Gravitational Entropy and Quantum Cosmology

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Abstract

We investigate the evolution of different measures of “Gravitational Entropy” in Bianchi type I and Lemaître-Tolman universe models.

A new quantity behaving in accordance with the second law of thermodynamics is introduced. We then go on and investigate whether a quantum calculation of initial conditions for the universe based upon the Wheeler-DeWitt equation supports Penrose’s *Weyl Curvature Conjecture*, according to which the Ricci part of the curvature dominates over the Weyl part at the initial singularity of the universe. The theory is applied to the Bianchi type I universe models with dust and a cosmological constant and to the Lemaître-Tolman universe models. We investigate two different versions of the conjecture. First we investigate a local version which fails to support the conjecture. Thereafter we construct a non-local entity which shows more promising behaviour concerning the conjecture.

I. INTRODUCTION

The physics of the arrow of time in our universe seems somehow to involve gravitation [1]. The origin of this arrow may be searched for in the initial state and early evolution of the universe [2].

The isotropy of the cosmic background radiation shows that the universe was to a very high degree in a state of thermodynamic equilibrium already 300.000 years after the Big
Bang. At the present time there are great thermal differences between matter in the stars and that in the interstellar clouds. Matter has developed away from a state of thermal equilibrium instead of towards it, in contradiction to what one could expect from the second law of thermodynamics. This is clearly due to gravity, which increases density fluctuations and hence increases the cosmic temperature differences.

One would like to incorporate this tendency of gravity to produce inhomogeneities into a generalized second law of thermodynamics. Then one needs a quantity representing the entropy of a gravitational field. This quantity should vanish in the case of a homogeneous field and obtain a maximal value given by the Bekenstein-Hawking [3,4] entropy of a black hole, for a gravitational equilibrium configuration in the form of the field of a black hole.

One suggestion in this connection was Penrose’s formulation of what is called the Weyl curvature conjecture (WCC) [5]. Wainwright and Anderson [6] express this conjecture in terms of the ratio of the Weyl and the Ricci curvature invariants,

$$P^2 = \frac{C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}}{R_{\mu\nu}R^{\mu\nu}}$$

According to the conjecture $P^2$ vanishes at the initial singularity of the universe. The physical content of the conjecture is that the initial state of the universe is homogeneous and isotropic. As pointed out by Rothman and Annnios [7,8] the entities $P^2$ and $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ are local entities in contrast to what we usually think of entropy. In this paper we will investigate various entities in the Bianchi type I and the Lemaitre-Tolman models. We will give a careful investigation of both the entities $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ and $P^2$, and show that both of these fail to describe a behaviour according to the WCC. We will therefore introduce a non-local entity which shows a more promising behaviour concerning the WCC. This entity is also constructed in terms of the Weyl tensor, and it has therefore a direct connection with the Weyl curvature tensor but in a non-local form. The local version of the WCC is a much stronger restriction on the cosmological models and it is not strange that it fails for a general model.

The vanishing of the Weyl curvature tensor at the initial singularity is a very special initial condition. Due to the evolution during the inflationary era, however, this condition may be relaxed. A much larger variety of initial conditions are consistent with the homogeneity of the universe, as observed in the cosmic background radiation, in inflationary universe models than in models without inflation. As we shall see from the quantum calculations, an inhomogeneous universe is more likely to be spontaneously created than a homogeneous one. A universe with a large cosmological constant is also more likely to be created than a universe with a small cosmological constant.

In sections II and III, respectively, we present a classical cosmological investigation of $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ and $P^2$ for the Bianchi type I and Lemaitre-Tolman universe models. Both of these entities fail to have the right entropic behavior. We therefore propose in section IV an entity which is non-local and shows a much better behaviour concerning the WCC. It should be noted however that this entity is by no means the true entropy for the gravitational field but it does have the right entropic behaviour. In section V we give a quantum cosmological treatment of the WCC in the stage set by the classical investigations.
II. THE WEYL CURVATURE CONJECTURE FOR A BIANCHI TYPE I
MODEL: LOCAL VERSION

Based on the results obtained in an earlier paper where the Bianchi type I minisuperspace model was investigated [9], we will consider the Weyl curvature conjecture (WCC) for a Bianchi type I model. Writing

\[ ds^2 = -dt^2 + e^{2\alpha} \left[ e^{2\beta} \right]_{ij} dx^i dx^j \]  

where \( \beta = \text{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \) the Weyl curvature invariant and the Ricci square turn out to be:

\[
C_{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} = 4 \left[ 12(\dot{\beta}_+^2 + \dot{\beta}_-^2)^2 + 3(\dot{\beta}_+^2 + \dot{\beta}_-^2) + 3\dot{\alpha}^2(\ddot{\beta}_+^2 + \ddot{\beta}_-^2) 
- 12\dot{\alpha}\dot{\beta}_+(\dot{\beta}_+^2 - 3\dot{\beta}_-^2) + 3\dot{\alpha} \frac{d}{dt}(\dot{\beta}_+^2 + \dot{\beta}_-^2) - \frac{d}{dt}\dot{\beta}_+(\ddot{\beta}_+^2 - 3\ddot{\beta}_-^2) \right]
\]

\[
R_{\mu\nu} R_{\mu\nu} = 36(\dot{\beta}_+^2 + \dot{\beta}_-^2)^2 + 90\dot{\alpha}^2(\ddot{\beta}_+^2 + \ddot{\beta}_-^2) + 36\dot{\alpha}^2 + 12\ddot{\alpha}^2 
- 36\dot{\alpha}(\dot{\beta}_+\dot{\beta}_+ + \dot{\beta}_-\dot{\beta}_-) + 6(\ddot{\beta}_+^2 + \ddot{\beta}_-^2) - 36\dot{\alpha}(\ddot{\beta}_+^2 + \ddot{\beta}_-^2) - 36\ddot{\alpha}\dot{\alpha}^2
\]

