Vanishing of cosmological constant in nonfactorizable geometry

T. Padmanabhan and S. Shankaranarayanan

IUCAA, Post Bag 4, Ganeshkhind, Pune 411 007, INDIA.
email: paddy@iucaa.ernet.in, shanki@iucaa.ernet.in

We generalize the results of Randall and Sundrum to a wider class of four-dimensional space-times including the four-dimensional Schwarzschild background and de Sitter universe. We solve the equation for graviton propagation in a general four dimensional background and find an explicit solution for a zero mass bound state of the graviton. We find that this zero mass bound state is normalizable only if the cosmological constant is strictly zero, thereby providing a dynamical reason for the vanishing of cosmological constant within the context of this model. We also show that the results of Randall and Sundrum can be generalized without any modification to the Schwarzschild background.

PACS numbers: 11.10Kk, 04.50.+h

Recent developments in string theory have shown that if matter fields are localized on a 3-brane in 1 + 3 + d dimensions, and gravity is allowed to propagate in the extra dimensions, then the extra dimensions can be large [1–4]. In this scenario, the Planck scale \( M_P \) is traded for the size of the extra dimensions felt by gravity and gauge coupling unification can occur at scales as low as a TeV.

Randall and Sundrum (RS, hereafter) [5] have shown that these extra dimensions in five-dimensional space-times need not be compact and that four-dimensional gravity can naturally arise at large distances. They showed that there exists a zero mass mode which looks like a four-dimensional graviton bound to the brane, together with a continuum of massive Kaluza-Klein modes arising from the linear fluctuations about the three-brane (four-dimensional Minkowski) background. The zero mode produces the standard 1/r gravitational potential along the brane, and the Kaluza-Klein modes give rise to corrections of order 1/r^3.

The corrections to the Newtonian gravitational potential \( V_N(r) \propto (m_1 m_2/r) \) have been investigated earlier by several authors from different points of view. Duff [6], for example, takes the Schwarzschild metric as the lowest order solution and has obtained quantum gravitational corrections. Since the lowest order corrections have to be linear in \( \hbar \), it is obvious from dimensional grounds that the correction will multiply \( V_N \) by a factor of the form \( [1 + a(G\hbar/c^3 r^2)] \) where \( a \) is a numerical coefficient. [While this is the leading quantum correction, it may be noted that the lowest order post-Newtonian approximation will give a correction of the form \( [1 + b(G(m_1 + m_2)/c^2 r)] \), where \( b \) is a numerical factor, which is a slower fall-off with distance]. Donoghue [7] has obtained similar results by treating gravity as an effective field theory. In the case of RS, there is no background Schwarzschild metric and they merely study the graviton perturbations around the flat four-dimensional spacetime. Their approach is essentially to look at the corrections to the graviton propagator arising from a set of continuum states with mass \( m > 0 \). The analysis by itself is classical and indeed, the corrections to \( V_N \) which they find does not depend on \( \hbar \) directly; of course, they provide an interpretation which is quantum mechanical. In contrast, much of the earlier work, concentrated on quantum gravitational corrections, have used the background Schwarzschild line element.

This raises the question: Is it possible to generalize the ideas of RS to a situation in which the four-dimensional metric is nontrivial (say, a Schwarzschild metric or de Sitter universe)? Will we get the same mass spectrum for the graviton modes and the same correction term to \( V_N(r) \)? The fact that Duff and Donoghue obtained similar results suggests that this could be the case — though it is far from obvious.

In this Letter, we show that the main results of RS have a simple mathematical origin and can indeed be generalized to a wider class of models. We will provide a general solution to the zero mass graviton mode in arbitrary background and — as an illustration — will work out explicitly the case that incorporates a spherically symmetric solution in four dimensions. (This will include as special cases, the Schwarzschild and de Sitter manifolds.) It is important to show that the properties of the graviton propagation, and the effective gravitational potential does not change under such a generalization. We shall provide exact solutions which demonstrate that such is indeed the case; these solutions also provide some insight into the structure of the solution and will possibly allow us to study — for example — models for black hole evaporation in this context.

