Schrödinger cat state of trapped ions in harmonic and anharmonic oscillator traps

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We examine the time evolution of a two level ion interacting with a light field in harmonic oscillator trap and in a trap with anharmonicities. The anharmonicities of the trap are quantified in terms of the deformation parameter $\tau$ characterizing the $q$-analog of the harmonic oscillator trap. Initially the ion is prepared in a Schrödinger cat state. The entanglement of the center of mass motional states and the internal degrees of freedom of the ion results in characteristic collapse and revival pattern. We calculate numerically the population inversion $I(t)$, quasi-probabilities $Q(t)$, and partial mutual quantum entropy $S(P)$, for the system as a function of time. Interestingly for small deformations of the trap, the population inversion pattern is seen to get better, in comparison with the zero deformation case. For $\beta = 3$ and 4, the best collapse and revival sequence is obtained for $\tau = 0.0047$ and $\tau = 0.004$ respectively. For large values of $\tau$ decoherence sets in accompanied by loss of amplitude of population inversion and for $\tau \sim 0.1$ the collapse and revival phenomenon disappear. Each collapse or revival of population inversion is characterized by a peak in $S(P)$ versus $t$ plot. During the transition from collapse to revival and vice-versa we have minimum mutual entropy value that is $S(P) = 0$. Successive revival peaks show a lowering of the local maximum point indicating a dissipative irreversible change in the ionic state. Improved definition of collapse and revival pattern as the anharmonicity of the trapping potential increases is also reflected in the Quasi- probability versus $t$ plots.

I. INTRODUCTION

Two-level ions trapped in a harmonic or anharmonic trap have turned into an important tool for understanding the time evolution of non classical states. Experimentally harmonic oscillator traps have been realized and various types of non-classical states of ions in trapped systems constructed [1] [2] with sufficient control over amplitudes, relative phases etc. of the component states. It is now possible to detect experimentally constructed non-classical states and measure the statistics of the quantum motion of the center of mass of the ion. The non-classical states constructed and detected include the Fock states, thermal states, Shrodinger cat states and squeezed states. The possibility of manipulating the entangled states of particle systems provides the basis for applications like quantum computations and quantum communications. These experimental advances have paved a way to an understanding of some of the basic aspects of quantum theory. A controlled manipulation of these states requires precise control of the Hamiltonian of the system. The form of the trap potential is usually controlled by the geometry of the system, the laser field intensities and relative phases. As such anharmonic traps with different potential shapes should also be viable experimentally. Another possible mechanism for introducing anharmonicities in the system is the coupling of oscillation modes of ions in orthogonal directions. The measurement of dynamical behavior of non-classical states in harmonic and anharmonic traps and a study of anharmonicity related new features makes an interesting study.

In our earlier work [3] on ions in harmonic and anharmonic traps, we examined the collapses and revivals of the population inversion and quasi-probabilities as a function of time for the initial state of the system having ion in it’s ground state and the center of mass motion described by a coherent state. This study brought out some very interesting new features in the dynamics of the system. In the present work, we consider a two-level ion initially in a Schrödinger cat state and examine the collapse and revival of population inversion and quasi-probabilities with a focus on the coherence of the system and anharmonicities of the trap. We also examine the time evolution of partial mutual entropy of the system to understand the improvement in collapse and revival pattern of the system for characteristic values of parameter quantifying the trap anharmonicities. The motivation for studying these states is provided by the possible use of these states in quantum computation.

The anharmonicity of the trap potential is modeled through a $q$-analog harmonic oscillator trap. Since Macfarlane [4] and Biedenharn [5] discussed $q$-analog of harmonic oscillator, a lot of work using these in various fields of physics
has been done. In our efforts to understand the physical nature of \( q \)-deformations in the context of pairing effect in nuclei [6–8], the \( q \)-deformation is found to simulate the nonlinearities of the problem caused by unaccounted part of the residual interaction. In the present context, we use the parameter \( q \) to quantify the anharmonicity of the trap potential. For small values of \( q \), the \( q \)-analog harmonic oscillator is essentially an anharmonic oscillator with \( x^6 \) anharmonicities [9].

We start with a description of the trapped ion-laser system Hamiltonian in section I for the cases of ion in a Harmonic Oscillator trap and ion in a \( q \)-analog harmonic oscillator trap. In section II the initial state and the equations governing it’s time evolution are given. Population inversion and quasi-probability are defined in section III. In section IV, we discuss the concept of partial mutual entropy for the system and the numerical results are analyzed in section V.

