Asymptotic behaviour of cylindrical waves interacting with spinning strings

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Abstract

We consider a family of cylindrical spacetimes endowed with angular momentum that are solutions to the vacuum Einstein equations outside the symmetry axis. This family was recently obtained by performing a complete gauge fixing adapted to cylindrical symmetry. In the present work, we find boundary conditions that ensure that the metric arising from this gauge fixing is well defined and that the resulting reduced system has a consistent Hamiltonian dynamics. These boundary conditions must be imposed both on the symmetry axis and in the region far from the axis at spacelike infinity. Employing such conditions, we determine the asymptotic behaviour of the metric close to and far from the axis. In each of these regions, the approximate metric describes a conical geometry with a time dislocation. In particular, around the symmetry axis the effect of the singularity consists in inducing a constant deficit angle and a timelike helical structure. Based on these results and on the fact that the degrees of freedom in our family of metrics coincide with those of cylindrical vacuum gravity, we argue that the analysed set of spacetimes represent cylindrical gravitational waves surrounding a spinning cosmic string. For any of these spacetimes, a prediction of our analysis is that the wave content increases the deficit angle at spatial infinity with respect to that detected around the axis.

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1 Introduction

The study of cylindrically symmetric spacetimes has received a lot of attention in general relativity [1, 2]. Cylindrical gravity possesses a non-trivial field content, and therefore provides a suitable arena where one can test and develop methods (such as solutions generating techniques [2] and quantisation procedures [3, 4]) which are capable of dealing with the infinite number of degrees of freedom of the gravitational theory. More importantly, cylindrical spacetimes have found application in describing (idealized) situations of interest in gravitational physics and astrophysics. One of these applications is the analysis of the propagation and interaction of gravitational waves.

The first family of exact solutions corresponding to time-dependent cylindrical waves in a vacuum seems to have been obtained by Beck [5]. This family was rediscovered by Einstein and Rosen in a systematic analysis of spacetimes which represent cylindrical or plane gravitational waves [6]. The waves considered by Einstein and Rosen display what
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has been called ‘whole-cylinder symmetry’ [7, 8], namely, they are not only cylindrically symmetric, but also linearly polarized, so that the metric can be written globally in a diagonal form [1]. The most general vacuum solution describing cylindrical waves was studied independently by Kompaneets and by Ehlers et al. [9]. These waves can be described by a gravitational model whose configuration space has two field-like degrees of freedom which are subject to generalized wave equations [10]. The general solution to such dynamical equations is not explicitly known.

A different type of physical phenomena that can be associated with cylindrical spacetimes are cosmic strings [11]. Cosmic strings are topological defects (characterized by a non-trivial homotopy group $\pi_1$) that could be formed during phase transitions predicted by grand unified theories [12]. In the particular case of a straight cosmic string with a non-vanishing mass per unit length, the exterior gravitational field is simply a vacuum cylindrical spacetime that presents a conical defect [11, 13]. The string may also possess spin, providing the exterior spacetime with an angular momentum [14, 15]. The existence of straight cosmic strings would have important consequences for astrophysics: they would give rise to discontinuities in the microwave background [16] and act as gravitational lenses [17]. It has also been argued that cosmic strings might have produced the density fluctuations that led to galaxy formation [18]. Nevertheless, recent observations indicate that the anisotropy of the cosmic microwave background has most probably originated in an inflationary scenario [19], although the possibility that cosmic strings are partially responsible for the formation of structure is still open [20].

In contradistinction to the properties displayed by cylindrical waves, whose spacetime is regular everywhere, a straight cosmic string can be described by a cylindrical spacetime whose symmetry axis is singular (quasi-regular, to be more precise [21]). This singularity corresponds to the line source provided by the string [22, 23]. The exterior is a conical geometry that possesses a deficit angle and an angular momentum which are proportional, respectively, to the mass and spin of the string [14, 15].

Another example of cylindrical spacetimes are those that represent cylindrical waves surrounding a straight cosmic string. The analysis of these spacetimes allows a simplified discussion of the interaction between gravitational waves and strings. A few (parameter-dependent) families of solutions of this type have been found explicitly [2, 24, 25, 26]. Essentially, these solutions have been constructed by applying the soliton technique of Belinski and Zakharov, which makes use of the existence of two commuting Killing vectors [2, 27]. The spacetimes obtained in this way generally present, at spatial infinity and/or around the line source, a dislocation in the direction of the axis [23, 28]. Exceptions are the Garriga-Verdaguer solutions [26], whose metric is diagonal, and a subfamily of the spacetimes discussed in [25]. A property that is shared by all of these solutions, as well as by the gravitational cylindrical waves and the straight cosmic string without spin (but not by the spinning string), is that the surface spanned by the rotational and translational Killing vectors admits an orthogonal surface, i.e. that the considered isometry group is orthogonally transitive [1]. In practice, this implies the vanishing of the metric components that mix the radial coordinate or the timelike one with the coordinates that describe the Killing trajectories.