A. The Weyl Curvature Tensor for a classical Bianchi type I model

In the classical theory we have no factor ordering problems. Inserting the classical expressions found in [9] and expressing them in terms of the volume element \( v = e^{3\alpha} \), we get the Weyl curvature invariant:

\[
C_{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} = \frac{16 A^4}{27 v^4} \left[ 2 + 2\sqrt{1 + \frac{3Mv}{A^2} + \frac{3\Lambda v^2}{A^2}} \cos 3\gamma + \frac{3Mv}{A^2} + \frac{3\Lambda v^2}{A^2} \right]
\]

where the \(-\) is for expanding and \(+\) is for contracting solutions. \( A \) is an anisotropy parameter, \( v \) is the 3-volume element, \( M \) is the total mass of the dust, \( \Lambda \) is a cosmological constant and \( \gamma \) is an angular variable which correspond to different Kasner solutions. Except for degenerate cases this curvature invariant diverges as \( \frac{1}{v^4} \) for small values of \( v \). For large values of \( v \) (if the classical equations allow it) this invariant tends to zero. If we write \( z = \frac{\ddot{v}}{A} = \pm \sqrt{1 + \frac{3Mv}{A^2} + \frac{3\Lambda v^2}{A^2}} \) we can write:

\[
C_{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} = \frac{16 A^4}{27 v^4} (1 - 2z \cos 3\gamma + z^2)
\]

As expected, the Einstein-deSitter case, \( A = 0 \), yields a zero Weyl tensor. The Weyl tensor is also zero in a few other cases. Let us from now on assume that \( A \neq 0 \). Then iff\(^1\) the Weyl tensor is zero, then \( \cos 3\gamma = \pm 1 \) and \( z = \pm 1 \). If \( \Lambda = M = 0 \) these solutions correspond to the line element:

\[
ds^2 = -dt^2 + A^2 t^2 dx^2 + dy^2 + dz^2
\]

\(^1\text{Iff}=\text{If and only if}\)
which is easily seen to be locally flat space, if we make the transformation:

\[
\begin{aligned}
T &= t \cosh Ax \\
X &= t \sinh Ax
\end{aligned}
\]

The last degenerate case where the Weyl tensor is zero, is only a special hypersurface in space time. It is the hypersurface where

\[
v = \frac{M}{-\Lambda}
\]

\((\Lambda < 0, M \neq 0)\).

All in all we can say that:

*The Weyl curvature invariant, \(C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}\), will for the family of Bianchi Type I universes with dust and a cosmological constant, diverge as \(\frac{1}{v^4}\) near the initial singularity, except for a set of models of measure zero.*

This seems to come in conflict with the WCC in its strongest form.

Wainwright and collaborators [6,10] suggested that it is not the Weyl curvature invariant, \(C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}\), that would represent a “Gravitational entropy”, but the invariant \(P^2 = \frac{C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}}{R^{\mu\nu}R_{\mu\nu}}\). Let us also check out this more restrictive form of the WCC.

The expression for \(P^2\) in a Bianchi type I model is

\[
P^2 = \frac{4}{27} \frac{A^4}{v^2} \frac{1 + z^2 - 2z \cos 3\gamma}{M^2 + 2\Lambda M v + 4\Lambda^2 v^2}
\]

If the universe contains dust, this entity diverges as \(\frac{1}{v^2}\) near the initial singularity. In the absence of dust, this entity diverges as the Weyl tensor; the Ricci square is just a constant. Even this more restrictive form of the WCC does not seem to agree with the classical Bianchi type I universe.

**B. The WCC: Local version**

What conclusions can we draw from these results? Both the entities \(C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}\) and \(P^2\) diverge near the origin. Thus the local version of the WCC is set in doubt. Based on the above investigations we have to conclude:

- If the WCC is meant to be correct in its strongest form, we have to conclude that the classical Bianchi type I model, does not give a good description of the initial stages of the universe. The models are too simple to take into account the true behaviour of the universe.

- The entity suggested by Wainwright and collaborators are more likely to describe a “gravitational entropy”.