II. THE HAMILTONIAN

A. Harmonic Oscillator trap

The system that consists of the two level ion moving in a harmonic oscillator trap potential and interacting with a classical single-mode light field of frequency \( \omega_l \) is described by the Hamiltonian,

\[
H = \frac{1}{2} \hbar \omega (a^\dagger a + a a^\dagger) + \frac{1}{2} \hbar \Delta \sigma_z + \frac{1}{2} \hbar \Omega (F \sigma^+ + F^* \sigma^-)
\]  

(1)

where \( a \) and \( a^\dagger \) are the usual creation and destruction operators for the oscillator, \( \Delta = \omega_0 - \omega_l \), is the detuning parameter and \( \Omega \) is the Rabi frequency of the system. The pseudo-spin operators of the two-level ion, \( \sigma^\dagger \) and \( \sigma_z \) operate on state vectors \( |g \rangle \) and \( |e \rangle \) where indices \( g \) and \( e \) stand for the ground and excited states of the two level ion. The operator \( F \) stands for \( \exp[i\epsilon k x] = \exp[i \epsilon (a^\dagger + a)] \) where \( k \) is the wave vector of the light field and \( x \) position quadrature of the center of mass. The parameter \( \epsilon = \sqrt{\frac{E_r}{E_i}} \) is a function of the ratio of the recoil energy of the ion \( E_r = \frac{\hbar^2 k^2}{2m} \) and the characteristic trap quantum energy \( E_i = \hbar \omega = \frac{\Delta}{\omega} \). The second term in the Hamiltonian refers to the energy associated with internal degrees of freedom of the ion, whereas the third term is the interaction of the ion with the light field. By using a special form of Baker-Hausdorff Theorem the operator \( F \) may be written as a product of operators i.e.e we use the equality

\[
F = \exp[i\epsilon (a^\dagger + a)] = e^{\frac{i\epsilon^2}{2}[a^\dagger, a]} e^{i\epsilon a^\dagger} e^{i\epsilon a}.
\]  

(2)

For harmonic oscillator trap one has

\[
[a, a^\dagger] = 1 ; \quad Na^\dagger - a^\dagger N = a^\dagger ; \quad Na - aN = -a.
\]  

(3)

The physical processes implied by the various terms of the operator

\[
F = \left[ \exp \left( \frac{-\epsilon^2}{2} \right) \sum_n \frac{(i\epsilon)^n a^{\dagger n}}{n!} \sum_k \frac{(i\epsilon)^k a^k}{k!} \right],
\]  

(4)

may be divided into three categories. The terms for \( n > k \) correspond to an increase in energy linked with the motional state of center of mass of the ion by \( (n - k) \) quanta. The terms with \( n < k \) represent destruction of \( (k - n) \) quanta of energy thus reducing the amount of energy linked with the center of mass motion. For \( (n = k) \), we have diagonal contributions. The contribution from a particular term containing operators \( a^{\dagger n} a^k \) is determined by the coefficient \( \exp \left( \frac{-\epsilon^2}{2} \right) \frac{(i\epsilon)^{n+k}}{n!k!} \).

B. \( q \)-analog harmonic oscillator trap

Next we consider a single two level ion having ionic transition frequency \( \omega_0 \) in a quantized \( q \)-analog quantum harmonic oscillator trap (\( q \)-deformed harmonic oscillator trap) interacting with a single mode travelling light field. The creation and annihilation operators for the trap quanta satisfy the following commutation relations,

\[
AA^\dagger - qA^\dagger A = q^{-N} ; \quad NA^\dagger - A^\dagger N = A^\dagger ; \quad NA - AN = -A.
\]  

(5)
Here $N$ is the number operator. The operators $A$ and $A^\dagger$ act in a Hilbert space with basis vectors $|n\rangle$, $n = 0, 1, 2, ...$, given by,

$$|n\rangle = \frac{(A^\dagger)^n}{([n]_q)!} |0\rangle$$

such that $N |n\rangle = n |n\rangle$. The vacuum state is $A |0\rangle = 0$. We define here $[x]_q$ as

$$[x]_q = q^x - q^{-x}$$

and the $q$-analog factorial $[n]_q!$ is recursively defined by $[0]_q! = [1]_q! = 1$ and $[n]_q! = [n]_q[\frac{n}{q} - 1]$. It is easily verified that

$$A^\dagger |n\rangle = |n+1\rangle$$

and

$$A |n\rangle = |n-1\rangle$$

and $N$ is not equal to $A^\dagger A$. Analogous to the harmonic oscillator one may define the $q$-momentum and $q$-position coordinate

$$P_q = i \sqrt{\frac{\hbar m \omega}{2}} (A^\dagger - A); \quad X_q = \sqrt{\frac{\hbar}{2m \omega}} (A^\dagger + A).$$