One of the authors has recently considered all possible cylindrically symmetric spacetimes that are solutions to the vacuum Einstein equations outside the axis, allowing them to contain a non-zero angular momentum [29]. Note that cylindrical waves and straight
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Cosmic strings are included in this family of solutions. For all of the spacetimes in this family, it has actually been shown that, once the radial coordinate is chosen orthogonal to the group trajectories, the shift vector has non-vanishing projection on the Killing orbits provided that the angular momentum differs from zero. Therefore, orthogonal transitivity does not hold in this case. In addition, it has also been shown that, in any of these solutions, there exists a dynamically conserved quantity that describes the total energy per unit length in the axis direction [29]. A similar energy density was known to exist for purely gravitational cylindrical waves [10, 30, 31]. The energy density for those waves is actually a non-polynomial function of the total C-energy introduced by Thorne [8], and turns out to be positive and bounded from above. Although the total C-energy ceases to be a constant of motion when the angular momentum differs from zero, the results of [29] generalize the expression of the energy in the presence of an axial singularity endowed with spin. In particular, the energy density continues to be bounded both from below and above.

In this paper, we will argue that, apart from a possible dislocation, the spacetimes considered in [29] can, in fact, be interpreted as describing the interaction of a spinning cosmic string with a cylindrical gravitational wave. In this sense, it is worth noting that the degrees of freedom outside the symmetry axis are just those corresponding to the cylindrical reduction of vacuum general relativity, namely, the field-like degrees of freedom of cylindrical waves. Furthermore, we will prove that the singular behaviour allowed on the axis produces a conical geometry which is endowed with a constant angular momentum and whose deficit angle does not vary in time. These are precisely the effects of a spinning cosmic string. So the intuitive picture that one gets is that of a stringy defect surrounded by an empty cylindrical spacetime which differs from the Minkowskian vacuum in that it generally contains time-dependent gravitational fields.

In order to attain this picture, we will investigate the form of the metric, both at spatial infinity and around the symmetry axis. We will show that these asymptotic metrics can indeed be understood as being created by a spinning cosmic string interacting in a non-linear way with a cylindrical wave. We will also see that the main effect of the gravitational content at spatial infinity is to increase the deficit angle caused by the string. To reach these conclusions, we will first carefully determine the boundary conditions which guarantee that the analysed cylindrical solutions are rigorously defined. These boundary conditions are crucial to show that the system possesses non-trivial solutions that are physically acceptable and admit a well defined notion of energy density. If such boundary conditions did not exist, our whole analysis would simply remain as a formal discussion devoid of physical content. Furthermore, it is precisely the knowledge of these boundary conditions what will allow us to obtain the asymptotic behaviour of the metric and discuss the physical phenomena that can take place in spacetime.

The paper is organized as follows. In section 2, we summarize the main results of [29]. This includes the expression of the metric in terms of the field-like degrees of freedom of the system, the dynamical equations that these fields satisfy and the Hamiltonian that generates the evolution. In section 3 we discuss boundary conditions that guarantee that the metric expressions are well defined. The stability of these conditions is studied in section 4. Once a set of dynamically stable conditions has been determined, we check in section 5 that they really lead to a consistent Hamiltonian formalism. We then analyse the behaviour of the metric at large and short distances from the axis in sections 6 and
Section 7, respectively. In section 7, we also discuss the physical interpretation of the solutions. Section 8 contains the conclusions and further discussion. Finally, two appendices are added.

2 Spinning spacetimes

In this section, we briefly review the main results of [29] concerning the Hamiltonian formalism for cylindrical spacetimes in vacuo that, in principle, do not include the axis of symmetry. The starting point of this analysis is the Hamiltonian formulation of vacuum general relativity for spacetimes with two commuting Killing vectors, one of them translational and the other one rotational. It is possible to introduce coordinates \( x^a = \{ z, \theta \} \) (with \( z \in \mathbb{R}, \theta \in S^1 \) and \( a = 1, 2 \)) adapted to these Killing isometries so that the metric is independent of \( x^a \). In addition, we will call \( x^3 = r > 0 \) the radial coordinate, denote spatial indices with Latin letters from the middle of the alphabet and adopt units such that \( 4G = c = 1 \), where \( G \) is the Newton constant per unit length in the \( z \) direction.

