On the scalar sector of the covariant graviton two-point function in de Sitter spacetime

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Abstract

We examine the scalar sector of covariant graviton two-point function in de Sitter spacetime. This sector consists of the pure-trace part and another part described by a scalar field. We show that it does not contribute to two-point functions of gauge-invariant quantities. We also demonstrate that the long-distance growth present in some gauges is absent in this sector for a wide range of gauge parameters.

1 Introduction

The covariant graviton two-point function in de Sitter spacetime (CGTF) has been studied by several authors \cite{1}–\cite{7}. (See, e.g., \cite{8} for a description of de Sitter spacetime.) It is known that the CGTF grows with the distance between the two points in some gauges \cite{2, 4}, although infrared divergences have been shown to be absent \cite{1}.\textsuperscript{1} In particular, it is known that the pure-trace part of the CGTF grows with the distance in a gauge where the traceless part is divergence-free \cite{4}. (We call this gauge the Landau gauge in this paper.)

\textsuperscript{1}See \cite{9} for a related observation.
Recently, a non-covariant physical graviton two-point function, which contains only the two physical polarizations, was computed in open de Sitter spacetime [10]. It was found that this two-point function does not grow as the distance between the two points increases. Subsequently, the present authors showed [11] that the logarithmic growth of the non-covariant physical two-point function in spatially-flat de Sitter spacetime [12] is a gauge artefact. These results suggest that the seemingly problematic long-distance behaviour in the pure-trace part of the CGTF might also be a gauge artefact. Interestingly, the one-loop effect in pure-trace external gravitational fields has been shown to vanish [6]. In fact one of the present authors (AH) showed under some assumptions [9] that the contribution from the pure-trace part and that from another scalar part of the CGTF together takes a pure-gauge form for a one-parameter family of gauges which includes the Landau gauge as a limit.\(^2\) (We will call these two parts the scalar sector.)

In this paper we prove this fact without making any assumptions, thus establishing that the scalar sector of the CGTF does not contribute to two-point functions of physical quantities at the tree level. We also emphasize that the mass of the scalar sector is gauge dependent and can be chosen to be a value for which there is no long-distance growth.

The rest of the paper is organized as follows. In section 2 we recall some essential facts about the scalar field theory in de Sitter spacetime as a preliminary. In section 3 we examine the field equations of linearized gravity in a one-parameter family of covariant gauges and identify the scalar sector, which will be the focus of this paper. Then, in sections 4 and 5 we present useful tools for calculating the Klein-Gordon inner product of mode functions, which is closely related to the symplectic product, and the commutators of annihilation and creation operators. In section 6 we use these tools to find the Klein-Gordon inner product of the mode functions in the scalar sector of linearized gravity. In section 7 we present our main result, i.e. the scalar sector of the CGTF, and make some remarks about the full CGTF. In Appendix A a technical identity used in section 5 is proved. In Appendix B the symplectic product of traceless scalar modes is calculated by a direct method without using the tool given in section 5. In Appendix C an explicit form of the scalar sector two-point function is presented for a particular value of the gauge parameter, where some simplification occurs.

The metric signature is \((-+++\)) and we set \(c = \hbar = 1\) throughout this paper.

\(^2\)Allen and Turyn [2] have also pointed out that the part which grows at large distances in these two parts of the CGTF is in a pure-gauge form in a particular gauge in the Euclidean approach.
2 Scalar field theory

Most of the discussions in this paper are independent of the explicit metric, but for definiteness we work mainly with the metric

\[ ds^2 = -dt^2 + H^{-2} \cosh^2 Ht \, dS_3^2, \]

where \( dS_3^2 \) is the line element of the unit 3-sphere. The Hubble constant is \( H \) and the cosmological constant is \( 3H^2 \) here.

Let us consider the scalar field theory with the Lagrangian density

\[ \mathcal{L}_S = -\frac{\sqrt{-g}}{2} \left[ \nabla_a \phi \nabla^a \phi + \frac{12H^2}{\alpha - 3} \phi^2 \right], \]

where \( \nabla_a \) is the covariant derivative in the background de Sitter spacetime, and the indices are raised and lowered by the de Sitter metric \( g_{ab} \) given by (1). The mass parameter, \( \frac{12H^2}{(\alpha - 3)} \), is given in a form which is useful later.\(^3\) The corresponding Euler-Lagrange equation is

\[ L^{(S)} \phi \equiv \left( \Box - \frac{12H^2}{\alpha - 3} \right) \phi = 0, \]

where \( \Box = \nabla_a \nabla^a \). We expand the field \( \phi \) as

\[ \phi = \sum_{l\sigma} \left( a_{l\sigma} \phi^{l(\sigma)} + a_{l\sigma}^\dagger \phi^{l(\sigma)} \right), \]

where the mode functions \( \phi^{l(\sigma)} \) are given by

\[ \phi^{l(\sigma)} = f_l(t) Y_{l\sigma} \]

with the \( Y_{l\sigma} \) being spherical harmonics on the unit 3-sphere with angular momentum \( l \). They satisfy \( \Delta Y_{l\sigma} = -l(l+2)Y_{l\sigma} \), where \( \Delta \) is the Laplace operator on the unit 3-sphere. (See, e.g., [13] for a concise description of spherical harmonics in higher dimensions.) The label \( \sigma \) distinguishes the spherical harmonics with the same value of \( l \). We require

\[ \int_{S^3} d\Omega \overline{Y_{l\sigma}} Y_{l'\sigma'} = \delta_{ll'} \delta_{\sigma\sigma'}. \]

The functions \( f_l(t) \) satisfy

\[ \left[ \frac{d^2}{dt^2} + 3H \tanh Ht \frac{d}{dt} + \frac{l(l+2)H^2}{\cosh^2 Ht} + \frac{12H^2}{\alpha - 3} \right] f_l(t) = 0. \]

\(^3\)The theory may not be well defined for some values of \( \alpha \). We avoid these values.
We normalize the functions $f_l(t)$ by requiring the following Wronskian:

$$W(f_l, f_l) \equiv f_l \frac{df_l}{dt} - f_l \frac{df_l}{dt} = - \frac{iH^3}{\cosh^3 Ht}. \quad (8)$$

We define the conjugate momentum vector $\pi^{(i)c}$ of a solution $\phi^{(i)}$ of equation (3) as

$$\pi^{(i)c} = - \nabla^c \phi^{(i)}. \quad (9)$$

Then, given any two solutions $\phi^{(1)}$ and $\phi^{(2)}$, the following current is conserved:

$$J^{(S)c}(\phi^{(1)}, \phi^{(2)}) = \pi^{(1)c} \phi^{(2)} - \phi^{(1)} \pi^{(2)c}. \quad (10)$$

The symplectic product of $\phi^{(1)}$ and $\phi^{(2)}$ is defined by

$$\Omega_S(\phi^{(1)}, \phi^{(2)}) \equiv \int_{\Sigma} d\Sigma \mathcal{J}^{(S)c}(\phi^{(1)}, \phi^{(2)}), \quad (11)$$

where $\Sigma$ is any Cauchy surface and where $d\Sigma = d\Sigma_n$ with $n$ being the future-pointing unit normal to $\Sigma$. Here, $d\Sigma = d\theta_1 d\theta_2 d\theta_3 \sqrt{\eta}$, where the $\theta_i$ are the coordinates on $\Sigma$ and $\eta$ is the determinant of the metric on $\Sigma$. The product $\Omega_S(\phi^{(1)}, \phi^{(2)})$ is independent of the Cauchy surface $\Sigma$ because the current $\mathcal{J}^{(S)c}$ is conserved. If we use a hypersurface with $t = \text{const}$ as $\Sigma$, then

$$\Omega_S(\phi^{(1)}, \phi^{(2)}) = \frac{\cosh^3 Ht}{H^3} \int_{S^3} dS \left[ \phi^{(1)} \frac{\partial}{\partial t} \phi^{(2)} - \phi^{(2)} \frac{\partial}{\partial t} \phi^{(1)} \right], \quad (12)$$

where $dS$ is the volume element of the unit 3-sphere. Now, define the Klein-Gordon inner product of two solutions $\phi^{(1)}$ and $\phi^{(2)}$ by

$$\langle \phi^{(1)} | \phi^{(2)} \rangle_S \equiv i \Omega_S(\phi^{(1)}, \phi^{(2)}). \quad (13)$$