The $q$-analog harmonic oscillator Hamiltonian is given by

$$H_{qho} = \frac{1}{2} \hbar \omega (AA^\dagger + A^\dagger A)$$

with eigenvalues

$$E_n = \frac{1}{2} \hbar \omega ([n + 1]_q + [n]_q).$$

We note that the trap states are not evenly spaced, the energy spacing being a function of deformation. Besides that as we move up in the number of vibrational quanta in the states the spacing between successive states increases. We choose $q = e^\tau$, where $\tau$ is a real or complex valued parameter. For $\tau = 0.0$ the harmonic oscillator trap is recovered. As such the parameter $\tau$ is a measure of the extent to which the harmonic oscillator potential trap is deformed. Using the Taylor expansion of the $q$-number $[n]_q$ in terms of powers of $\tau^2$ the energy eigenvalues may be rewritten for small $\tau$ as

$$E_n = \hbar \omega \left[ \left( n + \frac{1}{2} \right) \left( 1 \pm \frac{\tau^2}{24} \right) \mp \left( n + \frac{1}{2} \right)^3 \frac{\tau^2}{6} + ... \right].$$

Bonatsos et. Al [11] have shown that the potential giving a spectrum similar to that of Eq. (12) up to the order $\tau^2$ looks like

$$V(x) = \left( \frac{1}{2} \pm \frac{\tau^2}{8} \right) x^2 + \frac{\tau^2}{120} x^6,$$

which is an anharmonic oscillator with $x^6$ anharmonicities [9].

Using a nonlinear map given by Curtright and Zachos [12] one can express $H_{qho}$ in terms of the operators $a$ and $a^\dagger$ of the harmonic oscillator. To make this point clear we express the operators $A$ and $A^\dagger$ as

$$A = a f(N) \quad ; \quad A^\dagger = f(N) a^\dagger$$

where $f(N) = \left( \frac{[N]}{N} \right)^\dagger$ and $N = a^\dagger a$. We can also verify that

$$f(N) a^\dagger = a^\dagger f(N + 1) \quad ; \quad f(N) a = a f(N - 1)$$

The Hamiltonian of Eq. 10 can be rewritten as
\[ H_{qho} = \hbar \omega \left( f(N+1)^2 + f(N)^2 \right) \left( a^\dagger a + \frac{1}{2} \right) \]

which can be interpreted as a harmonic oscillator Hamiltonian with a frequency \( \omega_q(N) \) that depends on the quantum number \( n \) of the state in question.

The quadratures \( x \) and \( p \) are related to \( P_q \) and \( X_q \) through

\[
P_q = i \sqrt{\frac{m \hbar \omega}{2}} \left( (f(N) - f(N+1)) \frac{x}{\sqrt{2}} - (f(N) + f(N+1)) \frac{ip}{\sqrt{2}} \right)
\]

\[
X_q = \sqrt{\frac{\hbar}{2m \omega}} \left( (f(N) + f(N+1)) \frac{x}{\sqrt{2}} - (f(N) - f(N+1)) \frac{ip}{\sqrt{2}} \right)
\]

C. Hamiltonian for an ion interacting with light in a \( q \)-analog harmonic oscillator trap

The Hamiltonian for an ion interacting with light in a \( q \)-analog harmonic oscillator trap may now be written as

\[
H_q = \frac{1}{2} \hbar \omega (AA^\dagger + A^\dagger A) + \frac{1}{2} \hbar \Delta \sigma_z + \frac{1}{2} \hbar \Omega (F_q \sigma^+ + F_q^\dagger \sigma^-)
\]

where by analogy with Eq. 4 we choose

\[
F_q = e^{\left( -\frac{\omega^2}{2} \right)} e^{i\epsilon A^\dagger} e^{i\epsilon A},
\]

which in the limit \( q \to 1 \) reduces to \( F \), and can be expanded as [13],

\[
F_q = e^{\left( -\frac{\omega^2}{2} \right)} \sum_{n=0}^{\infty} (i\epsilon)^n A_{-1}^n \sum_{k=0}^{\infty} (i\epsilon)^k A^k.
\]