All of the gravitational constraints can be removed from the system by imposing a gauge fixing that makes use of the cylindrical symmetry. Let us first consider the momentum constraints associated with the Killing vectors, which have the form \( H_a = -2 \partial_r (h_{ai} \Pi^{ir}) \). Here, \( h_{ij} \) is the induced 3-metric and \( \Pi^{ij} \) its canonically conjugate momentum [29]. These constraints can be eliminated by requiring that the metric components \( h_{ar} \) vanish. In this gauge, the solution to the constraints \( H_a = 0 \) is given by \( \Pi^{ar} = h^{ab} c_b / 4 \) \((a, b = 1, 2)\), where the \( c_a \)s are two constants related to global properties of the spacetime. In particular, if the spacetime is regular everywhere, including the axis, the constants \( c_a \) must vanish. The stability of the gauge conditions, on the other hand, determines the value of the components \( N^a \) of the shift vector. After this partial gauge fixing, the system has two constraints and the gravitational degrees of freedom reside in the components \( h_{ab} \) and \( h_{rr} \) of the induced metric.

In fact, the constants \( c_z \) and \( c_\theta \) describe, respectively, the linear momentum in the \( z \) direction and the angular momentum contained in the spacetime, both of them expressed as linear densities. In order to prove this statement, it suffices to remember that, modulo surface terms, the generators of the asymptotic rotations (far from the axis) and the asymptotic translations in the \( z \) direction both have the integral form \( \int dr N^i H_i \), where the shift \( N^i \) vanishes in the region \( r \ll 1 \) and tends to \( \delta^i_a \) when \( r \to \infty \), with \( a = 1 \) for translations and \( a = 2 \) for rotations [30]. Taking into account the expression of \( H_a \), a simple integration by parts shows that the considered generators are differentiable on the gravitational phase space, and hence well defined, provided that the neglected surface terms coincide with the limit of \( 2h_{ai} \Pi^{ir} \) when \( r \) tends to infinity. With our gauge fixing, this limit is half the constant \( c_a \). Therefore, on solutions to the gravitational constraints, the values of the analysed generators, which are the linear and angular momentum densities under consideration [30, 32], are given in our system of units by the constant quantities \( c_z / 2 \) and \( c_\theta / 2 \), respectively. In the following, we will allow the possibility that the cylindrical spacetime contains a non-zero angular momentum, but will restrict our attention to the case in which \( c_z \) vanishes. Otherwise, our gauge-fixing procedure would lead to metric expressions plagued with divergences that would render them meaningless [29].

The gauge freedom corresponding to the only momentum constraint that remains on
the system can be removed by choosing as a radial coordinate the square root of the determinant of the metric on Killing orbits, since this metric degenerates on the axis and its determinant is supposed to possess a spacelike gradient. The canonical momentum of this determinant can then be determined by solving the radial constraint on the gauge section. The requirement of stability of the gauge condition, on the other hand, fixes the radial component of the shift, which turns out to be proportional to the momentum $P_w$ canonically conjugate to the metric variable $w = \ln \sqrt{h_{zz} h_{rr}}$. One can then make the shift component $N^r$ equal to zero by imposing that $P_w$ vanishes as a gauge condition for the Hamiltonian constraint. In this way, one eliminates all the gauge degrees of freedom from the system. The expression of the metric function $w$ can be obtained by solving the Hamiltonian constraint with $P_w = 0$. In addition, the dynamical stability of the gauge (i.e. $\dot{P}_w = 0$, where the overdot denotes the time derivative) determines the value of the lapse function, assuming that it approaches the unity in the limit $r \to \infty$.

After this complete gauge fixing, the system can be described by two canonically conjugate pairs of field-like degrees of freedom, which we will call $(v, P_v)$ and $(y, P_y)$. The reduced metric can be written in the form

$$ds^2 = e^{2w+y} \left[ -\tilde{N}^2 dt^2 + dr^2 \right] + e^y r^2 (d\theta + N^\theta dt)^2 + e^{-y} \left[ dz - v d\theta + (N^z - v N^\theta) dt \right]^2,$$  

where

$$e^{2w} = \frac{4\bar{E}[r] r^2}{c_\theta^2 + 4D r^2 - 2c_\theta^2 r^2 \int_0^r ds s^{-3} (\bar{E}[s] - 1)^2},$$