Then we find using (6) and (8)

$$\langle \phi^{(1)} | \phi^{(2)} \rangle_S = \delta_{l' l} \delta_{\sigma \sigma'}, \quad (14)$$

$$\langle \phi^{(1)} | \phi^{(2)} \rangle_S = 0. \quad (15)$$

The usual canonical quantization leads to the commutation relations

$$[a_{l\sigma}, a_{l'\sigma'}^\dagger] = \delta_{l' l} \delta_{\sigma \sigma'}, \quad (16)$$

with all other commutators vanishing. We define the vacuum state $\langle 0 \rangle$ by requiring $a_{l\sigma} |0\rangle$ for all $l$ and $\sigma$. There is some freedom in choosing $f_l(t)$ and, consequently, in choosing the vacuum state. The standard choice corresponds to the so-called Euclidean
vacuum [14], which is invariant under the de Sitter group. We use this vacuum, but the explicit form of $f_i(t)$ is not necessary here (see, e.g., [15]).

The two-point function of the scalar field $\phi$ is expressed as

$$\Delta_+(x, x') \equiv \langle 0 | \phi(x) \phi(x') | 0 \rangle = \sum_{l\sigma} \phi^{(l\sigma)}(x) \phi^{(l\sigma)}(x'),$$  \hspace{1cm} (17)

where $x$ and $x'$ are spacetime points. The explicit form of $\Delta_+(x, x')$ is well known [16]. For example, Allen and Jacobson [17] give it as

$$\Delta_+(x, x') = \frac{\Gamma(a_+) \Gamma(a_-)}{16\pi^2} F(a_+, a_-; 2; z),$$  \hspace{1cm} (18)

where

$$a_\pm = \frac{3}{2} \pm \left( \frac{9}{4} - \frac{M^2}{H^2} \right)^{1/2} \text{ with } M^2 = \frac{12H^2}{\alpha - 3},$$  \hspace{1cm} (19)

$$z = \cos^2 \left( \frac{\mu(x, x') H}{2} \right).$$  \hspace{1cm} (20)

Here, the function $\mu(x, x')$ is the spacelike geodesic distance between points $x$ and $x'$, and $F(a, b; c; z)$ is the hypergeometric function. The variable $z$ can be extended to the case where there is no spacelike geodesic between $x$ and $x'$. If we write the de Sitter metric as

$$ds^2 = \frac{1}{H^2\lambda^2} (-d\lambda^2 + dx^2),$$  \hspace{1cm} (21)

with $x = (x_1, x_2, x_3)$, then it is known that, for $x = (\lambda, x)$ and $x' = (\lambda', x')$,

$$z = \frac{(\lambda + \lambda')^2 - \|x - x'\|^2}{4\lambda\lambda'}. \hspace{1cm} (22)$$

(This can readily be inferred by comparing the two-point functions in [16] and [17].)

Hence, the large-distance limit corresponds to $z \to -\infty$. The large-distance limit of (18), which will be useful later, can be found as $|\Delta_+(x, x')| \sim (-z)^{-a_-}$ if $0 < M^2 \leq 9/4$ and $|\Delta_+(x, x')| \sim (-z)^{-3/2}$ if $9/4 \leq M^2$, up to a constant factor. Thus, if $M^2 > 0$, i.e. if $\alpha > 3$, the scalar two-point function tends to zero as $z \to -\infty$.

### 3 The scalar-sector mode functions

The Lagrangian density for the linearized gravity can be chosen as

$$\mathcal{L}_{\text{inv}} = \sqrt{-g} \left[ \frac{1}{2} \nabla_a h^{ac} \nabla^b h_{bc} - \frac{1}{4} \nabla_a h_{bc} \nabla^a h^{bc} + \frac{1}{4} (\nabla^a h - 2 \nabla^b h^a_b) \nabla_a h - \frac{1}{2} H^2 \left( h_{ab} h^{ab} + \frac{1}{2} h^2 \right) \right].$$  \hspace{1cm} (23)
with $h = h^a_a$. This Lagrangian density is invariant under the gauge transformation

$$h_{ab} \rightarrow h_{ab} + \nabla_a \Lambda_b + \nabla_b \Lambda_a$$

up to a total divergence. Therefore, one needs to fix the gauge for the canonical quantization of $h_{ab}$. For this purpose we add the following gauge-fixing term in the Lagrangian density:

$$L_{gf} = -\sqrt{-g} \left( \nabla_a h^a_b - \frac{1 + \beta}{\beta} \nabla^b h \right) \left( \nabla^c h_{cb} - \frac{1 + \beta}{\beta} \nabla_b h \right).$$ (24)

Then the Euler-Lagrange field equations for $L_{inv} + L_{gf}$ are

$$L_{ab}^{(T) cd} h_{cd} \equiv \frac{1}{2} \Box h_{ab} - \left( \frac{1}{2} - 1 \right) \left( \nabla_a \nabla_c h^c_b + \nabla_b \nabla_c h^c_a \right) + \left[ \frac{1}{2} - \frac{1 + \beta}{\alpha \beta} \right] \nabla_a \nabla_b h + \left[ \frac{(\beta + 1)^2}{\alpha \beta^2} - \frac{1}{2} \right] g_{ab} \Box h + \frac{1}{2} g_{ab} \left( 1 - \frac{2(1 + \beta)}{\alpha \beta} \right) \nabla_c \nabla_d h_{cd} - H^2 \left( h_{ab} + \frac{1}{2} g_{ab} h \right) = 0.$$ (25)

The scalar sector of the field $h_{ab}$ satisfying this equation can be extracted as follows [9]. First by taking the trace of equation (25) we find

$$\left[ \frac{4(1 + \beta)^2}{\alpha \beta^2} - \frac{1 + \beta}{\alpha \beta} - 1 \right] \Box h - 3H^2 h + \left( 1 - \frac{3}{\alpha} - \frac{4}{\alpha \beta} \right) \nabla^a \nabla^b h_{ab} = 0.$$ (26)

There is mixing between the trace $h$ and the traceless part of $h_{ab}$ in general. This mixing can be avoided by choosing the parameter $\beta$ as

$$\beta = \frac{4}{\alpha - 3}.$$ (27)

We make this choice in the rest of this paper. We also assume that $\alpha \neq 0, 3$ though we will consider the limit $\alpha \rightarrow 0$ in the concluding section. Equation (26) now reads

$$\left( \Box - \frac{12H^2}{\alpha - 3} \right) h = 0.$$ (28)

We define the traceless part of $h_{ab}$ by

$$h^{(l)}_{ab} = h_{ab} - \frac{1}{4} g_{ab} h.$$ (29)