Various terms in the expansion of this operator represent processes which might result in transitions of the center of mass from a given motional state, in the \( q \)-analog trap, to another, while loosing or gaining energy. The coefficient of the operators \( A_{-1}^n A^k \) is again \( \exp \left( -\frac{\epsilon^2}{2} \right) \frac{(i\epsilon)^{n+k}}{n!k!} \), the same as that in the corresponding term with operators \( a^\dagger a \) in Eq. 4. Intuitively this is the correct way of representing the interaction of the ion and the laser in a \( q \)-analog harmonic oscillator trap. This form of the operator shows that the energy exchange of the center of mass motion of the ion occurs as the ion moves up or down in the trap. Since \( E_{n+1} - E_n \) is \( n \) dependent, it implies an \( n \) dependent entanglement of the center of mass motion and the internal degrees of freedom of the two level atom. Using Eq. 14, we can rewrite Eq. 19 as

\[
F_q = e^{\left( -\frac{\omega^2}{2} \right)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (i\epsilon)^{n+k} \frac{(\epsilon f(N) a^\dagger)^n (\epsilon a f(N))^k}{n!k!}.
\]

Comparing with \( F \) we notice that the effective lamb Dicke parameter in a \( q \)-analog trap, for the loss and gain of motional state energy in an interaction process, is \( \epsilon f(N) \) where \( N \) is the number operator. With the \( q \)-analog of Glauber coherent state defined as

\[
|\alpha_q\rangle = \frac{1}{\sqrt{\exp_q|\alpha|^2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{|n|_q!}} |n\rangle,
\]

the calculated value of \( q \langle \alpha | f(N) | \alpha \rangle_q \) is 1.0004, for the choice \( \alpha = 4 \), \( \epsilon = 0.05 \) and \( \tau = 0.003 \).

It is important to recall at this point that the operator \( (A^\dagger + A) \) is not a self adjoint operator. The vectors \( |m\rangle \) are analytic vectors of the operators \( (A^\dagger)^n A^k \) but not of the operators \( A^k (A^\dagger)^n \). One can easily verify that the matrix elements \( \langle m | e^{\left( -\frac{\omega^2}{2} \right)} \sum_{n=0}^{\infty} (i\epsilon)^n A_{-1}^n \sum_{k=0}^{\infty} (i\epsilon)^k A^k | m \rangle \) are not well defined. On the other hand the matrix elements of the operator in Eq. 18 are well defined as discussed in Ref. [13]. From the form of the operator in Eq. 19 it is clear that the product \( F_q \sigma^+ \) contains all the processes in which the atom is excited from the ground state to the
excited state while at the same time some of the energy quanta in the center of mass motion are lost or gained. It is also important to note that $X_q$ is not the true position coordinate but is related to $x$ and $p$ in some complex way. As such the operator $\exp[i\epsilon(A^\dagger + A)]$ cannot be used to represent the laser-ion interaction.

The expression for the matrix element of the operator $F_q$ between the states $\langle m |$ and $| n \rangle$ for $m \leq n$ is given by,

$$
\langle m | F_q | n \rangle = \frac{e^{-|\epsilon|^2/2} (i\epsilon)^{n-m} [m]_q^{1/2}}{[n]_q^{1/2}} \sum_{k=0}^{m} \frac{(e^{2k}(1)^{k}[n]_q)!}{k!(n-m+k)!(m-k)_q!}
$$

III. SCHRÖDINGER CAT STATE

Initially the ion is prepared in a Schrodinger Cat State with equal probabilities of finding the ion in its ground state and in the excited state coupled to a coherent state describing the state of motion of the center of mass of the ion in the anharmonic trap. The experimental technique for producing trapped ion in a state with these characteristics has been described in many experimental papers and such states have been produced [14,1].

The initial state of the trapped ion in a $q$-analog harmonic oscillator trap can be expressed as

$$
\Psi(t=0) = |g,\beta\rangle_q + e^{i\phi} |e,-\beta\rangle_q
$$

where $|g,\beta\rangle_q$ is given by

$$
|g,\beta\rangle_q = \frac{1}{\sqrt{\exp[|\beta|^2]}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{[n]_q!}} |g,n\rangle
$$

The relative phase between two possible internal states is determined by $\phi$, in the present study we choose $e^{i\phi} = 1$.