We will follow the work of RS closely to provide easy comparison. We begin with a general metric in five dimensions of the form

\[
\text{ds}^2 = g_{ab} \text{d}x^a \text{d}x^b = \exp(-2a(y)) \left[ g^{(4)}_{\mu\nu} \text{d}x^\mu \text{d}x^\nu \right] - dy^2, \tag{1}
\]

with the condition that it satisfies the full five dimensional Einstein’s equation with a five dimensional cosmological constant. The metric signature we adopt is \((+ - - - -)\). We use the lowercase Latin letters for the
We will now proceed to study the plane wave gravitons, $h_{\mu\nu}$, propagating in the above space-time. Denoting the perturbed metric by $g_{ab} = g_{ab} + h_{ab}$ and using the gauge

$$h_{55} = h_{5\mu} = 0, \quad \nabla^\mu h_{\mu\nu} = 0, \quad h_{\mu}^\mu = 0, \quad (2)$$

it is easy to see that $h_{\mu\nu}$ can be written as plane wave gravitons, i.e., $h_{\mu\nu} = e^{i\omega t} \Phi$ where $e_{\mu\nu}$ is the polarization tensor. The equation satisfied by $\Phi$ can be separated with the ansatz $\Phi(x^\mu, y) = A(x^\mu)Z(y)$. Substituting into the wave equation, separating the variables using a constant $m^2$, we find that $A$ satisfies the standard wave equation for a particle of mass $m$ while $Z$ satisfies the equation:

$$\frac{d^2 Z}{dy^2} + (-4\dot{a}^2(y) + 2\ddot{a}(y) + m^2 \exp[2a(y)]) Z = 0, \quad (3)$$

where the dot denotes the derivative with respect to $y$. This reduces to equation $(8)$ of RS, when we use their solution $a(y) = k|y|$. We are interested in the allowed range of values for $m$ and whether we can get an acceptable solution for $m = 0$. By inspection, it is clear that this equation has a solution for $m = 0$, given by

$$Z = \exp[-2a(y)]. \quad (4)$$

In fact, this is precisely the ground state wave function which RS obtain (after a series of algebraic transformations!) for their special case of $a(y) = k|y|$. The physical meaning, mathematical simplicity and generality of the result is hidden by: (i) their transformations and (ii) the fact that they never give $\psi(y)$ but only $\dot{\psi}(z)$ in their paper. [Note that Eq. (4) is a valid solution to Eq. (3) with $m = 0$ as long as $a(y)$ is continuous even if its derivative is discontinuous at the origin.]

This is the first result of this Letter and shows that the existence of a zero mass graviton is a very general result and does not require much of the extra assumptions in RS except that $Z$ should be well behaved and normalizable as a function of $y$, in the relevant range. The question arises as to the conditions under which we will obtain a normalizable function for $Z(y)$. Such an analysis for a general $a(y)$ is complicated and hence we will illustrate it explicitly for a special case. We shall solve the five dimensional Einstein’s equations assuming that the four dimensional spacetime is spherically symmetric and the metric is of the form

$$ds^2 = \exp(-2a(y)) \left[ A(r) dt^2 - B(r) dr^2 - r^2 d\Omega^2 \right] - dy^2, \quad (5)$$

where, $d\Omega^2$ is the angular line element and $a(y)$, $A(r)$ and $B(r)$ need to be determined via the five-dimensional Einstein’s equations. We consider the latter to be of the form

$$G_{ab} = \Lambda g_{ab} \quad (6)$$

with possible non-zero vacuum energy density $\Lambda$ in five dimensions. Inserting the ansatz $(5)$ for the metric, the only non-vanishing components of the Einstein tensor, $G_{\mu\nu}$, are the diagonal components. The Einstein’s equation, for $(00)$ and $(11)$ components, reduces to