By substituting this in (25) with the choice (27) we find

$$\frac{1}{2} \Box h^{(l)}_{ab} - \left( \frac{1}{2} - \frac{1}{2\alpha} \right) \left( \nabla_a \nabla_c h^{(l)c}_b + \nabla_b \nabla_c h^{(l)c}_a \right) + \left( \frac{1}{4} - \frac{1}{4\alpha} \right) g_{ab} \nabla_c \nabla_d h^{(l)cd} - H^2 h^{(l)}_{ab} = 0.$$ (30)
Next, define a scalar field \( B^{(d)} \) by
\[
B^{(d)} = \frac{(\alpha - 3)^2}{36\alpha H^4} \nabla^a \nabla^b h^{(l)}_{ab} .
\] (31)

By using the fact that the Riemann tensor takes the form \( R_{abcd} = H^2(g_{ac}g_{bd} - g_{ad}g_{bc}) \), we find
\[
\left( \square - \frac{12H^2}{\alpha - 3} \right) B^{(d)} = 0 .
\] (32)

(Notice that the two fields \( h \) and \( B^{(d)} \) satisfy the same equation as the scalar field \( \phi \) discussed in the previous section.) Let us write \( h_{ab} \) as
\[
h_{ab} = h^{(r)}_{ab} + h^{(d)}_{ab} ,
\] (33)
where
\[
h^{(d)}_{ab} = \left( \nabla_a \nabla_b - \frac{3H^2}{\alpha - 3} g_{ab} \right) B^{(d)} .
\] (34)

The tensor \( h^{(d)}_{ab} \) is traceless because of equation (32). By taking the divergence of this equation twice, we find
\[
\nabla^a \nabla^b h^{(d)}_{ab} = \frac{36\alpha H^4}{(\alpha - 3)^2} B^{(d)} = \nabla^a \nabla^b h^{(l)}_{ab} .
\] (35)

As a result we have \( \nabla^a \nabla^b h^{(r)}_{ab} = 0 \).

Thus, the field \( h_{ab} \) can be decomposed as
\[
h_{ab} = h^{(r)}_{ab} + h^{(d)}_{ab} + h^{(t)}_{ab} ,
\] (36)
where
\[
h^{(d)}_{ab}(x) = \sum_{l\sigma} \left[ b_{l\sigma} h^{(d,l\sigma)}_{ab}(x) + b_{l\sigma}^\dagger h^{(d,l\sigma)}_{ab}(x) \right] ,
\] (37)
\[
h^{(t)}_{ab}(x) = \sum_{l\sigma} \left[ c_{l\sigma} h^{(t,l\sigma)}_{ab}(x) + c_{l\sigma}^\dagger h^{(t,l\sigma)}_{ab}(x) \right] ,
\] (38)
with
\[
h^{(d,l\sigma)}_{ab} = \left( \nabla_a \nabla_b - \frac{3H^2}{\alpha - 3} g_{ab} \right) \phi^{(l\sigma)} ,
\] (39)
\[
h^{(t,l\sigma)}_{ab} = \frac{1}{4} g_{ab} \phi^{(l\sigma)} .
\] (40)

We call the field \( h^{(d)}_{ab} + h^{(t)}_{ab} \) the scalar sector of \( h_{ab} \).
4 The Klein-Gordon inner product and the commutators

The commutators of annihilation and creation operators in (37) and (38) can be found using a formula relating them to the Klein-Gordon inner product. This formula [see equation (50)] has appeared in various forms in the literature (see, e.g., [18]), but since it is not widely known, we present its derivation in this section. We closely follow Wald [19], who treats the case with a scalar field. The discussions here can be applied to other fields in other spacetimes with little modification.

We first define the symplectic product of mode functions. Suppose that the dynamics of a symmetric tensor field $h_I$ with $I = (a, b)$ in de Sitter spacetime (or any other globally-hyperbolic spacetime) is described by a quadratic Lagrangian density (such as $L_{\text{inv}} + L_{\text{gf}}$ in the previous section),

$$L = \frac{\sqrt{-g}}{2} \left[ T^{abIJ} \nabla_a h_I \nabla_b h_J + S^{IJ} h_I h_J \right], \quad (41)$$

where $\nabla_a$ is the covariant derivative appropriate for the field $h_I$. The functions $T^{abIJ}$ and $S^{IJ}$ depend in general on the spacetime point and satisfy the symmetry properties $T^{abIJ} = T^{baJI}$ and $S^{IJ} = S^{JI}$. [The indices $I = (c, d)$ and $I' = (d, c)$ are treated as independent if $c \neq d$ though the field satisfies $h_{cd} = h_{dc}$. The symmetry property of the field is reflected on the tensors $T^{abIJ}$ and $S^{IJ}$. Thus, if $I = (c, d)$ and $I' = (d, c)$, then, for example, $T^{abIJ} = T^{a'b'J}$ and $S^{IJ} = S^{I'J}$. Repeated indices indicate the usual tensor contraction. For example, $h^I h_I = h^{ab} h_{ab}$.] The conjugate momentum tensor of $h_I$ is defined by

$$\pi^{cI} = \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial (\nabla_c h_I)} = T^{cblJ} \nabla_b h_I. \quad (42)$$

The Euler-Lagrange equations read

$$L^{IJ} h_J \equiv -\nabla_c \pi^{cI} + S^{IJ} h_J = -\nabla_c (T^{cblJ} \nabla_b h_I) + S^{IJ} h_J = 0. \quad (43)$$

Now, suppose that the field $h_I$ is expanded in terms of mode functions $h^{(\kappa)}_I(t, \theta)$, with $\theta$ representing space coordinates in general and $\kappa$ being the label for mode functions, as

$$h_I(t, \theta) = \sum_\kappa A_\kappa^{(\kappa)}(t, \theta). \quad (44)$$

(The label $\kappa$ will be discrete in our case, but the following argument applies even if $\kappa$ is continuous with little modification.) The field $h_I$ is quantized by imposing the
equal-time commutation relations,

\[
\begin{align*}
[h_I(t, \theta), h_J(t, \theta')] &= [n_a \pi^{aI}(t, \theta), n_b \pi^{aJ}(t, \theta')] = 0 \\
[n_a \pi^{aJ}(t, \theta), h_I(t, \theta')] &= i \delta^I_J \delta^3(\theta, \theta'),
\end{align*}
\]

(45)

where the three-dimensional \( \delta \)-function satisfies

\[
\int d\Sigma \delta^3(\theta, \theta') f(\theta') = f(\theta)
\]

(46)

for any smooth and compactly-supported function \( f(\theta) \). We have defined \( \delta^I_J = \frac{1}{2} (\delta^c_a \delta^d_b + \delta^d_a \delta^c_b) \). Note here that, if \( \pi^{tJ} \) is the time component of \( \pi^{aJ} \) with the metric (1), then \( n_a \pi^{aJ} = -\pi^{tJ} \). Thus, commutation relation (45) can also be written as

\[
[h_I(t, \theta), \pi^{tJ}(t, \theta')] = i \delta^I_J \delta^3(\theta, \theta').
\]

From these commutation relations we find for any modes \( h^{(\kappa)}_I \) and \( h^{(\kappa')}_I \)

\[
\left[(h^{(\kappa)}|h)_{T}, (h|h^{(\kappa')})_{T}\right] = (h^{(\kappa)}|h^{(\kappa')})_{T},
\]

(47)

where \( (h^{(\kappa)}|h)_{T} \) is the Klein-Gordon inner product of the mode \( h^{(\kappa)}_I(t, \theta) \) and the field \( h_I(t, \theta) \) itself. Let us define the matrix \( M^{\kappa\kappa'} \) by

\[
M^{\kappa\kappa'} \equiv (h^{(\kappa)}|h^{(\kappa')})_{T}.
\]

(48)