The state of the system at a time $t$,

$$
\Psi(t) = \sum_m g_m(t) |g,m\rangle + \sum_m e_m(t) |e,m\rangle
$$

is a solution of the time dependent Schrodinger equation

$$
H_q \Psi(t) = i\hbar \frac{d}{dt} \Psi(t).
$$

In the state $|g,m\rangle$ ($|e,m\rangle$) the two level ion in it’s ground (excited) state is coupled to number state $|m\rangle$ of the anharmonic oscillator. The probability amplitudes $g_m(t)$ and $e_m(t)$ satisfy the following set of coupled equations

$$
\frac{dg_m(t)}{dt} = \frac{1}{2} g_m(t) (\omega([m+1]_q + [m]_q) - \Delta) + \frac{1}{2} \Omega \sum_n e_n(t) \langle g,m | F^\dagger_q \sigma^+ | e,n \rangle
$$

$$
\frac{de_m(t)}{dt} = \frac{1}{2} e_m(t) (\omega([m+1]_q + [m]_q) + \Delta) + \frac{1}{2} \Omega \sum_n g_n(t) \langle e,m | F_q \sigma^+ | g,n \rangle
$$

In the limit $q \to 1$ we recover the system that gives the dynamics of two-level ion in a harmonic oscillator trap. By rescaling $\epsilon$, $\omega$, and $t$, the parameter $\Omega$ can be eliminated from Eq. (26).

IV. POPULATION INVERSION AND QUASI-PROBABILITY

The entanglement of motional degrees of freedom of the center of mass of the ion with it’s internal degrees of freedom manifests itself in the well known collapse and revival of population inversion. The population inversion is defined as

$$
I(t) = P_g(t) - P_e(t),
$$
that is the difference between the probability of finding the system in the ground state, $P_g(t)$, and the probability of finding the system in the excited state, $P_e(t)$. The time evolution of population inversion is examined for different values of trap anharmonicities for the cases when coherent state is characterized by parameter $\beta = 3, 4$. The time $t$ is expressed in units of $\frac{\Omega^2}{\pi}$.

We have also calculated numerically and plotted as a function of $\alpha_r$ and $\alpha_i$ the Quasi-Probability function $Q(\alpha)$ defined as

$$Q(\alpha) = \frac{1}{\pi} \langle \Psi(t) | \rho_\alpha | \Psi(t) \rangle.$$  (28)

where $\rho_\alpha = |\alpha\rangle_q \langle \alpha|.$

V. THE PARTIAL QUANTUM MUTUAL ENTROPY

Dynamical behavior of the system can be better understood in terms of quantum mutual entropy. Furuichi and co-workers [15] have applied the concept of quantum mutual entropy to dynamical change of state of the atom in Jaynes-Cummings model (JCM) [16]. The analogy between JCM and ion in a trap system allows us to extend the concept to understand the entanglement of the internal degrees of freedom of ion and its motional degrees of freedom.

We next outline the calculation of quantum mutual entropy $I$ for the system at hand prepared initially in state given in (22). It is found that $I$ can be decomposed into a part determined by the relative populations of the system and another that depends on the coherences. The partial mutual entropy used by us in our numerical study is essentially a measure of mutual entropy due to populations developed in the system with the passage of time.

With the initial state of the trapped ion expressed as in (22), The density operator for the internal states of the ion at $t = 0$ is

$$\rho_{\text{ion}} = \lambda_1 |g\rangle \langle g| + \lambda_2 |e\rangle \langle e|.$$  (29)

The state of motion of the ionic center of mass at $t = 0$ is represented by a coherent state $|\beta\rangle_q$ for ion in ground state and by coherent state $|\beta\rangle_q$ for ion in excited state. where

$$|\beta\rangle_q = \frac{1}{\sqrt{\exp_q(|\beta|^2)}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle_q .$$  (30)

The state of the system at a time $t$ is given by Eq. (24).