- Including a positive cosmological constant the universe is unconditionally driven to isotropization and \(\frac{C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}}{R^{\mu\nu}R_{\mu\nu}} \rightarrow 0\) as \(v \rightarrow \infty\).
III. THE WEYL CURVATURE CONJECTURE FOR THE LEMAÎTRE-TOLMAN MODELS: LOCAL VERSION

Choosing universal time gauge we can write the Lemaître-Tolman (L-T) models as:

\[ ds^2 = -dt^2 + Q^2 dr^2 + \frac{1}{Q^2} (d\theta^2 + \sin^2 \theta d\phi^2) \]  \hspace{1cm} (6)

where \( Q = Q(r, t) \) and \( R = R(r, t) \). Classically the metric will be a solution of the Einstein field equations where the spacetime contains a cosmological constant \( \Lambda \) and dust. The energy-momentum tensor for dust in co-moving coordinates is given by \( T_{\mu\nu} = \text{diag}(\rho, 0, 0, 0) \). The classical equations will now turn into

\[ R' = F Q \]

where \( F = F(r) \) is an arbitrary function, and

\[ \frac{1}{2} R R'' + \frac{1}{2} (1 - F^2) R - \frac{\Lambda}{6} R^3 = m \]  \hspace{1cm} (7)

The function \( m = m(r) \) is given by the integral \( m(r) = \int_0^r 4\pi \rho R^2 R' \, dr \). The square root of the Weyl curvature scalar in the L-T models is:

\[ \sqrt{C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}} = \frac{2}{\sqrt{3}} \left| \frac{1}{R} \left( \frac{R''}{R} - \frac{R^2}{R^2} - \frac{Q' R'}{QR} \right) - \frac{\ddot{R}}{R} + \frac{\dot{Q}}{Q} \frac{\dot{Q} R}{QR} + \frac{\ddot{R}}{R^2} \right| \]  \hspace{1cm} (8)

Using the results obtained in [11] we will investigate the WCC in the L-T models.

The classical value

Inserting eq. (7) and the classical solution \( R' = F Q \) into eq. (8) we can write the Weyl scalar for all of the L-T models as:

\[ \sqrt{C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}} = 4 \frac{\sqrt{3}}{3 \pi R^3} \left| 1 - \frac{m' R}{3R'' m} \right| \]  \hspace{1cm} (9)

If we assume that we can write \( R(r, t) = a(t) r \) (which is possible in the FRW and the deSitter models), the Weyl tensor will be zero for models where

\[ m \propto r^3 \]

The FRW models do have this behaviour, thus their Weyl tensor vanishes.

The Schwarzschild black hole case (which is now in Lemaître coordinates) has \( m' = 0 \) and the Weyl tensor squared consists solely of the Kretschmann scalar. The square root of the Weyl scalar diverges as \( R^{-3} \).

Let us define a mean dust density function \( \bar{\rho} \) by the relation:

\[ m(r) = \frac{4}{3} \pi \bar{\rho} R^3 \]  \hspace{1cm} (10)
Then we can write the Weyl curvature invariant as:

\[ C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} = \frac{16^2}{3}\pi^2(\bar{\rho} - \rho)^2 \]

\(m(r)\) is a function of \(r\) only, so

\[ \frac{\dot{\rho}}{\rho} = -3\frac{\dot{R}}{R} \]

The physical meaning of the Weyl curvature invariant is now clear; it measures the difference between the actual dust density and the mean dust density. If \(\rho = \bar{\rho}\) everywhere, we have a FRW universe with a cosmological constant, thus a zero Weyl tensor. If the Weyl curvature invariant should be ever increasing then the function \(d \equiv \bar{\rho} - \rho\) has to obey: \(dd > 0\).

Let us check out the Weyl tensor more explicitly. Near the initial singularity \((R \to 0)\) we can write the solution of the L-T model for all the models approximately as:

\[ R \approx \left(\frac{9}{2m}\right)^{\frac{1}{3}}(t - t_0(r))^\frac{2}{3} \tag{11} \]

which leads to

\[ 4\pi(\bar{\rho} - \rho) = \frac{3m}{R^3} \cdot \frac{2t'_0}{2t'_0 - \frac{m'}{m}(t - t_0)} \]

To avoid intersecting world-lines we have to assume \(t'_0(r) < 0\) [12]. The above entity will therefore diverge as \(R \to 0\) unless \(t'_0 = 0\) which is the FRW case. Thus, the Weyl scalar invariant will diverge when \(R \to 0\) in the generic L-T model.

Let us also check out the scalar suggested by Wainwright and collaborators. The Ricci square in the classical L-T model is:

\[ R^{\mu\nu}R_{\mu\nu} = 4\Lambda^2 + 16\pi\rho\Lambda + 64\pi^2\rho^2 \]

We can therefore write

\[ P^2 = \frac{4}{3}\left(\frac{1 - \frac{\kappa}{\rho}}{\Lambda^{2}(\frac{4\pi\rho}) + 1}\right)^2 \tag{12} \]

In the small \(R\) limit we can use the preceding solution. The \(\Lambda\) term may be neglected (\(\rho\) diverge), so near the initial singularity the entity behaves as

\[ P^2 \propto \left(\frac{2mt'_0}{m'(t - t_0)}\right)^2 \tag{13} \]

Thus, we have to conclude that (unless \(t'_0 = 0\))

\[ \frac{\partial P^2}{\partial t} \biggr|_{R \to 0} < 0 \tag{14} \]
Bonnor [13] considered the model $1 - F^2 > 0$ with a zero cosmological constant upon which he studied the behaviour of the same entity. He did however reach the opposite result. But as he clearly points out, he inserted the condition $t'_0 = 0 \text{ by hand}$. He assumed an isotropic initial singularity, thus demanding that the Weyl tensor is zero initially. As we see, with a more general assumption the entity suggested by Wainwright and collaborators will initially be a decreasing function of time.