By taking the Klein-Gordon inner product of (44) with the mode \( h^{(\kappa)}_{ab} \) we find

\[
(h^{(\kappa)}|h)_{T} = M^{\kappa\kappa'} A_{\kappa'}. \]

(49)

By substituting this in (47) and multiplying both sides by the inverse matrix \((M^{-1})^{\kappa\kappa'}\) (assuming that it exists), we find

\[
[A_\kappa, A_{\kappa'}] = (M^{-1})^{\kappa\kappa'}. \]

(50)

5 Symplectic product as a spacetime integral

In this section we present a tool which allows us to compute the symplectic product of mode functions as a four-dimensional integral. This is a straightforward generalization of a similar formula known for the scalar field theory [19]. In fact, equation (60), which is the main result in this section, can be used for a scalar field by deleting the indices \( I \) and \( J \).
Let $h_I^{(1)}$ and $h_I^{(2)}$ be two solutions of equation (43) and $\pi^{(1)cI}$ and $\pi^{(2)cI}$ be the corresponding conjugate momentum tensor. Then the current defined by

$$J^{(T)c}(h_I^{(1)}, h_I^{(2)}) = \pi^{(1)cI} h_I^{(2)} - h_I^{(1)} \pi^{(2)cI}$$

(51)

can readily be seen to be conserved [20] by using equation (43). The symplectic product of the two solutions $h_I^{(1)}$ and $h_I^{(2)}$ is defined on $\Sigma$ by

$$\Omega_T(h_I^{(1)}, h_I^{(2)}) \equiv \int_\Sigma d\Sigma_c J^{(T)c}(h_I^{(1)}, h_I^{(2)}) .$$

(52)

This product is independent of the choice of the Cauchy surface as in the scalar case. The Klein-Gordon inner product is defined by

$$(h_I^{(1)}|h_I^{(2)})_T \equiv i \Omega_T(h_I^{(1)}|h_I^{(2)}) .$$

(53)

We assume that, for given Cauchy data $(h_I, \pi^{cI}n_c)$ on a Cauchy surface $\Sigma$, there is a unique solution of $L^{IJ} h_J = 0$ in spacetime. Then, we have unique advanced and retarded Green functions, $G^{(T,A)}_{IJ}(x, y)$ and $G^{(T,R)}_{IJ}(x, y)$, of the operator $L^{IJ}$, where $x$ and $y$ are spacetime points, satisfying

$$L^{IJ}_x G^{(T,A/R)}_{JK}(x, y) = \delta^I(x, y) \delta^J_K$$

(54)

and

$$G^{(T,A)}_{IJ}(x, y) = 0 \text{ if } x \text{ is not in the past of } y ,$$

$$G^{(T,R)}_{IJ}(x, y) = 0 \text{ if } x \text{ is not in the future of } y .$$

(55)

(56)

We have defined $\delta^I(x, y)$ by

$$\int dV_y \delta^I(x, y) f(y) = f(x)$$

(57)

for any smooth and compactly supported function $f(x)$, where

$$dV_y = d^4 y \sqrt{-g(y)} .$$

Now, define the advanced-minus-retarded Green function as

$$E^{(T)}_{IJ}(x, y) \equiv G^{(T,A)}_{IJ}(x, y) - G^{(T,R)}_{IJ}(x, y) .$$

(58)

Note that this is a (generalized) solution of equation (43) unlike the retarded and advanced Green functions themselves. Therefore a field $h_I^{(2)}(x)$ given by

$$h_I^{(2)}(x) = \int dV_y E^{(T)}_{IJ}(x, y) \bar h^{(2)J}(y) ,$$

(59)
where \( \tilde{h}^{(2)j}(y) \) is smooth and compactly supported, satisfies \( L^{IJ} h^{(2)}_I = 0 \). One can in fact show as in the scalar case that any smooth solution \( h^{(2)}_I(x) \) whose support on a Cauchy surface is compact can be expressed in this manner, though the choice of \( \tilde{h}^{(2)j}(y) \) is not unique.\(^4\) A proof of this fact is given in Appendix A for completeness. Equation (59) allows us to write the symplectic product as a spacetime integral. One can express the symplectic product of any smooth solution \( h^{(1)}_I \) of \( L^{IJ} h^{(1)}_J \) (with compact support on a Cauchy surface) and the solution \( h^{(2)}_I \) given in (59) as

\[
\Omega_T(h^{(1)}, h^{(2)}) = \int dV_x \ h^{(1)I}(x) \tilde{h}^{(2)}_I(x) .
\]  

(60)

We will prove this formula in the rest of this section.

It is convenient to define the advanced and retarded parts of the solution \( h^{(2)}_I \) as

\[
h^{(2),A/R}_I(x) = \int dV_y \ G^{(A/R)}_{IJ}(x, y) \tilde{h}^{(2)J}(y) .
\]  

(61)

Then, we have \( h^{(2)}_I = h^{(2),A}_I - h^{(2),R}_I \). If we evaluate the symplectic product on a Cauchy surface \( \Sigma_+ \) such that the support of \( \tilde{h}^{(2)I} \) is contained in its past, only the retarded part \( h^{(2),R}_I \) contributes. Therefore

\[
\Omega_T(h^{(1)}, h^{(2)}) = -\int_{\Sigma_+} d\Sigma_c \left( \pi^{(1)cI} h^{(2),R}_I - h^{(1)}_I \pi^{(2,R)cI} \right) ,
\]  

(62)

where \( \pi^{(2,R)cI} \) is the conjugate momentum tensor of \( h^{(2),R}_I \). Now, consider another Cauchy surface \( \Sigma_- \) such that the support of \( \tilde{h}^{(2)I} \) is contained in its future. Then, since \( h^{(2),R}_I(x) \) vanishes on \( \Sigma_- \), we can add

\[
\int_{\Sigma_-} d\Sigma_c \left( \pi^{(1)cI} h^{(2),R}_I - h^{(1)}_I \pi^{(2,R)cI} \right)
\]  

on the right-hand side of (62). The resulting integral on the right-hand side of equation (62) can be converted into a volume integral by using Gauss’ divergence theorem, with the volume being the region between the two Cauchy surfaces \( \Sigma_- \) and \( \Sigma_+ \). Thus, we obtain\(^5\)

\[
\Omega_T(h^{(1)}, h^{(2)}) = \int dV_x \ \nabla_c \left( \pi^{(1)cI} h^{(2),R}_I - h^{(1)}_I \pi^{(2,R)cI} \right) .
\]  

(63)

By using the definition (42) of \( \pi^{cd} \), we find

\[
\pi^{(1)cI} \nabla_c h^{(2),R}_I - \nabla_c h^{(1)}_I \cdot \pi^{(2,R)cI} = 0 .
\]  

(64)

\(^4\)Note that, since the Cauchy surfaces are compact in de Sitter spacetime, all solutions are compactly supported on any Cauchy surface.

