Radial and elliptic flow in non-central heavy ion collisions can constrain the effective Equation of State(EoS) of the excited nuclear matter. To this end, a model combining relativistic hydrodynamics and a hadronic transport code(RQMD [17]) is developed. For an EoS with a first order phase transition, the model reproduces both the radial and elliptic flow data at the SPS. With the EoS fixed from SPS data, we quantify predictions at RHIC where the Quark Gluon Plasma(QGP) pressure is expected to drive additional radial and elliptic flow. Currently, the strong elliptic flow observed in the first RHIC measurements does not conclusively signal this nascent QGP pressure. Additional measurements are suggested to pin down the EoS.

1. By colliding heavy nuclei at the SPS and RHIC accelerating facilities, physicists hope to excite hadronic matter into a new phase consisting of deconfined quarks and gluons – the Quark Gluon Plasma(QGP) [1]. After the collision, the produced particles move collectively or flow and this flow may quantify the effective Equation of State(EoS) of the matter. In central PbPb collisions at the SPS, a strong radial flow is observed [2]. The matter develops a collective transverse velocity approaching (1/2)c. In non-central collisions, a radial and an elliptic flow are observed [3–5]. Since in non-central collisions the initial nucleus-nucleus overlap region has an elliptic shape, the initial pressure gradient is larger along the impact parameter and the matter moves preferentially in this direction [6].

The phase transition to the QGP influences both the radial and elliptic flows. QCD lattice simulations show an approximately 1st order phase transition [7]. Over a wide range of energy densities $\epsilon = 0.5 - 1.4 \text{GeV/fm}^3$, the temperature and pressure are nearly constant. Over this range then, the ratio of pressure to energy density $p/\epsilon$, decreases and reaches a minimum at a particular energy density known as the softest point, $\epsilon_{sp} \approx 1.4 \text{GeV/fm}^3$ [8]. When the initial energy density is close to $\epsilon_{sp}$, the small pressure (relative to $\epsilon$) cannot effectively accelerate the matter. However, when the initial energy density is well above $\epsilon_{sp}$, $p/\epsilon$ approaches 1/3, and the larger pressure drives collective motion [8,9]. At a time of $\sim 1 \text{fm/c}$, the energy densities at the SPS($\sqrt{s_{NN}} = 17 \text{GeV}$) and RHIC ($\sqrt{s_{NN}} = 130 \text{GeV}$) are very approximately 4 and 7 $\text{GeV/fm}^3$ respectively [10,11]. Based on these experimental estimates, the hard QGP phase is expected to live significantly longer at RHIC than at the SPS. The final flows of the produced particles should reflect this difference. In this paper we pose the question: Can both the radial and elliptic flow at the SPS and RHIC be described by a single effective EoS?

Since the various hadron species have different elastic cross sections, they freezeout (or decouple) from the hot fireball at different times [12]. Because flow builds up over time, it is essential to model this differential freezeout. It was ignored in previous hydrodynamic simulations of non-central heavy ion collisions and elliptic flow was over-predicted flow by a factor of two [13,14].

2. The Hydro to Hadrons(H2H) model will be described in detail elsewhere [15]. Other authors have previously constructed a similar model for central collisions [16]. The model evolves the QGP and mixed phases as a relativistic fluid, but switches to a hadronic cascade (RQMDv2.4 [17]) at the beginning of the hadronic phase to model differential freezeout. The computer code consists of three distinct components. Assuming Bjorken scaling, the first component solves the equations of relativistic hydrodynamics in the transverse plane [6] and constructs a switching surface at a temperature, $T_{switch} = 160 \text{MeV}$. The second component generates hadron on the switching surface using the Cooper-Frye formula [18] with a theta function rejecting backward going particles [19,20]. Finally, the third component (RQMD) sequentially re-scatters the generated hadrons until freezeout.

![FIG. 1. The pressure versus the energy density(\epsilon) for different EoSs (see text). EoSs with Latent Heats 0.4 GeV/fm$^3$, 0.8 GeV/fm$^3$,... are labeled as LH4, LH8,...](image-url)
FIG. 2. The transverse mass slope ($T_{slope}$) as a function of the total charged particle multiplicity in PbPb collisions at an impact parameter of $b=6$ fm (see also [9,16]). For consistency with the elliptic study in Fig. 3, we show $b=6$ fm although the NA49 data points [21] are for $b=0$ fm. For all EoSs at the SPS, the proton slope parameters at $b=6$ fm are $\approx 7$ MeV smaller than at $b=0$ fm, as for the $b=0$ LH8 curve. The difference is negligible for $p^-$. For all EoSs, $T_{slope}$ increases with the collision energy [9,16]. For a soft EoS (e.g. LH\infty) the increase is small and for a hard EoS (e.g. LH8) the increase is large. At RHIC multiplicities, the difference between the slope parameters is large and easily experimentally observable.