To see what is actually happening if we push $t'_0(r)$ towards zero in the case $1 - F^2 > 0$, we define a new time variable $T = t - t_0(r)$. Now a dot means derivative with respect to $T$. Close to $T = 0$ we can write

$$R^3 = \frac{9}{2} m T^2 g(r, T)$$

where $g(r, T) \approx 1 - \epsilon(r) T^2$ and $\epsilon(r) > 0$. The entity $P^2$ now becomes

$$P^2 \propto \frac{1}{(gm')^2} \left[ g' - t'_0 \left( \frac{2mg}{T} + mg \right) \right]^2$$

(15)

For $t_0(r)$ small but non-zero $P^2$ will diverge at the origin. As $T$ increases $P^2$ will rapidly decrease, until the $g'$ part becomes dominant in eq. (15). $P^2$ reaches a minimum before it will increase again. As $t'_0$ is driven towards zero, this minimum approaches zero, and at $t'_0 = 0$, the minimum will be at $T = 0$ where $P = 0$. We see that the behaviour at the initial singularity is drastically altered near $t'_0 = 0$.

**Large $R$ limit**

In the large $R$ limit and in the presence of a cosmological constant, we have the deSitter-like solutions. As $R$ heads for infinity, we can approximate these solutions as $R \propto e^{Ht}$. The ratio $\frac{\bar{\rho}}{\rho}$ is then approximately constant. The Wainwright entity will therefore go as

$$P^2 \propto R^{-6} \approx e^{-6Ht}$$

It is exponentially decreasing as a function of $t$, and at the end of the inflationary era, it is close to zero.

In a $1 - F^2 = 0$ matter dominated universe, we can set $\Lambda \approx 0$ to obtain the solutions of eq. (11). The behavior of the entity $P^2$ is now exactly the same as for the small $R$ limit, but the relation (14) will hold for any $R$. Even in this case the WCC is set in doubt.

**IV. GRAVITATIONAL ENTROPY: REVISITED**

As we have seen the entity $P^2$ diverges badly at the initial singularity, and decreases as $(t - t_0)^{-2}$ shortly after the initial singularity in the L-T model. Firstly this seems rather contra-intuitive. The Weyl tensor squared measures the square of the difference between the mean dust density $\bar{\rho}$ and the actual dust density $\rho$. However, both of these densities diverge at the initial singularity. Shortly after the Big Bang both are reduced as the universe expands, and $\rho$ is only reducing as $(t - t_0)^{-1}$ compared to $\bar{\rho} \propto (t - t_0)^{-2}$. If the gravitational
entropy is infinite at the Big Bang singularity, the total entropy also has to be infinite, which is contrary to the second law of thermodynamics.

Let us consider a finite 3-volume $V$ in our space time. According to the first law of thermodynamics the matter entropy $S_M$, the internal energy $U$ and the number of particles $N$ in $V$ will evolve as:

$$TdS_M = dU + pdV - \mu dN \quad (16)$$

If this volume is co-moving then $dN = 0$. If the matter content is dust then $p = 0$. There are a couple of things to note. Firstly in a dense dust cloud, we expect the internal energy of the dust to be high, thus we expect the entropy to be high. Secondly the entropy is an increasing function of the volume. Let us therefore consider a co-moving volume $V$ in our spacetime. What could the expression for a gravitational entropy be? From the previous sections we noticed that an L-T model with $\bar{\rho} = \rho$ is homogeneous. The quantity $\bar{\rho} - \rho$ is an inhomogeneity measure in the L-T models. In the absence of a cosmological constant we notice that

$$P = \frac{2}{\sqrt{3}} \left( \frac{\bar{\rho} - \rho}{\rho} \right) \quad (17)$$

The sign of $P$ is here chosen so that the configuration $\bar{\rho} < \rho$ is associated with a $P < 0$ while the more realistic configuration $\bar{\rho} > \rho$ has $P > 0$. The matter entropy is a function of the volume $V$, thus the gravitational entropy ought to be so too. We therefore consider the entity defined by:

$$S = \int_V PdV \quad (18)$$

Introducing co-moving coordinates $x^i$ we write $dV = \sqrt{h}d^3x$ where $\sqrt{h}$ is the 3-volume element. The integration range is now constant as a function of time and if we integrate over a unit coordinate volume which is so small that the integrand is approximately constant, we may write

$$S = \int_V PdV \approx P\sqrt{h}$$

In both articles [9,11] we considered co-moving dust so we can use these results directly. We will therefore investigate the entity

$$\Pi = P\sqrt{h} = \pm \sqrt{h} \left( \frac{C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}}{R_{\mu\nu}R^{\mu\nu}} \right)^{1/2} \quad (19)$$

In the L-T model we can choose the sign $\pm$ as mentioned above. We will however consider only the case where $\Pi \geq 0$.

A. Behaviour of $\Pi$ in the L-T model

In the L-T model the entity $\Pi$ turns into
\[ \Pi = \frac{P}{F} R^2 R' \]  

(20)

In the small \( R \) limit we can ignore the \( \Lambda \) term. If \( \Lambda = 0 \), \( \Pi \) is simply

\[ \Pi = \frac{2R^3 |\dot{\rho}|}{\sqrt{3F} \rho} \]

In the limit \( R \to 0 \) we get:

\[ \Pi = \frac{8}{3\sqrt{3}} \frac{m|m'_0|}{Fm'} (m'(t - t_0) - 2t'_0) \]  

(21)

which is finite for \( t - t_0 = 0 \) and increasing thereafter. This entity has the right behaviour.