The time evolution of the internal state of the ion can also be represented by a continuous mapping

$$\Lambda_t^{\text{ion}} \rho_{\text{ion}} = tr_{c.m.} (U_t \rho_0 U_t)$$  (31)

where $\rho_0$ is the initial state of the system, $U_t = \exp\left(-i\frac{tH}{\hbar}\right)$ a unitary operator and the mapping $\Lambda_t^{\text{ion}}$ maps the initial internal state $\rho_{\text{ion}}$ to the final internal state of the ion. We represent the time evolution of initial constituent ionic states $|g\rangle \langle g|$ and $|e\rangle \langle e|$ through the maps

$$\Lambda_t^{\text{ion}} |g\rangle \langle g| = tr_{c.m.} (U_t |g, \beta\rangle \langle g, \beta| U_t)$$  (32)

and

$$\Lambda_t^{\text{ion}} |e\rangle \langle e| = tr_{c.m.} (U_t |e, -\beta\rangle \langle e, -\beta| U_t) .$$  (33)

The correlation between the initial ionic state and its final state containing information about the time evolution of each constituent ionic states is given in terms of the mutual quantum entropy defined as

$$I = \sum_{i=1}^{2} \lambda_i S (\Lambda_t^{\text{ion}} |i\rangle \langle i| , \Lambda_t^{\text{ion}} \rho_{\text{ion}}) ,$$  (34)

where the relative quantum entropy

$$S (\Lambda_t^{\text{ion}} |i\rangle \langle i| , \Lambda_t^{\text{ion}} \rho_{\text{ion}}) = tr (\Lambda_t^{\text{ion}} |i\rangle \langle i| \log (\Lambda_t^{\text{ion}} |i\rangle \langle i|)$$

$$- \log (\Lambda_t^{\text{ion}} \rho_{\text{ion}}))$$  (35)
Substituting Eq. (43) in the definition of quantum mutual entropy for the system, we have

\[ S(\Lambda_{g,\rho_{ion}}) = \sum_{n} \left( g_{n}(t) \right)^{2} \log \left( g_{n}(t) \right) + \sum_{n} \left( e_{n}(t) \right)^{2} \log \left( e_{n}(t) \right) + 2 * \text{Re} \left[ C_{ge}(t) \log \left( \frac{C_{ge}(t)}{C_{ge}(t)} \right)^{*} \right] \]  

Besides that the system at a time \( t \) is characterized by the following non-diagonal matrix element of the operator \( tr_{ion}\rho(t) \) (coherence),

\[ C_{ge}(t) = \frac{\langle g | tr_{ion}\rho(t) | e \rangle}{\langle g | g \rangle} \]

We also define the populations and coherences for the system prepared in initial state \( |g_{i}, \beta_{i}\rangle \) (\( i = 1 \)) and the initial state with the ion in the excited state \( |e_{i}, -\beta_{i}\rangle \) (\( i = 2 \)). The state of the system at a time \( t \) is now given by

\[ \psi^{(i)}(t) = \sum_{m} g_{m}^{(i)}(t) |g_{m}, m\rangle + \sum_{m} e_{m}^{(i)}(t) |e_{m}, m\rangle \]

with the populations defined as

\[ P_{g}^{(i)}(t) = \frac{\langle g | tr_{ion}\rho^{(i)}(t) | g \rangle}{\langle g | g \rangle} \]

\[ + 2 * \text{Re} \left[ C_{ge}(t) \log \left( \frac{C_{ge}(t)}{C_{ge}(t)} \right)^{*} \right] \]

The coherences for these initial states are given by

\[ C_{ge}^{(i)}(t) = \frac{\langle g | tr_{ion}\rho^{(i)}(t) | e \rangle}{\langle g | g \rangle} \]

We can verify that in terms of the populations and the coherences defined above the relative quantum entropy, \( S(\Lambda_{t}^{ion} | i \rangle \langle i | , \Lambda_{t}^{ion} \rho_{ion}) \) (Eq. 35) for the system starting in initial state given by Eq. (29) is

\[ S(\Lambda_{t}^{ion} | i \rangle \langle i | , \Lambda_{t}^{ion} \rho_{ion}) = \sum_{i} \lambda(i) \left[ P_{g}^{(i)}(t) \log \left( \frac{P_{g}^{(i)}(t)}{P_{g}(t)} \right) + P_{e}^{(i)}(t) \log \left( \frac{P_{e}^{(i)}(t)}{P_{e}(t)} \right) \right] \]