\(^5\)One needs to be careful about the sign here because the time component of \( n_c \) is negative.
The dot here indicates that the derivative operator $\nabla_c$ is applied only on $h^{(1)}_I$. Dots will be used in this manner throughout this paper.) Hence, equation (63) can be written as

$$\Omega_T(h^{(1)}, h^{(2)}) = \int dV_x \left( \nabla_c \pi^{(1)} c^I \cdot h^{(2),R}_I - h^{(1)}_I \nabla_c \pi^{(2),R} c^I \right). \quad (65)$$

By substituting the field equation $\nabla_c \pi^{(1)} c^I = S^{IJ} h^{(1)}_J$ and using the symmetry $S^{IJ} = S^{JI}$, we find

$$\Omega_T(h^{(1)}, h^{(2)}) = - \int dV_x h^{(1)}_I \left( \nabla_c \pi^{(2),R} c^I - S^{IJ} h^{(2),R}_J \right)$$

$$= \int dV_x h^{(1)}_I L^{IJ} h^{(2),R}_J$$

$$= \int dV_x h^{(1)}_I \tilde{h}^{(2),I} \quad (66)$$

as required. The last equality holds because

$$L^{IJ}_x h^{(2),R}_J(x) = \int d^4V_x L^{IJ}_x G^{(T,R)}_{JK}(x,y) \tilde{h}^{(2),K}(y)$$

$$= \tilde{h}^{(2),I}(x). \quad (67)$$

### 6 Calculation of the Klein-Gordon inner product

Since the two-point function of the scalar field is well known, we only need to find the commutators of relevant annihilation and creation operators in order to determine the scalar sector of the two-point function. For this purpose, we compute the Klein-Gordon inner product of the relevant mode functions and then use equation (50).

Let $G^{(S,A/R)}(x)$ be the advanced/retarded Green function for the operator $L^{(S)}$ defined by (3) and let $E^{(S)}(x,y) = G^{(S,A)}(x,y) - G^{(S,R)}(x,y)$. The scalar version of the equality (59) reads

$$\phi^{(l\sigma)}(x) = \int dV_y E^{(S)}(x,y) \tilde{\phi}^{(l\sigma)}(y), \quad (68)$$

where $\tilde{\phi}^{(l\sigma)}(x)$ is a smooth and compactly-supported function. We write $\phi^{(l\sigma)}(x) = \phi^{(A,l\sigma)}(x) - \phi^{(R,l\sigma)}(x)$, where

$$\phi^{(A/R,l\sigma)}(x) = \int dV_y G^{(S,A/R)}(x,y) \tilde{\phi}^{(l\sigma)}(y). \quad (69)$$

We first examine the pure-trace modes $h^{(t,l\sigma)}_{ab}$ defined by (40). We obtain by a straightforward calculation

$$L^{(T)}_{abcd} \left( \frac{1}{4} g_{ca} F \right) = \frac{\alpha - 3}{16} g^{ab} L^{(S)} F \quad (70)$$
for any function $F$, where $L^{(T)}_{abcd}$ is defined by (25). We substitute $\phi^{(A/R,l\sigma)}(x)$ for $F(x)$ in this equation and multiply by the advanced/retarded Green function $G^{(T,A/R)}_{x,abcd}(z, x)$ and integrate over $x$. Then we integrate by parts so that the operator $L^{(T)}_{cdef}$ is applied on $G^{(T,A/R)}_{abcd}(z, x)$. There will be no boundary terms because the common support of $G^{(T,R)}_{abcd}(z, x)$ and $\phi^{(R,l\sigma)}(x)$ with fixed $z$ is compact, and similarly for $G^{(T,A)}_{abcd}(z, x)$ and $\phi^{(A,l\sigma)}(x)$. Then we use

$$I^{(T)}_{x,abcd}G^{(T,A/R)}_{cdef}(z, x) = \frac{1}{2}\left(\delta^b_c \delta^d_a + \delta^a_d \delta^b_c \right) \delta^4(z, x),$$

which follows from the equality $G^{(T,A)}_{abcd}(x, y) = G^{(T,R)}_{cdab}(y, x)$. Thus, we find

$$\frac{1}{4} g_{ab} \phi^{(A/R,l\sigma)}(z) = \frac{\alpha - 3}{16} \int dV_x G^{(T,A/R)}_{abcd}(z, x) \left(g^{cd}(x) \tilde{\phi}^{(l\sigma)}(x)\right).$$

From this we immediately have

$$\frac{1}{4} g_{ab} \phi^{(l\sigma)}(z) = \frac{\alpha - 3}{16} \int dV_x E^{(T)}_{abcd}(z, x) \left(g^{cd}(x) \tilde{\phi}^{(l\sigma)}(x)\right).$$

Let $h^{(1)}_{ab}$ be any smooth solution of equation (25). Then, by equation (60) we find

$$\Omega_T(h^{(1)}_{ab}, h^{(t,l\sigma)}_{ab}) = \frac{\alpha - 3}{16} \int dV_x h^{(1)}_{ab}(x) g^{ab}(x) \tilde{\phi}^{(l\sigma)}(x).$$

Therefore, we have $\Omega_T(h^{(1)}_{ab}, h^{(t,l\sigma)}_{ab}) = 0$ if $h^{(1)}_{ab}$ is traceless. On the other hand,

$$\Omega_T(h^{(t,l\sigma)}, h^{(t,l\sigma)}) = \frac{\alpha - 3}{16} \int dV_x \phi^{(l\sigma)}(x) \tilde{\phi}^{(l\sigma)}(x).$$

By the scalar version of equation (60) we see that the integral on the right-hand side of this equation is equal to $\Omega_S(\phi^{(l\sigma)}, \phi^{(l\sigma)})$. Hence

$$\Omega_T(h^{(t,l\sigma)}, h^{(t,l\sigma)}) = \frac{\alpha - 3}{16} \Omega_S(\phi^{(l\sigma)}, \phi^{(l\sigma)}).$$

The same formula is valid if we replace $h^{(l\sigma)}$ by its complex conjugate. Thus,

$$\left(h^{(t,l\sigma)} | h^{(t,l\sigma)}\right)_T = \frac{\alpha - 3}{16} \left(\phi^{(l\sigma)} | \phi^{(l\sigma)}\right)_S = \frac{3 - \alpha}{16} \delta_{ll'} \delta_{\sigma\sigma'}$$

and $\left(h^{(l,l\sigma')} | h^{(t,l\sigma)}\right)_T = 0$.

Equation (77) and the fact that the pure-trace modes $h^{(t,l\sigma)}_{ab}$ are orthogonal to traceless solutions with respect to $(\cdot | \cdot)_T$ can also be shown directly as follows. For a general mode
function $h_{ab}^{(1)}$, the conjugate momentum tensor $\pi^{(1)cab}$ is

$$\pi^{(1)cab} = -\frac{1}{2} \nabla^c h_{ab}^{(1)} + \frac{1}{2} \left(1 - \frac{1}{\alpha}\right) \left[ g^{ca} \nabla_d h_{db}^{(1)} + g^{cb} \nabla_d h_{da}^{(1)} \right] + \frac{1}{4} \left(1 - 1 \alpha\right) \left\{ g^{ab} \nabla_d h_{dc}^{(1)} + \frac{1}{2} \left[ g^{ac} \nabla_b h_{dc}^{(1)} + g^{bd} \nabla_a h_{dc}^{(1)} \right] \right\} + \frac{1}{2} \left[ \frac{(\alpha + 1)^2}{16 \alpha} \right] g^{ab} \nabla^c h_{ab}^{(1)}. \quad (78)$$

Hence, for a pure-trace mode $h_{ab}^{(t, l\sigma)}$ we have

$$\pi^{(t, l\sigma) cab} = 3 - \alpha \frac{1}{16} \sqrt{-g} g^{ab} \nabla^c \phi^{(t, l\sigma)} . \quad (79)$$

By substituting these expressions in $J(T) c (h_{ab}^{(1)}, h_{ab}^{(t, l\sigma)})$ we obtain

$$J(T) c (h_{ab}^{(1)}, h_{ab}^{(t, l\sigma)}) = 3 - \alpha \frac{1}{16} (h_{ab}^{(1)} \nabla^c \phi^{(t, l\sigma)} - \nabla^c h_{ab}^{(1)} \cdot \phi^{(t, l\sigma)}) . \quad (80)$$

By integrating this over a Cauchy surface, we obtain (76).