In the deSitter limit with \( \Lambda > 0 \) the universe is expected to approach homogeneity. In this limit the \( \Pi \) will go as

\[ \Pi = \frac{2}{\sqrt{3F\Lambda}} \left( \frac{\bar{\rho} - \rho}{\rho} \right) \]  

(22)

Since \( \bar{\rho} \) will approximately have the same time evolution as \( \rho \) in the deSitter limit, \( \Pi \) will asymptotically evolve towards a constant. The solutions in the deSitter limit can be written as

\[ R = \exp\left( \sqrt{\frac{2}{3}} t + \frac{m}{3} f(r) \right) \]

where \( f(r) \) is some function of \( r \) determined by eq. (7). The expression for \( \Pi \) is now

\[ \Pi = \frac{2}{\sqrt{3F\Lambda}} m^2 m^2 f'(r) \]

The physically realistic cosmologies has \( f'(r) > 0 \) thus \( f(r) \) is an increasing function of \( r \). The relation between \( f(r) \), \( m \), \( F(r) \) and \( t_0(r) \) is not calculated here but can be found by using the result obtained by Zecca [14]. The qualitative behaviour is, however, expressed in the present relation.

It appears as if the deSitter state is a stable state. Recall that for a stable thermodynamical state, the entropy will be constant. Interestingly since \( \Pi \neq 0 \) in the deSitter limit, some of the information of the initial state is present.

The evolution of the entity \( \Pi \) for the L-T model can be summarized:

**Large \( \Lambda \) and ever expanding:** In the initial epoch the dust dominates and \( \Pi \) is increasing linearly in \( t \). The universe is becoming more and more inhomogeneous. After the universe has grown considerably, the cosmological constant becomes dominant, \( \Pi \) stops growing and if the cosmological constant is large enough, it decreases asymptotically towards a constant value. The universe is smoothened out.

**Small \( \Lambda \) and ever expanding:** Again the dust dominates initially. The cosmological constant is too small to make \( \Pi \) decreasing. The entity \( \Pi \) is ever increasing but is bounded from above by a relatively large constant value.

**Zero \( \Lambda \) and ever expanding:** The \( \Pi \) will again be ever increasing and will asymptotically move towards a function \( f(t) = c + bt^p \) where \( c \) and \( b \) are constants and \( p = 3 \) iff \( F^2 > 1 \) and \( p = 1 \) iff \( F^2 = 1 \).
• **Recollapsing universe:** Due to the dust term, the final singularity will not be symmetric to the initial singularity. Hence this entity is asymmetric in time for a recollapsing universe.

In the L-T models we see that $\Pi$ behave just like the WCC suggests.

### B. $\Pi$ in the Bianchi type I model

We can now use the same entity $\Pi$ for the Bianchi type I model which we “derived” for the L-T model. The expression for $\Pi$ in the Bianchi type I model is

$$
\Pi = \sqrt{\hbar} P = \frac{2A^2}{3\sqrt{3}} \left( \frac{1 + z^2 - 2z \cos 3\gamma}{M^2 + 2\Lambda M v + 4\Lambda^2 v^2} \right)^{\frac{1}{2}}
$$

where $\sqrt{\hbar} = v$. Near the initial singularity we can make a Taylor expansion of $\Pi^2$ to first order in $v$:

$$
\Pi^2 \approx \frac{8}{27} \frac{A^4}{M^2} (1 - \cos 3\gamma) \left( 1 + \left( \frac{3M}{2A^2} - \frac{2\Lambda}{M} \right) v \right)
$$

Interpreting $m = \frac{M}{A}$ as the dust density [9] in coordinate space, we see that as $v \to 0$, $\Pi \to \frac{2A}{3\sqrt{3}m}$. But $\Pi$ only increases iff $3m^2 > 4\Lambda$. Thus if $m \neq 0$ this inequality is satisfied for $\Lambda \leq 0$. Investigating the classical solutions for $\Lambda > 0$ more closely we see that iff $3m^2 = 4\Lambda$, the volume element $v(t)$ can be written as a part of the exponential map. Iff $3m^2 > 4\Lambda$ and $3m^2 < 4\Lambda$, $v(t)$ can be written as a part of the hyperbolic cosine and hyperbolic sine respectively. Thus if the solutions expand too rapidly, the dust term does not manage to increase the value of $\Pi$ immediately after the initial singularity.

Also in the Bianchi type I case the deSitter limit will yield a constant value of $\Pi$:

$$
\lim_{v \to \infty} \Pi = \frac{A}{3\Lambda^{\frac{1}{2}}}
$$

In the absence of a cosmological constant the result is simply:

$$
\Pi = \frac{2A}{3\sqrt{3}m} \left( (2 + 3mt)(1 - \cos 3\gamma) + \frac{9}{4} m^2 t^2 \right)^{\frac{1}{2}}
$$

In this case $\Pi$ is monotonically increasing. For large $t$ we have $\Pi \approx \frac{A}{\sqrt{3}} t$, which is according to the same power-law as in the “flat” and $\Lambda = 0$ case of the L-T model.

