Omega_2 \ldots d\Omega_5 \ \sin((N_1+1)\psi_5) \sin((N_2+1)\psi_4) \sin((N_3+1)\psi_3) \sin((N_4+1)\psi_2) \\
K_{N_5}(y_2, y_5) K_{N_6}(y_2, y_4) K_{N_7}(y_2, y_3) K_{N_8}(y_3, y_4) K_{N_9}(y_4, y_5),
\]

where \( d\Omega_i \) is the normalized measure on the 2-sphere \( \psi_i = \) constant. Now we bound the Barrett-Crane amplitude using that \( K_N \leq N + 1 = \sqrt{\Delta_N} \), namely

\[
|A_v(N_1, \ldots, N_{10})| \leq \left( \frac{2}{\pi} \right)^4 \sqrt{\Delta_{N_1} \Delta_{N_2} \Delta_{N_3} \Delta_{N_4} \Delta_{N_5} \Delta_{N_{10}}} \ \prod_{i=1}^{4} \int_0^\pi d\psi_i \ \sin((N_i+1)\psi_i) \sin(\psi_i) \\
\leq \sqrt{\Delta_{N_1} \Delta_{N_2} \Delta_{N_3} \Delta_{N_4} \Delta_{N_5} \Delta_{N_{10}}},
\]

The argument is obviously independent of the choice of the six colors in (12). Weaker versions of the bound in which more colors are included also hold. In particular if we directly bound the \( K_N \)'s in (7) we obtain that the absolute value of the amplitude is bounded by the square root of the product of the ten dimensions. More in general, let \( I_{10}(k) \), with \( k \) taking the values 0, 1, 2, 3, 4, be an arbitrary subset of \( \{1, 2, \ldots, 10\} \) with \( 10 - k \) elements. Then the following bound holds for any \( I_{10}(k) \)

\[
|A_v(N_1, \ldots, N_{10})| \leq \prod_{i \in I_{10}(k)} \sqrt{\Delta_N} \quad \forall \ I_{10}(k), \ k = 0, 1, 2, 3.
\]

For \( k = 4 \) we recover (12).

In [15], Barrett and Williams have analyzed the asymptotic behavior of the oscillatory part of the amplitude, in connection to the classical limit of the theory. We add here some information on the asymptotic behavior of the
where the bounds for the sum on the RHS of the last equation converge.

First, we have \( \Delta_{\text{f}} \), defined in (5), of arbitrary Feynman diagrams of the model. Inserting the inequalities (18) and (13), both with \( \Delta_{\text{f}} \)’s are such that \( \Delta_{\text{f}} = 0, \) in the definition of \( A(J) \), we obtain a bound for the amplitude of an arbitrary pentavalent 2-complex \( J \). Namely,

\[
|A(J)| \leq \prod_{f \in J} \sum_{N_f} (\Delta_{N_f})^{1-2n_f+n_f/2+n_f/4} = \prod_{f \in J} \sum_{N_f} (\Delta_{N_f})^{1-n_f},
\]

where \( n_f \) denotes the number of edges of the face \( f \). The term \( \Delta_{N_f}^{1-n_f} \) in (19) comes from various contributions. First, we have \( \Delta_N \) from the face amplitude. Second, we have \( \Delta_N^{-2n_f} \) and \( \Delta_N^{n_f/2} \) from the denominators of the \( n_f \) edge amplitudes (6) and the bounds for the corresponding numerators (18) respectively. Finally, we have \( \Delta_N^{n_f/2} \) from the bounds for the \( n_f \) vertex amplitudes (13).

If the 2-complex contains only faces with more than one edge, then the previous bound for the amplitude is finite. More precisely, if all the \( n_f \)’s are such that \( n_f \geq 2 \) then \( 1-n_f \leq -1 \), and using the fact that \( \Delta_N = (N + 1)^2 \) the sum on the RHS of the last equation converges.

On the other hand, if some of the \( n_f \) are equal to 1, then the right hand side of (19) diverges, and therefore for this case we need a stricter bound, involving stronger inequalities. This can be done as follows. Notice that every time a 2-complex contains a face whose boundary is given by a single edge, the same edge must be part of the boundary of another face, bounded by more than one edge. In Fig. 1 an elementary vertex of a 2-complex containing such a face is shown. The thick lines represent edges converging at the vertex, each of them is part of the boundary of four faces. To visualize those faces we have drawn in thin lines the intersection of the vertex diagram with a 3-sphere. There is a face bounded by a single edge. Its intersection with the sphere is denoted by \( S \). Notice that the surface intersecting the sphere in \( S' \) will have a boundary defined by at least two edges. Notice also that a single vertex can have a maximum of two such peculiar faces.
Therefore, using (18) for \( k = 1 \) we can choose to bound the numerator in (6) with the colors corresponding to the three adjacent faces like \( S' \) in Fig. 1. If \( N_1 \) denotes the color of the face \( S \) then we construct the bound for the amplitude using

\[
\Delta_{N_1,...,N_4} \leq \sqrt{\Delta_{N_2} \Delta_{N_3} \Delta_{N_4}}.
\]  

One of the square roots would be sufficient to bound the edge amplitude, but the symmetry in the previous expression simplifies the construction of the bound for the amplitude of an arbitrary 2-complex. Then use (13) for \( k = 1 \) or \( k = 2 \) to bound the vertex amplitude corresponding to a vertex containing one (respectively two) face(s) whose boundary is given by only one edge. In this way, we can exclude the color corresponding to these “singular” faces from the bounds corresponding to the vertex and the denominator of the edge amplitude. Thus these faces contribute to the bound as \( \Delta_N \) (face amplitude) times \( \Delta_N^{-2} \) (from the denominator of the single edge amplitude), i.e., as \( \Delta_N^{-1} \).

We keep using (18), and (13) for \( k = 0 \) for faces with \( n_f > 1 \) and vertices containing no faces with \( n_f = 1 \). If we denote by \( \{ f_{(n_f > 1)} \} \) and \( \{ f_{(n_f = 1)} \} \) the set of faces with more than one edge and one edge respectively, the general bound is finally given by

\[
|A(J)| \leq \prod_{f_{(n_f > 1)} \in J} \sum_{N_f} (\Delta_{N_f}^{-1})^{1-n_f} \prod_{f_{(n_f = 1)} \in J} \sum_{N_f} (\Delta_{N_f}^{-1})^{-1} \leq \prod_{f \in J} \sum_{N_f} (\Delta_{N_f}^{-1})^{-1} = (\zeta(2) - 1)^{F_f} \approx (0.6)^{F_f},
\]

where \( F_f \) denotes the number of faces in the 2-complex \( J \), and \( \zeta \) denotes the Riemann zeta function. This concludes the proof of the finiteness of the amplitude for any 2-complex \( J \).

V. DISCUSSION

Equation (21) proves that there are no divergent amplitudes in the field theory defined by (2). This field theory was defined in [8] as a model for euclidean quantum gravity based on the implementations of the constraints that reduce \( SO(4) \) BF theory to euclidean GR. The corresponding BF topological theory is divergent after quantization: regularization is done ad-hoc by introducing a cut-off in the colors or, more elegantly, by means of the quantum deformation of the group \( (SO(4) \to SO_q(4)) \). Remarkably, the implementation of the constrains that give the theory the status of a quantum gravity model automatically regularizes the amplitudes.

Another encouraging result comes from the fact that according to equation (21) contributions of Feynman diagrams decay exponentially with the number of faces. This might be useful for studying the convergence of the full sum over 2-complexes.

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