4. **Elliptic flow** is quantified experimentally with the number elliptic flow parameter, $V_2 = \langle \cos(2\Phi) \rangle$; here $\Phi$ is the angle around the beam measured relative to the impact parameter and $\langle \rangle$ denotes an average over single particle distribution, $\pi^\pm$. The momentum elliptic flow, $A_2 = (P_T^2 \cos(2\Phi))/(P_T^2)$ is also used [6]. $V_2$ and $A_2$ measure the response of the fireball to the spatial deformation of the overlap region, which is usually quantified in a Glauber model by the eccentricity, $\epsilon = (\langle y^2 - x^2 \rangle)/(\langle x^2 + y^2 \rangle)$ [6]. Since the response ($V_2$ and $A_2$) is proportional to the driving force ($\epsilon$), the ratio $V_2/\epsilon$ is used to compare different impact parameters and nuclei [24,25].

In Fig. 3(a), the number elliptic flow ($V_2$) is plotted as a function of charged particle multiplicity at an impact parameter of 6 fm. Before studying the energy dependence, look at the magnitude of the elliptic flow at the SPS. For LH8, the stars show the pion $V_2$ when the matter is evolved as a fluid until a decoupling temperature of $T_f = 120$ MeV; they further illustrate the excessive elliptic flow typical of pure hydrodynamics. Once a cascade is included, LH8 (the squares) is only $\approx 20\%$ above the data – a substantial improvement. Typically in hydrodynamic calculations, the freezeout temperature $T_f$ is adjusted to fit the proton $P_T$ spectrum. However, protons are driven by a pion “wind” and decouple from the fireball 5 fm/c after the pions on average. This pion wind accounts for the strong proton flow at the SPS and is certainly not described by ideal hydrodynamics [16,15]. In order to match the observed proton flow, hydrodynamic calculations must decouple at low freezeout temperatures, $T_{frz} \approx 120$ MeV/c. Decoupling at low temperatures has three consequences for elliptic flow: First the reduction of elliptic flow due to resonance decays is small $\approx 15\%$, compared to $\approx 30\%$ in the H2H model. Second, compared to a cascade, the hydrodynamics generates twice as much elliptic flow during the late cool hadronic stages of the evolution. Third, the elliptic flow is not reduced by random thermal motion. By including the pion “wind”, and more generally by decoupling differentially, we can simultaneously describe the radial and elliptical flow data at the SPS.

The energy dependence of $V_2$ is the central issue. As seen in Fig. 3, the H2H model predicts an increase in elliptic flow by a factor $\approx 1.4$ and is in reasonable agreement with SPS and RHIC flow data. This result was presented prior to the publication of RHIC data [20]. In contrast, UrQMD, a hadronic cascade based on string dynamics, predicts a decrease by a factor of $\approx 2$ [26]. This is because the UrQMD string model has a Hagedorn limiting temperature, resulting in a super-soft EoS at high energies [27]. For pure hydrodynamics as illustrated by
the stars, $V_2$ is approximately constant [13] (but see [14]). For HIJING [28], a model which considers only the initial parton collisions, $V_2$ is $\approx 0$ [29]. The first RHIC data clearly contradict these models.

The increase in $V_2$ is now used to constrain the EoS of the excited matter. The QCD phase diagram has two distinguishing features. It is soft at low energy densities and subsequently hard at high energies. A RG EoS (the open squares) has no softness and the elliptic flow is clearly too strong both at the SPS and RHIC. The entire family of EoSs, LHS through LH$\infty$, reproduces the elliptic flow data in both energy regimes. Counter-intuitively, as the latent heat is increased $V_2$ first decreases and then increases. In the final count, LHS and LH$\infty$ have roughly the same $V_2$. However, they correspond to two very different hydrodynamic solutions. For LHS, the EoS shifts from hard to soft and the matter explodes, forming two shells of outgoing matter [30]. For LH$\infty$, the EoS is just soft and the matter burns slowly inward, evaporating particles into RQMD. For LH$\infty$ the fireball lives a long time [8] and this is responsible for the significant $V_2$. At RHIC, the fireball lifetime for LH$\infty$ is $\approx 16$ fm/$c$ in contrast to $\approx 9$ fm/$c$ for LHS. For LH$\infty$, $V_2$ builds up slowly but the driving force, the spatial asymmetry $\epsilon$, is only slowly destroyed. Over the long lifetime of the system, the driving force is slowly integrated, and the very soft EoS builds up a large $V_2$. Since LH$\infty$ (the closed circles) is slightly pathological, we compare it to LHS+Soft QGP (the open circles), which has a small speed of sound and which builds up just a little transverse motion. As seen in Fig. 3(a), even a little transverse motion reduces the lifetime and $V_2$ of LH$\infty$ significantly.