To summarize our investigation of $\Pi$ in the Bianchi type I model we can say that the entity $\Pi$ at the initial singularity is a constant determined by the inverse of the dust density. For $3m^2 > 4\Lambda$ it will increase immediately after the initial singularity. In most cosmological considerations it is assumed that the cosmological constant is small. The exception is in the inflationary era in which the vacuum energy dominates over all other matter degrees of freedom. The inflationary era will smooth out anisotropies as well as inhomogeneities, and the behavior of $\Pi$ in this case is therefore expected. In the deSitter limit $\Pi$ will asymptotically move towards a constant. Large $\Lambda$ means small value, while small $\Lambda$ corresponds to a large
value. This is in full agreement with the L-T models. It is also interesting that the entity Π is very sensitive to different matter configurations. This makes it a lot easier to check whether the Π has the right entropic behaviour.

It also appears to us that the Kasner universe is a sort of “Worst case scenario”. It has a horrendous initial singularity, a cigar-shaped singularity where the solution space is completely disconnected from the isotropic FRW universe. Also the apparently flat case, cos3γ = 1, possess a severe and devastating non-smooth singularity at t = 0. In this case the singularity is that of a cone and is a true singularity in the smooth structure of the space time [9]. The entity Π is for v = 0 a continuous function in γ so the cos3γ = 1 case is not particularly different from the cases cos3γ ≠ 1. It seems that while the Schwarzschild black hole case serves as an upper bound in the inhomogeneous case the Kasner universe serves as an upper bound for the anisotropic cosmologies.

In figure 1 we have plotted Π as a function of V ≡ v in the case of a positive cosmological constant. By investigating the entity Π we note that there is one particular case where Π is a constant as a function of V. It is the case where $3m^2 = 4\Lambda$ and $\cos 3\gamma = \frac{1}{2}$ and it corresponds to a line element that can be written [9]:

$$
\begin{align*}
 ds^2 &= -\frac{d\tau^2}{(1 - \sqrt{3\Lambda}\tau)^2} \\
 &\quad + \frac{1}{(1 - \sqrt{3\Lambda}\tau)^{\frac{4}{3}}} \left[ \tau^{\frac{4}{3}\cos(\frac{\pi}{9})} dx^2 + \tau^{\frac{4}{3}\cos(\frac{2\pi}{9})} dy^2 + \tau^{\frac{4}{3}\cos(\frac{7\pi}{9})} dz^2 \right] 
\end{align*}
$$

(24)

In figure 2 we have plotted Π$^2$ as a function of V in the case of a negative cosmological constant. The expanding solutions can be mapped onto the contracting solutions by changing the sign of cos3γ. Thus if cos3γ = 0 the entity Π will be symmetric with respect to the turning point where the universe turns from an expanding to a contracting phase. We note that only if cos3γ > 0 the function Π will be larger in the contracting phase than in the expanding phase. The case where cos3γ = 0 is symmetric with respect to the turning point of the evolution of the Bianchi type I universe. Thus one might say that this case is the most symmetric of the Bianchi type I universes, while the cases cos3γ = ±1 are the most asymmetric cases concerning the time evolution$^2$.

$^2$If we look at the spacetimes these solutions correspond to, we note something interesting. The case cos3γ = 1 Kasner universe corresponds to the conar-like universe described in [9]. The (local) symmetry group of the spatial section of all these related spacetimes (with matter content) can be expanded to $\mathbb{R} \times \text{Sym}(E^2) = \mathbb{R} \times \text{ISO}(2)$. This is also the case for the solutions described by cos3γ = −1. These solutions lie just on the opposite side of the Kasner circle from the conar-like universes cos3γ = 1. Thus these solutions (the conar-like universe cos3γ = 1, and the “anticonar”-like universe cos3γ = −1) have actually a larger (local) symmetry group than the other solutions cos3γ ≠ ±1.
V. QUANTUM COSMOLOGY AND THE WEYL CURVATURE CONJECTURE

In the previous section we investigated the entity $\Pi$ in the classical cosmologies of the L-T model and in the Bianchi type I model. We saw that is had a very promising behaviour. What can the theory of Quantum Cosmology say about its behaviour? Is a small initial value of $\Pi$ more likely than a large value? Again we will use the results from the two papers [9,11] and we will start by investigating the L-T models.

A. The L-T model

The full wave function for the universe is a linear combination of particular solutions of the Wheeler-DeWitt equation:

$$\Psi = \sum_i C(i) \rho(i) \Psi_i$$

where $i$ runs over some index set. In the paper [11] we considered semi-classical tunneling wave functions. The universe was tunneling from a matter-dominated universe which was classically confined to a finite size, into a deSitter like universe. After the tunneling across the classically forbidden region the universe described vacuum-dominated models, similar to an inflationary model of the universe. Thus in the semi-classical approximation the wave function will have a superspace current which will flow asymptotically towards deSitter-like solutions.

The actual expectation values from entities like $C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$, $P^2$ and $\Pi$ are not explicitly obtained for these models, because the actual calculations suffer from “endless” expressions and highly time-consuming quantities. We will therefore give a more general description of the evolution of the Weyl tensor for the L-T models.

Let us ask the question: Is it more likely for a universe with small Weyl tensor to tunnel through the classical barrier than a universe with a large Weyl tensor? If this was the case this would have been in agreement with the WCC. The question is difficult to answer in general but we shall make some simple considerations in order to shed some light upon it.