$$\frac{1}{r^2} - \frac{1}{r^2 B(r)} + \frac{B'(r)}{r^2 B^2(r)} = \exp[-2a(y)]R(y) \quad (7)$$

$$\frac{1}{r} \left[ \frac{1}{r^2} - \frac{B'(r)}{r^2} + \frac{A'(r)}{r A(r)} \right] = \exp[-2a(y)]R(y) \quad (8)$$

where,

$$R(y) = \Lambda + 6\dot{a}^2(y) - 3\ddot{a}(y), \quad (9)$$

and the prime denotes derivative with respect to $r$. Combining the two equations, we obtain $B(r) = 1/A(r)$. Substituting for $B(r)$, and solving the resulting equations, along with the (22) and (33) components of the Einstein’s equation, gives $A(r)$ to be

$$A(r) = 1 - \frac{C}{r} - \frac{\lambda}{3} r^2, \quad (10)$$

where $C$ and $\lambda$ are the constants of integration. This four-dimensional metric is the well known Schwarzschild - de Sitter metric for the choice of $C > 0$, where $\lambda$ is the four-dimensional cosmological constant, in the sense that the four dimensional metric with $A(r)$ given by Eq. (10) corresponds to a four dimensional space-time with this cosmological constant. [We use the term cosmological constant in four dimensions in the above sense and it should not be confused with the other possible ways of defining the cosmological constant — for example, from the brane tension etc. Note that the sign of $\lambda$ is still undetermined.] Substituting the form of $A(r)$ in the original equations (along with the (55) component of the Einstein’s equation), the differential equation for $a(y)$ becomes

$$\frac{d^2 a(y)}{dy^2} = \frac{\lambda}{3} \exp(2a(y)). \quad (11)$$

It is clear that the conformal factor will have only the $\lambda$ dependence and will be independent of $C$. [Normally the four dimensional space-time can have a non-vanishing cosmological constant only when there is a source in the right hand side of the four dimensional Einstein’s equations. In our case, if we write the five dimensional $G_{ab}$ in terms of four dimensional Einstein tensor $G_{\mu\nu}$ and extra terms arising from the fifth dimension, it is possible to show that the effective source for $G_{\mu\nu}$ exactly that corresponding to a four dimensional cosmological constant $\lambda$.]

Solving the Eq. (11), it is easy to obtain the form of $a(y)$ such that it reduces to the RS result of $a(y) = k|y|$ when $\lambda = 0$. We get

$$\exp[-2a(y)] = \exp[2k|y|] \left[ \exp[-2k|y|] - (\lambda/12k^2) \right]^2 \quad (12)$$
with $k$ being a constant related to $\Lambda$ by $\Lambda = -6k^2$. This shows that $\Lambda < 0$ for an acceptable solution. Equations (12), (10) with the result $A(r) = 1/B(r)$ completely determine the metric. The modulus sign in $|y|$ will make the derivatives of $a(y)$ discontinuous at the origin $y = 0$ which can be taken to be the location of the membrane as in the RS case.

Eq. (12) allows us to draw an important conclusion which is the second key result of this Letter. Note that the conformal factor $Z = \exp[-2a(y)]$ depends on $\lambda$ but not on $C$. In the limit of $\lambda \to 0$ the conformal factor for the four-dimensional world line element is same as in the RS model. Thus, the original analysis of RS can be generalized without any modifications to the case in which the four dimensional spacetime is described by Schwarzschild line-element [$\lambda = 0$, $C > 0$ in equation (10)] as well suggesting that the zeroth order gravitational interaction, in the form of Schwarzschild line element, gets “corrected” by the conformal factor. This could possibly be the reason why the one-loop corrections to Schwarzschild metric in the earlier analysis of Duff [6] also gives similar result.