The difference between the LHS and LH$\infty$ is further clarified in Fig. 3(b) by plotting the momentum elliptic flow, $A_2$. $A_2$ increases rapidly for LHS and levels off for LH$\infty$. Thus, the super soft EoS LH$\infty$ generates number elliptic flow $V_2$, but not momentum elliptic flow, $A_2$. Eventually pressure wins over lifetime and LHS produces a larger $V_2$ and $A_2$ than LH$\infty$. The $P_T$ dependence of $V_2$ did not discriminate between LHS and LH$\infty$ and both EoSs agreed with the STAR data within experimental errors [15]. For both EoSs, the pressure, is greater than $p_c \approx 75$ MeV/fm$^3$ during the initial phases. Therefore, the current STAR data, while not conclusively signaling the asymptotic pressure in the QGP, does indicate a substantial degree of equilibration for energy densities larger than $\epsilon_c \approx 0.4$ GeV/fm$^3$.

5. Impact Parameter Dependence. In Fig. 4, $V_2$ for LHS as a function of the number participants $(N_p)$ is compared to data. Different EoSs show a similar participant (or b) dependence. The agreement is good at RHIC where the multiplicity is high. For ideal hydrodynamics, $V_2 \propto \epsilon \propto (N_p^{max} - N_p)$ [6]. In the low density limit, since the response is proportional to the number of collisions, $V_2 \propto \epsilon \propto (N_p^{max} - N_p) N_p$. Therefore, $V_2$ has a different $N_p$ (or b) dependence in the hydrodynamic and low density limits [24,25]. At RHIC, except in very peripheral collisions, the $N_p$ dependence is clearly linear and strongly supports the hydrodynamic limit [5]. At the SPS, the $N_p$ dependence may not be clearly linear, but it also does not follow the low density limit. Two-pion correlations, may change the data analysis [31], reduce the elliptic flow in peripheral collisions, and improve the low density agreement. Working against the low density limit, the rapidity dependence of $V_2$ [4] does not follow the $dN/dy$ multiplicity distribution as might naively be expected. Estimates based on the low density limit do correctly predict an increase in $V_2$ with energy/multiplicity [25]. However, given the reasonable success of the current
model, it may be premature to conclude that the early evolution at the SPS is not hydrodynamic.

6. Summary and Discussion. By incorporating differential freezeout, the Hydro to Hadrons[H2H] model simultaneously reproduces the radial and elliptic flow at the SPS and RHIC. At the SPS, the radial flow demands an EoS with a latent heat \( LH \gtrsim 0.8 \text{GeV/fm}^3 \), while elliptic flow demands an EoS with a latent heat \( LH \lesssim 0.8 \text{GeV/fm}^3 \). Further, in contrast to string and collision-less parton models, the increase in \( V_2 \) at RHIC is naturally explained using hydrodynamics. This challenges the prevailing view [5,25] that the SPS is in the low density regime and that the increase in \( V_2 \) represents a transition to the hydrodynamic regime. However, the increase in \( V_2 \) does not uniquely signal the asymptotic QGP pressure. Indeed, at RHIC collision energies, a very soft EoS can have the same \( V_2 \) as an EoS with a well developed QGP phase. This EoS is not academic since softness can mimic non-equilibrium phenomena [22].

To reveal the underlying EoS and the burgeoning QGP pressure, the collision energy should be scanned from the SPS to RHIC. If the prevailing low density view of the SPS is correct, a transition in the \( b \) dependence of elliptic flow should be observed over the energy range [24,25]. In addition, \( V_2 \) and especially \( A_2 \) have a different collision energy dependence for different EoSs (Fig. 3). Taken with the radial flow (Fig. 2), this experimental information would help settle the EoS of hot hadronic matter.

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