We assume that the dust density near the origin of the coordinates is larger than further out. The homogeneous mass function goes as $m_h = \frac{4}{3}\pi \rho_h R_h^3$ where $\rho_h$ is a constant in $r$. If the dust density is larger near the origin of the $r$-coordinate than for larger values of $r$ then $m(r) \geq m_h(r)$ if we demand equality only at $r = 0$ and at $r = r_{\text{max}}$ (we only look at closed universes). This will not in general change the size of the universe so we can look at the effects from $m(r)$ alone. The wavefunction of the inhomogeneous universe is similar to the wavefunction of a point particle moving in a potential $V(R) = \frac{k^2}{2m} \left[ -mR + \frac{1}{2}(1 - F^2)R^2 - \frac{4}{3}R^4 \right]$ with zero energy. Since $m(r)$ is greater in general for an inhomogeneous universe than for a homogeneous universe, we see that the potential barrier will be smaller for an inhomogeneous universe than for a homogeneous universe. Thus apparently an inhomogeneous universe will

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3The geometric expression for the Ricci square in terms of the line element 6, filled after consulting the Mathematica package CARTAN, over 40 -forty- pages of print!
tunnel more easily through the classical barrier than the homogeneous universe. Since an inhomogeneous universe will have a larger Weyl tensor than an almost homogeneous one, we see that universes with large Weyl tensor tunnel more easily than those with a small Weyl tensor.

If we look at the tunneling amplitude concerning effects from the Λ term, it is evident that larger Λ will yield a larger tunneling probability. In the initial era inhomogeneities will increase the value of Π. We saw that an inhomogeneous state will tunnel more easily through the potential barrier than a homogeneous state. The largest tunneling probability amplitude thus occurs for universes with a large cosmological constant and large local inhomogeneities. From a classical point of view the value of Π initially was large (but increasing thereafter), but as the universe entered the inflationary era the cosmological constant was relatively big so the value of Π at the end of the inflationary era is relatively small.

In the initial epoch the universe is not believed to be dust dominated. The dust does not exert any pressure and dust particles do therefore not interact with each other. A more probable matter content is matter which has internal pressure. Even though gravitation tends to make the space inhomogeneous, internal pressure from the matter will try to homogenize the space. Since dust is the only matter source in our model, the model only indicates the tendency for gravitation itself to create inhomogeneities.

Since the universes tunnel into a deSitter-like state, the cosmological constant will rapidly dominate the evolution. The larger the cosmological constant the lower will the entity Π be after the inflationary era ends.

B. The Bianchi type I model

The general solution of the WD-equation for the Bianchi type I models is (no scalar field):

$$\Psi(v, \beta) = \int d^2 k \left[ C(\vec{k}) \psi(\vec{v}) \rho(\vec{k}) e^{i \vec{\beta} \cdot \vec{k}} \right]$$

where $$\psi(\vec{v}) = v^{-\frac{5}{2}} W_{\mu}(2Hv)$$ is a particular solution of the WD equation (with dust), $$C(\vec{k})$$ is a “normalizing constant”, and $$\rho(\vec{k})$$ is a distribution function in momentum space. This distribution function satisfies the equation:

$$\int d^2 k |\rho(\vec{k})|^2 = 1$$

There is still a factoring problem in turning the classical entities to operators. Let us first investigate the expectation value of $$C^{\alpha \beta \gamma \delta} C_{\alpha \beta \gamma \delta}$$. Working in the small $$v$$ limit it does not seem possible to avoid the horrendous $$\frac{1}{v^4} \left( \frac{\partial^2}{\partial \beta_1^2} + \frac{\partial^2}{\partial \beta_2^2} \right)^2$$ which will contribute with a term:

$$\int d^2 \beta \Psi^* \frac{1}{v^4} \left( \frac{\partial^2}{\partial \beta_1^2} + \frac{\partial^2}{\partial \beta_2^2} \right)^2 \Psi = \frac{1}{v^4} \int d^2 k |C(\vec{k})|^2 |\psi(\vec{v})|^2 |\rho(\vec{k})|^2 |\vec{k}|^4$$
Unless we have a delta-function distribution at \( \vec{k} = 0 \); \( \rho(\vec{k}) = \delta^2(\vec{k}) \), the contribution from this term to the Weyl Curvature invariant will diverge as \( v^{-4} \). Thus, we have to conclude that, in the small \( v \) limit the expectation value of \( C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} \) goes as:

\[
\langle C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} \rangle \propto \frac{1}{v^4}
\]  

just as in the classical case.

Investigating the invariant \( R^{\mu\nu}R_{\mu\nu} \), we notice that things are not so easy. The Ricci square also has a term which presumably would contribute with a \( k^4 \) term. However, looking at the classical expression we see that the Ricci square is independent of the anisotropy parameter \( A \). This indicates that at the classical level all terms involving the anisotropy parameters, have to cancel exactly. This is not the case quantum mechanically. In the quantum case operators do not necessarily commute. Hence we may have contributions from terms which classically would cancel each other. In other words, the fact that the classical vacuum has \( R_{\mu\nu} = 0 \), does not mean that the quantum vacuum has \( \hat{R}_{\mu\nu} = 0 \).