The condition on the four-dimensional cosmological constant $\lambda$ is more interesting. The ground state wave function $Z = \exp[-2a(y)]$ in (12) is not normalizable for $\lambda \neq 0$ and hence we do not get a massless [$m = 0$ in equation (3)] graviton for $\lambda \neq 0$. An examination of the general solution to (11) confirms this conclusion. Using $Z = \exp[-2a(y)]$, the first integral to (11) can be written as:

$$\frac{dZ}{dy} = \pm \left(4\beta_1 Z^2 + \frac{4\lambda}{3} Z \right)^{1/2},$$

where $\beta_1$ is the constant of integration. For $\beta_1 < 0$, the solution is oscillatory with nodes and hence is not of interest. For the case $\beta_1 = k^2 > 0$, we obtain the solution to be

$$Z = -\frac{\lambda}{6k^2} + \frac{1}{16k^2} \exp(\pm 2k(y - y_0))$$
$$+ \frac{\lambda^2}{9k^2} \exp(\mp 2k(y - y_0)),$$

where $y_0$ is the constant of integration. In the case of $\lambda = 0$, the wave function ($Z$) is normalizable and reduces to the ground state wave function obtained by RS with a suitable choice of the signs for $y > 0$ and $y < 0$ [We take the solution to be varying as $\exp(-2ky)$ for $y > 0$ and $\exp(2ky)$ for $y < 0$ with the membrane being located at $y = 0$]. However, when $\lambda \neq 0$, the wave function is not bounded as $|y| \to \infty$ (for any combination of signs in the argument of the exponential) and hence is not normalizable for non-zero $\lambda$. This is because the third term on the right hand side of Eq. (14) (which is non-zero when $\lambda \neq 0$) comes with an argument to the exponential having a different sign compared to the second term. This shows clearly that the nature of the solution for $Z(y)$ — which acts as the ground state wave-function for zero mass graviton mode — is very different when $\lambda \neq 0$ compared to the case considered by RS.

To conclude, we have shown that the results of RS can be generalized for the Schwarzschild black hole. But the presence of non-zero cosmological constant in four dimensions modifies the RS results. The presence of a non-zero cosmological constant does not provide a normalizable ground state wave function corresponding to the zero mass graviton. Hence, we have obtained a dynamical state wave function within the context of these models.

An interesting alternative scenario would be to use the model by RS in Ref. [4]. In this scenario, we can set up two 3-branes where the 3-branes are extended in the $x_\mu$ directions and are located at some fixed points in the $y$ axis and thus restricting the extra dimensions to be compactified. [In this model, it is assumed that the branes do not contribute to the energy momentum tensor.] By restricting the extra dimensions to be compactified, we can obtain normalizable zero mass gravitons. Such an analysis leads to two different situations depending on whether: (i) $\lambda > 12k^2$ or (ii) $\lambda < 12k^2$. The first possibility, even if $k \approx TeV$, will give a large cosmological constant. The other case, which is more plausible, gives us the upper bound on the compactification scale(radius) of the extra dimensions. These issues are under investigation.

Finally, we would like to mention the following curious fact: In conventional four dimensional general relativity, the cosmological constant appears as a mass term in the linearized spin-2 wave equation. Vanishing of the cosmological constant is required for this equation to be interpreted as representing the massless gravitons. However, it is possible to have arbitrarily small four dimensional cosmological constant with an arbitrarily long range of interaction for gravity. Our analysis here shows that even an arbitrarily small cosmological constant will make the ground state wave-function (corresponding to a massless graviton) to be non-normalizable, requiring the cosmological constant to strictly vanish. Whether there exists a deeper connection between the two results is not clear and is under investigation.

We thank Naresh Dadhich for fruitful discussions and for drawing us into the fifth dimension. We thank K. Subramanian for comments on the earlier draft of the paper. S.S. is being supported by the Council of Scientific and Industrial Research, India.