Next, we derive a relation similar to (77) for the traceless modes $h_{ab}^{(d, l\sigma)}$ in the scalar sector. The key equation is

$$L^{(T)abcd} \left( \nabla^c \nabla_d - \frac{1}{4} g_{cd} \Box \right) B = \frac{3 - \alpha}{4 \alpha} \left( \nabla^a \nabla^b - \frac{1}{4} g^{ab} \Box \right) L^{(S)} B \quad (81)$$

for any function $B$. This can be derived by using the identities

$$\Box \nabla_a \nabla_b B = 8 H^2 (\nabla_a \nabla_b - \frac{1}{4} g_{ab} \Box) B + \nabla_a \nabla_b \Box B \quad (82)$$

and

$$\nabla_a \Box \nabla_b B = 3 H^2 \nabla_a \nabla_b B + \nabla_a \nabla_b \Box B . \quad (83)$$

Then, by a procedure similar to that which led to (73) we find from (81)

$$h_{ab}^{(d, l\sigma)} = \frac{3 - \alpha}{4 \alpha} \int d^4 x E^{(T)}_{abcd} \left[ \left( \nabla^c \nabla^d - \frac{3 H^2}{\alpha - 3} g^{cd} \right) \tilde{\phi}^{(d, l\sigma)}(x) \right] , \quad (84)$$

where $h_{ab}^{(d, l\sigma)}$ is defined by (39). This implies that the symplectic product of any mode function $h_{ab}^{(1)}$ and the mode $h_{ab}^{(d, l\sigma)}$ is

$$\Omega_T (h_{ab}^{(1)}, h_{ab}^{(d, l\sigma)}) = \frac{3 - \alpha}{4 \alpha} \int d^4 x h_{ab}^{(1)}(x) \left( \nabla^a \nabla^b - \frac{3 H^2}{\alpha - 3} g^{ab} \right) \tilde{\phi}^{(l\sigma)}(x) . \quad (85)$$

If the mode $h_{ab}^{(1)}$ is pure trace, then we can show that this vanishes by integrating by parts and using (28). If $h_{ab}^{(1)}(x)$ is traceless, then

$$\Omega_T (h_{ab}^{(1)}, h_{ab}^{(d, l\sigma)}) = \frac{3 - \alpha}{4 \alpha} \int d^4 x h_{ab}^{(1)}(x) \nabla^a \nabla^b \tilde{\phi}^{(l\sigma)}(x) . \quad (86)$$
Integration by parts gives
\[ \Omega_T(h^{(1)}, h^{(d,lσ)}) = \frac{3 - \alpha}{4\alpha} \int dV_x \nabla^a \nabla^b h^{(1)}_{ab}(x) \tilde{\phi}^{(lσ)}(x), \]  
which is zero if \( \nabla^a \nabla^b h^{(1)}_{ab} = 0 \). On the other hand we have
\[ \nabla^a \nabla^b h^{(d,lσ)}_{ab} = \frac{36\alpha H^4}{(\alpha - 3)^2} \phi^{(lσ)}. \]  
Hence,
\[ \Omega_T(h^{(d,lσ)}, h^{(d,t'σ')}) = \frac{9H^4}{3 - \alpha} \int dV_x \phi^{(lσ)}(x) \tilde{\phi}^{(t'σ')}(x) \]
\[ = \frac{9H^4}{3 - \alpha} \Omega_S(\phi^{(lσ)}, \phi^{(t'σ')}). \]  
The same formula is valid if we replace \( \phi^{(lσ)} \) by its complex conjugate. Thus,
\[ (h^{(d,lσ)}|h^{(d,t'σ')})_T = \frac{9H^4}{3 - \alpha} \phi^{(lσ)}|\phi^{(t'σ')} \]
\[ = \frac{9H^4}{3 - \alpha} \delta_{tt'}\delta_{σσ'} \]  
and \( (h^{(d,lσ)}|h^{(d,t'σ')})_T = 0 \). Equation (89) can also be derived directly without using the four-dimensional form of the symplectic product. We present this derivation in Appendix B.

\section{Scalar sector of the graviton two-point function}

The Klein-Gordon inner product computed in the previous section can be used together with equation (50) to find the commutators of annihilation and creation operators in the scalar sector. First of all, since the modes \( h^{(d,lσ)}_{ab} \) and \( h^{(t,lσ)}_{ab} \) are orthogonal to any traceless mode \( h^{(1)}_{ab} \) satisfying \( \nabla^a \nabla^b h^{(1)}_{ab} = 0 \) with respect to \( \cdot|\cdot \)\_T, the operators \( b_{lσ}, b_{lσ}^\dagger, c_{lσ} \) and \( c_{lσ}^\dagger \) commute with the operator \( h^{(r)}_{ab}(x) \) according to (50). Also it is clear that the operators \( b_{lσ} \) and \( b_{lσ}^\dagger \) commute with \( c_{lσ} \) and \( c_{lσ}^\dagger \). Thus, the graviton two-point function can be written as
\[ \langle 0|h_{ab}(x)h^{(r)}_{a'b'}(x')|0 \rangle = \Delta^{(r)}_{ab'a'b'}(x, x') + \Delta^{(s)}_{ab'a'b'}(x, x'), \]  
where
\[ \Delta^{(r)}_{ab'a'b'}(x, x') = \langle 0|h^{(r)}_{ab}(x)h^{(r)}_{a'b'}(x')|0 \rangle, \]
\[ \Delta^{(s)}_{ab'a'b'}(x, x') = \langle 0|h^{(t)}_{a'b'}(x)|0 \rangle + \langle 0|h^{(d)}_{ab}(x)h^{(d)}_{a'b'}(x')|0 \rangle. \]
The commutators of annihilation and creation operators in the scalar sector are the inverse of the Klein-Gordon inner product of the corresponding mode functions according to (50). Thus,

\[
[b_{l\sigma}, b_{l'}^{\dagger}] = \frac{3 - \alpha}{9H^4} \delta_{ll'} \delta_{\sigma\sigma'}, \quad (94)
\]

\[
[c_{l\sigma}, c_{l'}^{\dagger}] = \frac{16}{\alpha - 3} \delta_{ll'} \delta_{\sigma\sigma'}, \quad (95)
\]

with all other commutators vanishing. Hence the scalar sector of the graviton two-point function is

\[
\Delta_{aba'b'}^{(s)}(x, x') = \frac{3 - \alpha}{9H^4} \left( \nabla_a \nabla_b - \frac{3H^2}{\alpha - 3} g_{ab} \right) \left( \nabla_{a'} \nabla_{b'} - \frac{3H^2}{\alpha - 3} g_{a'b'} \right) \Delta_+(x, x')
\]

\[
+ \frac{16}{\alpha - 3} \times \frac{1}{16} g_{ab} g_{a'b'} \Delta_+(x, x')
\]

\[
= \left( \frac{3 - \alpha}{9H^4} \nabla_a \nabla_b \nabla_{a'} \nabla_{b'} + \frac{1}{3H^2} \nabla_a \nabla_{a'} g_{ab} \nabla_{b'} \Delta_+ \right) \Delta_+(x, x') \quad (96)
\]

As we have seen, the function \(\Delta_+(x, x')\) decreases at large distances if \(\alpha > 3\). Hence the scalar-sector two-point function \(\Delta^{(s)}(x, x')\) decreases at large distances for these values of \(\alpha\). Note also that each term in the scalar sector is pure gauge either at point \(x\) or \(x'\). Hence, there will be no contribution from this sector to the two-point function of a gauge-invariant quantity. This is in agreement with the expectation that the extra modes introduced in the theory by gauge fixing should not contribute to physical quantities at the tree level. We present an explicit form of the scalar-sector two-point function for \(\alpha = 9\), where some simplification occurs.