We assume that \( (\xi_j) \) is a set of factor-ordering parameters which represents the “true” quantum mechanical system in such a way that \( \xi_j = 0 \) represents the classical system. With this parameterization of the factor ordering we would expect the Ricci square expectation value to be:

\[
\langle R^{\mu\nu}R_{\mu\nu} \rangle = 4\Lambda^2 + 2\Lambda \frac{M}{v} + \frac{M^2}{v^2} + \frac{f_j(v)}{v^4}\xi_j + \mathcal{O}(\xi_j^2)
\]

where \( f_j \) is some function of \( v \) which has the property: \( v \approx 0, \quad f_j(v) \approx constant \). Thus for small \( v \) and \( \xi_j \) the expectation value would behave as

\[
\langle R^{\mu\nu}R_{\mu\nu} \rangle \propto \frac{f_j(0)}{v^4}\xi_j
\]

and

\[
\frac{\langle C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta} \rangle}{\langle R^{\mu\nu}R_{\mu\nu} \rangle} \propto \frac{1}{f_j(0)\xi_j} \cdot constant
\]  

The Weyl square divided by the Ricci squared is in general finite as \( v \to 0 \) for a quantum system. The actual value at \( v = 0 \) is however strongly dependent on the factor-ordering. As the factor ordering parameters approach zero, the value will diverge. Quantum effects in the early epoch are essential for the behaviour of this entity near the initial singularity. As \( v \to 0 \) we expect the quantum effect to be considerable, thus expecting that the factor-ordering parameters \( \xi_j \) to be large.

As indicated already in section II the quantum mechanical expectation value of \( \hat{P} \) will be lower and presumably finite in the initial singularity. Therefore the expectation value of \( \hat{\Pi} \) is also presumed to be considerably lower in the initial stages than its classical counterpart.

Comparing different tunneling amplitudes in the Bianchi type I model is difficult and more speculative because the Bianchi type I universe has no classically forbidden region for \( \Lambda \geq 0 \). This causes the lowest order WKB approximation to be purely oscillatory. The lowest order WKB wavefunction will therefore be approximately constant. In the paper [9] we did however construct under some assumptions a wavefunction which clearly peaked at small values of the anisotropy parameter. Thus these wavefunctions predicts universes that have a relatively low value of \( \Pi \).
VI. CONCLUSION AND DISCUSSION

When Penrose first suggested the Weyl curvature conjecture, the conjecture seemed reasonable because of the tendency for gravitation to clump matter together and form inhomogeneities. Several investigations of WCC showed that things were not so simple and some authors set the WCC in doubt. Most of these authors investigated local entities like $P^2$. In a more recent paper by Rothman [8] it appears as if Penrose even tried to withdraw the statement. But as we have shown in this paper there is no reason to withdraw the statement as long as we stick to non-local entities. Also Rothman and Anninos [7,8] investigated a non-local entity$^4$ and their conclusion is similar to ours$^5$. Even if we cannot say that the Weyl tensor is directly linked to the gravitational entropy, we have shown that a certain non-local entity which is constructed from it has an entropic behaviour, and reflects the tendency of the gravitational field to produce inhomogeneities.

In the inflationary era the non-local entity $\Pi$ asymptotically evolved towards a constant. This is believed to be the correct behaviour in an inflationary era. The massive amount of vacuum energy tends to smooth out any inhomogeneities or anisotropies which is present from the pre-inflationary era. That this entity did not tend to zero (even though the comoving volume expanded exponentially) could be interpreted to tell that $\Pi$ contained some of the information from the pre-inflationary era. There were some seeds of the inhomogeneities left which could be the seeds needed to form galaxies as we see them today.

ACKNOWLEDGEMENTS

We would like to thank S. A. Johannessen for great support in the final stages of the work.

\footnote{They used the volume of the phase space, similar to what we do for classical thermodynamical systems.}

\footnote{When this paper was almost finished a paper by Pelvas and Lake [15] appeared, where they also investigated local entities like $P^2$. They also concluded that these entities cannot be a measure of gravitational entropy.}
REFERENCES

(a) $\cos 3\gamma = -1$

(b) $\cos 3\gamma = \frac{1}{2}$

(c) $\cos 3\gamma = 1$
FIG. 1. Plots of $\Pi(V)$ for the Bianchi type I model with different values of $\cos 3\gamma$. The cosmological constant is set to $\Lambda = \frac{3}{4}$ and in each plot graphs with $m = \frac{1}{2}, 1$ and 2 are drawn. We have also chosen $A = \frac{3}{2}\sqrt{3}$ so that for all of the above cases $\Pi \rightarrow 1$ in the limit $V \rightarrow \infty$. 
(a) \( \cos 3\gamma = -1 \)

(b) \( \cos 3\gamma = 1 \)

(c) \( \cos 3\gamma = -\frac{1}{2} \)
FIG. 2. Plots of $II(V)^2$ for the Bianchi type I model with different values of $\cos 3\gamma$. The cosmological constant is set to $\Lambda = -\frac{3}{4}$ and in each plot graphs with $m = \frac{1}{2}, 1$ and 2 are drawn. The expanding phase can be mapped onto the contracting phase by changing the sign of $\cos 3\gamma$. $A$ is chosen to be equal to $\frac{3}{2}\sqrt{3}$. 