Substituting Eq. (43) in the definition of quantum mutual entropy for the system, we have

\[ I = \sum_{i=1,2} \lambda(i) \left[ P_{g}^{(i)}(t) \log \left( \frac{P_{g}^{(i)}(t)}{P_{g}(t)} \right) + P_{e}^{(i)}(t) \log \left( \frac{P_{e}^{(i)}(t)}{P_{e}(t)} \right) \right] \]

\[ + 2 * \text{Re} \left[ C_{ge}^{(i)}(t) \log \left( \frac{C_{ge}^{(i)}(t)}{C_{ge}(t)} \right)^{*} \right] \]

We can split \( I \) in to two distinct parts, one depending on populations and the other on off diagonal coherences that is

\[ I = S(P) + S(C) \]
\[
S(P) = \sum_{i=1,2} \lambda(i) \left[ P_g^{(i)}(t) \log \left( \frac{P_g^{(i)}(t)}{P_g(t)} \right) + P_e^{(i)}(t) \log \left( \frac{P_e^{(i)}(t)}{P_e(t)} \right) \right].
\] (46)

We have calculated numerically \(S(P)\) the partial mutual quantum entropy for the cases \(\beta = 3, 4\) and used the \(S(P)\) peaks to pinpoint the \(t\) values for which the system is the most correlated and the \(t\) values where it shows the least correlation. Essentially \(S(P)\) is used as a measure of entanglement of the system.

VI. RESULTS

The population inversion as a function of a rescaled time parameter \(t^{(\Omega t)}/(2\pi)\) for two level ions interacting with light field in a harmonic oscillator trap and a \(q\)-deformed oscillator trap for the cases \(\beta = 3, 4\) is displayed in Figures 1 and 2 for three different values of deformation parameter \(\tau\) (\(q = \exp(\tau)\) with \(\tau\) real). The set of parameters used in the numerical calculation is \(\Omega = \hbar = 50, \tau = \hbar = 0.05\), and \(\Delta = \hbar = -50\). The maximum value of \(n\) in Eq. (24) is restricted to \(n = 32\). For both \(\beta\) values the collapse and revival pattern was found to improve in definition with increase in the anharmonicity of the trap potential, measured by parameter \(\tau\). In Figures 1a and 2a, we display the collapse and revival for ion in a harmonic oscillator trap that is \(\tau = 0.0\). Both for \(\beta = 3\) and 4 first revival and second revival are easily identified. For \(\beta = 3\) the second collapse, which is barely identifiable is succeeded by a general loss of coherence, whereas for \(\beta = 4\) the second collapse is followed by another revival which is quite wide. For \(\beta = 3\) and 4, the best collapse and revival sequence is obtained for \(\tau = 0.0047\) and \(\tau = 0.004\) respectively as shown in Figures 1b and 2b. We can identify four revivals in each case. The collapses and revivals for \(\beta = 4\) are remarkably sharp. Further increase in \(\tau\) results in a loss of contrast between successive revival peaks. Figures 1c and 2c show the loss of coherence as the anharmonicity of the trap potential is increased. We have also found that for a large value of \(\tau\), different for each \(\beta\) value, not only the collapses and revivals of population inversion vanish but also the time dependence of population inversion disappears in all cases.

In Figs. (3-4) we plot the partial quantum mutual entropy \(S(P)\) as a function of time for initial states with \(\beta = 3, 4\). A careful examination shows that each collapse as well as revival of population inversion is characterized by a peak in \(S(P)\) versus \(t\) plot. During the transition from collapse to revival and vice-versa we have minimum mutual entropy value that is \(S(P) = 0\). In the \(q\)-deformed oscillator trap for the cases \(\beta = 3, \tau = 0.0047\) and \(\beta = 4, \tau = 0.004\) we notice relatively higher \(S(P)\) peaks and well defined transition periods(collapse \(\Leftrightarrow\) revival) in comparison with the harmonic oscillator trap(\(\tau = 0.0\)). The regions without continuous minimum \(S(P)\) do not show well defined collapses or revivals in the population inversion plots. In Table I, the \(t\) values for successive peaks in \(S(P)\) for the cases \(\beta = 3, \tau = 0.0047\) and \(\beta = 4, \tau = 0.004\) are displayed. It is evident from figures 3 and 4 that \(S(P)\) decreases with the Rabi oscillations. Table I demonstrates that the successive revival peaks show a lowering of the local maximum point indicating a dissipative irreversible change in the ionic state.

Table I. The values of \(t\) for successive peaks in \(S(P)\) for the cases \(\beta = 3, \tau = 0.0047, \beta = 4, \tau = 0.0\) and \(\beta = 4, \tau = 0.004\).