Let us comment on the limit \(\alpha \to 0\). In this limit the field satisfies

\[
\nabla^b h_{ab} = \frac{1}{4} \nabla_a h. \quad (97)
\]

Note that the two-point function \(\Delta^{(s)}(x, x')\) remains in a pure-gauge form in this limit. However, the modes \(h_{ab}^{(d,lr)}\) satisfy \(\nabla^a \nabla^b h_{ab}^{(d,lr)} = 0\). (As a result, some intermediate results such as equation (87) are ill-defined.) Therefore, if one imposed the condition (97) from the start without taking the \(\alpha \to 0\) limit of nonzero \(\alpha\), the field would be decomposed as \(h_{ab} = h_{ab}^{(r)} + h_{ab}^{(t)}\) with \(\nabla^a \nabla^b h_{ab}^{(r)} = 0\). Therefore, the cancellation of physical contributions from the fields \(h_{ab}^{(d)}\) and \(h_{ab}^{(t)}\) could be overlooked.

We have not calculated \(\Delta^{(r)}_{aba'b'}(x, x')\), which is of course necessary to find the full two-point function. The full two-point function can most easily be calculated by extending

\(^6\text{Related remarks were made in [9] and [2].}\)
the work of Allen and Turyn [2] in the Euclidean approach. (They specialize to the choice \(\alpha = 1\) and \(\beta = -2\). Since the mass of the modes in the scalar sector for this choice is \(-6H^2\), their two-point function grows badly at large distances in the scalar sector.) Our preliminary results with arbitrary values of \(\alpha\) and \(\beta\) show that it is impossible to construct a two-point function which does not grow at large distances. However, the results in [10] and [11] imply that this growth in CGTF is also pure-gauge. We are currently investigating if this fact can be verified directly.

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Appendix A. Spacetime-integral expression of mode functions

Let \(\chi(t)\) be a smooth function of time with metric (1) which takes the value one for \(t < T_1\) and zero for \(T_2 < t\) (where \(T_1 < T_2\)). For any given solution \(h_I\) of the equations \(L^{IJ}h_J = 0\) with a compact support \(S\) on a Cauchy surface \(\Sigma\), let \(\tilde{h}^I(x) = L^{IJ} |\chi h_J(x)|\). This is zero if \(t > T_2\) because \(\chi = 0\) there. It is also zero if \(t < T_1\) because \(L^{IJ} (\chi h_J) = L^{IJ} h_J = 0\). Thus, the tensor \(\tilde{h}^I(x)\) has support in a subset of the causal future and past of \(S\), i.e. in the set of points which can be connected by a null or timelike curve with \(S\), with \(T_1 \leq t \leq T_2\). Thus, the support of \(\tilde{h}^I(x)\) is compact.

Now, if the point \(x\) is in the region with \(t < T_1\), then

\[
\int dV_y E^{(T)}_{IJ}(x, y) \tilde{h}^J(y) = \int dV_y G^{(T,A)}_{IJ}(x, y)L^{JK}_y \chi h_K(y). \tag{A1}
\]

We integrate by parts using (43) and apply \(L^{JK}_y\) on \(G^{(T,A)}_{IJ}(x, y)\). By taking into account the fact that \(L^{JK}_y G^{(T,A)}_{IJ}(x, y) = \delta^K_I \delta^4(x, y)\), which follows from \(G^{(T,A)}_{IJ}(x, y) = G^{(T,R)}_{IJ}(y, x)\), we have

\[
\int dV_y E^{(T)}_{IJ}(x, y) \tilde{h}^J(y) = \chi h_I(x). \tag{A2}
\]

However, since we are assuming that \(x\) is in the region with \(t < T_1\), the right-hand side is equal to \(h_I(x)\) itself. Then, by uniqueness of the solution, the left-hand side must equal \(h_I(x)\) over the whole spacetime.
Appendix B. Direct derivation of symplectic product for traceless scalar modes

For the traceless scalar mode \( h^{(d,l\sigma)}_{ab} \) given by (39), define \( A_{a}^{(l,\sigma)} = \nabla^{b} h^{(d,l\sigma)}_{ab} \). Then

\[
A_{a}^{(l,\sigma)} = \frac{3\alpha H^{2}}{\alpha - 3} \nabla_{a} \phi^{(l\sigma)}.
\]  

The conjugate momentum tensor can be found from (78) as

\[
\pi^{(d,l\sigma)\, cab} = -\frac{1}{2} \nabla^{c} \nabla^{a} \nabla^{b} \phi^{(l\sigma)} + \frac{3(\alpha - 1) H^{2}}{2(\alpha - 3)} \left( g^{ca} \nabla^{b} \phi^{(l\sigma)} + g^{cb} \nabla^{a} \phi^{(l\sigma)} \right) 
+ \text{ (terms proportional to } g^{ab}).
\]  

If \( h^{(2)}_{ab} \) is any traceless mode, then after a tedious but straightforward calculation we find

\[
h^{(d,l\sigma)}_{ab} \pi^{(2)\, cab} - \pi^{(2)\, cab} h^{(2)}_{ab} = \left( 3 - \alpha \right) H^{2} \frac{4\alpha}{\alpha - 3} \Omega_{S}(\phi^{(l\sigma)}, \Phi^{(2)}) + \frac{1}{2} \nabla_{a} K^{ac},
\]  

where \( \Phi^{(2)} = \nabla^{a} A^{(2)}_{a} = \nabla^{a} \nabla^{b} h^{(2)}_{ab} \) and where

\[
K_{ac} = \nabla^{b} \phi^{(l\sigma)} \cdot \nabla_{a} h^{(2)}_{cb} - \nabla^{b} \phi^{(l\sigma)} \cdot \nabla_{c} h^{(2)}_{ab} + h^{(2)}_{cb} \nabla_{c} \nabla^{b} \phi^{(l\sigma)} - h^{(2)}_{cb} \nabla_{c} \nabla^{b} \phi^{(l\sigma)} 
+ \phi^{(l\sigma)} \left( \nabla_{c} A^{(2)}_{a} - \nabla_{a} A^{(2)}_{c} \right) - A^{(2)}_{a} \nabla_{c} \phi^{(l\sigma)} + A^{(2)}_{c} \nabla_{a} \phi^{(l\sigma)} 
- \left( 1 - \frac{1}{\alpha} \right) \left[ \phi^{(l\sigma)} \left( \nabla_{c} A^{(2)}_{a} - \nabla_{a} A^{(2)}_{c} \right) - 2 A^{(2)}_{a} \nabla_{c} \phi^{(l\sigma)} + 2 A^{(2)}_{c} \nabla_{a} \phi^{(l\sigma)} \right].
\]  

Note that the tensor \( K_{ac} \) is anti-symmetric. Therefore,

\[
\int d\Sigma_{c} \nabla_{a} K^{ac} = 0.
\]

Thus, equation (B5) implies

\[
\Omega_{T}(h^{(d,l\sigma)}, h^{(2)}) = \frac{(3 - \alpha) H^{2}}{4\alpha} \Omega_{S}(\phi^{(l\sigma)}, \Phi^{(2)}).
\]  

If \( \Phi^{(2)} = \nabla_{a} \nabla_{b} h^{(2)ab} = 0 \), then \( \Omega_{T}(h^{(1)}, h^{(2)}) = 0 \). If \( h^{(2)}_{ab} = h^{(d,l\sigma)}_{ab} \), then

\[
\Phi^{(2)} = \nabla_{a} \nabla_{b} h^{(d,l\sigma)ab} = 36\alpha H^{2} \frac{(\alpha - 3)^{2}}{(\alpha - 3)} \phi^{(l\sigma)}.
\]  

By substituting this in (B7) we obtain equation (89).
Appendix C. The scalar sector of the two-point function with $\alpha = 9$

Note that for $\alpha = 9$ we have $12H^2/(\alpha - 3) = 2H^2$. Therefore we have the conformally-coupled massless scalar field. In this case the scalar two-point function (17) is

$$\Delta_+(x, x') = \frac{H^2}{16\pi^2} F(2, 1; 2; z) = \frac{H^2}{16\pi^2} \frac{1}{1 - z}. \quad (C1)$$

The two-point function $\Delta^{(s)}_{aba'b'}(x, x')$ can be found from (96) using (17)

$$\nabla_a n_b = H \cot H \mu (g_{ab} - n_a n_b), \quad (C2)$$

$$\nabla_a n_{b'} = -\frac{H}{\sin H \mu} (g_{ab'} + n_a n_{b'}), \quad (C3)$$

$$\nabla_a g_{bc'} = H(1 - \cos H \mu) \frac{1}{\sin H \mu} (g_{ab} n_{c'} + g_{ac} n_{b'}). \quad (C4)$$

The result is

$$\Delta^{(s)}_{aba'b'}(x, x') = T_1(z) n_a n_b n_{a'} n_{b'} + T_2(z) (g_{a'b'} n_a n_b + g_{ab} n_{a'} n_{b'})$$

$$+ T_3(z) (g_{a'b'} n_{a'} n_{b'} + g_{b'a'} n_a n_{b'} + g_{ab'} n_b n_{a'})$$

$$+ T_4(z) (g_{a'b'} g_{b'} + g_{b'a'} g_{a'}) + T_5(z) g_{ab} g_{a'b'}. \quad (C5)$$

where $n_a = \nabla_a \mu(x, x')$ is the tangent vector at point $x$ to the spacelike geodesic joining points $x$ and $x'$ (if there is such a spacelike geodesic). The bi-tensor $g_{ab'}$ is the parallel propagator, i.e. for any vector $X^{a'}$ at point $x'$, $g_{ab'} X^{a'}$ is the vector at point $x$ obtained by parallelly transporting $X^{a'}$ along the geodesic [17]. The coefficients $T_i(z)$ are given by

$$T_1(z) = \frac{H^2}{24\pi^2} \left[ \frac{4}{z - 1} + \frac{24}{(z - 1)^2} + \frac{24}{(z - 1)^3} \right], \quad (C6)$$

$$T_2(z) = -\frac{H^2}{24\pi^2} \left[ \frac{1}{z - 1} + \frac{4}{(z - 1)^2} + \frac{3}{(z - 1)^3} \right], \quad (C7)$$

$$T_3(z) = \frac{H^2}{24\pi^2} \left[ \frac{2}{(z - 1)^2} + \frac{3}{(z - 1)^3} \right], \quad (C8)$$

$$T_4(z) = \frac{H^2}{24\pi} \left[ \frac{1}{2(z - 1)^3} \right], \quad (C9)$$

$$T_5(z) = \frac{H^2}{24\pi} \left[ \frac{1}{(z - 1)^2} + \frac{1}{2(z - 1)^3} \right], \quad (C10)$$

where $z = \cos^2 \left( \frac{\mu(x, x') H}{2} \right)$. As we have seen in section 2 the variable $z$ tends to $-\infty$ as the coordinate distance $r = ||x - x'||$ in the coordinates for metric (21) tends to infinity.
In the present case the coefficients $T_i$ tend to zero like $r^{-2}$. In the rest of this appendix we calculate the components of the tangent vector $n_a$ and the parallel propagator $g_{ab'}$ and see explicitly that they are bounded as $r$ as $r \to \infty$. Thus, we see the two-point function in (C5) indeed goes to zero as $r \to \infty$.

Let us define

$$\chi(x, x') \equiv \cos H \mu = \frac{\lambda^2 + \lambda'^2 - r^2}{2\lambda \lambda'}.$$  (C11)

We extend the function $\mu(x, x')$ with $x = (\lambda, x)$ and $x' = (\lambda', x')$ in the coordinates for metric (21) to the region with $\chi > 1$ by

$$e^{iH\mu} = \chi + \sqrt{\chi^2 - 1}.$$  (C12)

We then have

$$n_a = -\frac{i}{H\sqrt{\chi^2 - 1}} \nabla_a \chi$$  (C13)

and

$$\nabla_a \chi = -\frac{Hr}{\lambda'} V_a + H \left( \frac{\lambda}{\lambda'} - \chi \right) t_a,$$  (C14)

where the vectors $V_a$ and $t_a$ at point $x = (\lambda, x)$ are defined by [12]

$$V^0 = 0 \ , \ V^i = \frac{H \lambda (x^i - x'^i)}{r}$$  (C15)

(“0” refers to the $\lambda$-component) and

$$t^a = -H \lambda \left( \frac{\partial}{\partial \lambda} \right)^a.$$  (C16)

The unit vector $V_a$ is spacelike and the vector $t_a$ is a future-pointing unit normal to the Cauchy surface with $\lambda$ = const, if we let the variable $\lambda$ decrease towards the future. (We define vectors $V_{a'}$ and $t_{a'}$ at $x'$ in a similar manner. We note that $\lambda^{-1}V_i = -(\lambda')^{-1}V_{i'}$ in components.) Thus, we obtain

$$n_a = \frac{ir}{\lambda' \sqrt{\chi^2 - 1}} V_a - \frac{i}{\sqrt{\chi^2 - 1}} \left( \frac{\lambda}{\lambda'} - \chi \right) t_a.$$  (C17)

We note that, since $\chi$ grows like $r^2$ as $r$ increases, the components of $n_a$ remain finite as $r \to \infty$.

The expression for the parallel propagator $g_{ab'}$ can be found from equation (C3). Thus, we have

$$g_{ab'} = \frac{1}{H^2} \nabla_a \nabla_{b'} \chi - \frac{1}{H^2(\chi + 1)} \nabla_a \chi \cdot \nabla_{b'} \chi.$$  (C18)
Let us define a bi-tensor $P_{ab}$ by $t^a P_{ab} = t^b P_{ab} = 0$ and
\[ P_{ij'} = \frac{1}{\lambda\lambda' H^2} \delta_{ij'} . \] (C19)

Then
\[ H^{-2} \nabla_a \nabla_{b'} \chi = P_{ab'} + \frac{r}{\lambda} t_a V_{b'} + \frac{r}{\lambda'} t_{b'} V_a + \left( \chi - \frac{\lambda}{\lambda'} - \frac{\lambda'}{\lambda} \right) t_a t_{b'} . \] (C20)

Hence, the bi-tensor $g_{ab'}$ can be written as
\[ g_{ab'} = P_{ab'} + \frac{1}{\chi + 1} \left( \chi - \frac{\chi'}{\chi} - \frac{\lambda}{\chi'} - 1 \right) t_a t_{b'} - \frac{r^2}{(\chi + 1)\lambda\lambda'} V_a V_{b'} \]
\[ + \left( \frac{1}{\chi} + \frac{1}{\chi'} \right) \frac{r}{\chi + 1} (t_a V_{b'} + t_{b'} V_a) . \] (C21)

It is clear that each component of this bi-tensor is bounded as $r \to \infty$. 

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References