<table>
<thead>
<tr>
<th>(\beta = 3, \tau = 0.0047)</th>
<th>(\beta = 4, \tau = 0.0)</th>
<th>(\beta = 4, \tau = 0.004)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t )</td>
<td>(S(P)) Peak type</td>
<td>(t )</td>
</tr>
<tr>
<td>0.0</td>
<td>0.693 Initial state</td>
<td>0.0</td>
</tr>
<tr>
<td>58.0</td>
<td>0.628 revival</td>
<td>58.0</td>
</tr>
<tr>
<td>114.0</td>
<td>0.314 collapse</td>
<td>114.0</td>
</tr>
<tr>
<td>175.0</td>
<td>0.285 revival</td>
<td>175.0</td>
</tr>
<tr>
<td>234.0</td>
<td>0.220 collapse</td>
<td>234.0</td>
</tr>
<tr>
<td>295.0</td>
<td>0.181 revival</td>
<td>295.0</td>
</tr>
<tr>
<td>360.0</td>
<td>0.234 collapse</td>
<td>360.0</td>
</tr>
<tr>
<td>415.5</td>
<td>0.175 revival</td>
<td>415.5</td>
</tr>
</tbody>
</table>

Next we plot in Fig. 5 the Quasi-probabilities for \(\beta = 4\) at \(t = 0.0, 85.8, 171.4, 266.8, 388.2\) and 447.6 to visualize the time evolution of ionic center of mass in a harmonic oscillator trap(\(\tau = 0.0\)). As expected, the coherent states \(|g, \beta\rangle\) and \(|e, -\beta\rangle\) each splits into two and move in a sense opposite to each other in phase space. At \(t = 85.8\) there are two somewhat distorted coherent states characterized by almost equal average position coordinate(\(\alpha_e \simeq 0\)), and momenta with opposite signs. This point corresponds to a revival peak in the population inversion plot and the first peak \(S(P)\) versus \(t\) plot( Fig. 4a). A second revival similarly occurs at \(t = 266.8\). The collapse at \(t = 171.4\) corresponds to a situation similar to the initial state. At 388.2 and 447.6 the quasi-probability is spread out in the phase space. For \(\tau = 0.004\) quasi-probability contour plots at \(t = 0.0, 67.8, 133.2, 201.2, 266.6,\) and 336.8 are presented.
in Fig. 6. Characteristic spreading of the Quasi-probabilities in the phase space is the result of non-linearities of the trap, however the initial state configuration is still recovered signalling collapse and revival phenomenon.

An improvement in the definition of collapse and revival pattern as the anharmonicity of the trapping potential increases, reflected in Population inversion plots, the Quasi- probability plots as well as the partial relative entropy plots is an extremely interesting feature. It is markedly clear in $\beta = 4$ case for the set of parameters used to characterize the ion-laser in trap system. Looking together at the population inversion and $S(P)$ plots we verify that the onset of collapse and revival is characterized by $S(P) = 0$. A well defined collapse and revival pattern is characterized by wide $S(P) = 0$ regions. Every collapse or revival corresponds to a peaked $S(P)$. Successive revival peaks show a lowering of the local maximum point which is an indicator of a dissipative irreversible change in the ionic state same being true for $S(P)$ peaks that indicate collapse of the population inversion.

Acknowledgments

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FIG. 1. Population Inversion $I(t)$ versus $t$ for $\beta = 3$ and $\tau = 0.0, 0.0047, 0.008$. 
FIG. 2. Partial mutual quantum entropy $S(P)$ versus $t$ for $\beta = 3$ and $\tau = 0.0, 0.0047, 0.008$. 
FIG. 3. Population Inversion $I(t)$ versus $t$ for $\beta = 4$ and $\tau = 0.0, 0.004, 0.008$. 
FIG. 4. Partial mutual quantum entropy $S(P)$ versus $t$ for $\beta = 4$ and $\tau = 0.0, 0.004, 0.008$. 
FIG. 5. Quasi-probability plots for $\beta = 4$ and $\tau = 0.0$ at $t = 0.0, 85.8, 171.4, 266.8, 388.2$ and 447.6.
FIG. 6. Quasi- probability plots for $\beta = 4$ and $\tau = 0.004$ at $t = 0.0, 67.8, 133.2, 201.2, 266.6, \text{ and } 336.8.$