$$\tilde{N} = F_\infty e^{-2w} \bar{E}[r], \quad F_\infty = \frac{e^{w_\infty}}{\bar{E}_\infty}, \quad \bar{E}[r] = \exp \left( \int_0^r \bar{H} \right),$$

$$\bar{H} = \frac{2}{r} \left[ \frac{(r \partial_r y)^2}{4} + \frac{(\partial_r v)^2}{4} e^{-2y} + P_y^2 + P_v^2 r^2 e^{2y} \right],$$

and the shift vector is

$$N^\theta = c_\theta F_\infty \left\{ \frac{1}{2r^2} + \int_r^\infty \frac{ds}{s^3} (\bar{E}[s] - 1) \right\}, \quad N^z = c_\theta F_\infty \int_r^\infty \frac{ds}{s^3} v \bar{E}[s].$$

In these equations, $w_\infty$ and $\bar{E}_\infty$ are, respectively, the values taken by $w$ and $\bar{E}[r]$ in the limit $r \to \infty$. To arrive at these expressions, the shift vector has been chosen to vanish when $r$ tends to infinity and the value of $y$ in this limit has been set equal to zero by an appropriate scaling of $r$, $z$, $c_\theta$ and the fields $(v, P_v)$. Finally, the parameter $D$, which determines the sub-leading terms in the metric function $e^{2w}$ around the axis, has been assumed to be a positive constant. This assumption is necessary if one wants that the dynamics of the gauge-fixed system is generated (at least formally) by a Hamiltonian [29]. We will show in section 5 that the constancy of $D > 0$ is, in fact, guaranteed by the boundary conditions that the fields must satisfy.

The dynamical evolution is dictated by the equations

$$\dot{v} = 2F_\infty P_v r e^{2y-2w} \bar{E}[r],$$

$$\dot{y} = 2F_\infty \frac{P_y}{r} e^{-2w} \bar{E}[r],$$
\[ \dot{P}_v = F_\infty \partial_r \left( \frac{\partial_v e^{-2y - 2w}}{2r} \hat{E}[r] \right), \text{ (2.8)} \]
\[ \dot{P}_y = F_\infty \partial_r \left( \frac{\partial_y y}{2} r e^{-2w} \hat{E}[r] \right) - \frac{F_\infty}{2r} e^{-2w} \hat{E}[r] \left[ 4P_v^2 r^2 e^{2y} - (\partial_r v)^2 e^{-2y} \right], \text{ (2.9)} \]

which are generated in the reduced system, via Poisson brackets, by the Hamiltonian \( H_R = 1 - e^{-w} \). Employing equation (2.2), one explicitly obtains
\[ H_R = 1 - \sqrt{C_D} \leq 1, \text{ (2.10)} \]

We are assuming that the integrals involved in these expressions converge; we will return to this point in section 3. An additive constant in the Hamiltonian has been fixed by requiring that the energy vanishes for Minkowski spacetime (i.e. when \( D = \hat{E}_\infty = 1, c_\theta = 0 \)).

Note that, since the reduced Hamiltonian must be real to define an acceptable time evolution, the functional expression (2.10) implies that the value of \( H_R \), which provides the energy density per unit length in the axis direction, is bounded from above by unity. The value \( H_R = 1 \) cannot be reached, because when \( e^{-w} \) vanishes the metric is ill-defined, according to our equations. On the other hand, taking into account that \( \dot{H} \) is a non-negative function on the reduced phase space, one can see that \( \hat{E}[r] \geq 1 \) for all values of \( r > 0 \). As a particular consequence, \( \Omega \) is non-negative. It is then easy to find a lower bound for the energy density, provided that the positive parameter \( D \) is fixed. One gets that \( H_R \geq (1 - \sqrt{D}) \). The lower bound is reached on solutions with vanishing momenta \( P_v \) and \( P_y \) and constant fields \( v \) and \( y \), because it is only then that \( \dot{H} \) vanishes and \( \hat{E}[r] = 1 \). Remembering that the value of \( y \) when \( r \to \infty \) has been set equal to zero and assuming that there exists no dislocation in the direction of the axis \([23, 28]\), we are then left with only one spacetime. This spacetime, which minimizes the energy density and can therefore be regarded as a background for the solutions with parameters \( D \) and \( c_\theta \), is precisely the region without closed timelike curves (CTCs) in the exterior of a spinning cosmic string \([29]\).

As we have briefly commented above, all the integrals over \( r \in \mathbb{R}^+ \) involved in the metric expressions must converge; otherwise, our previous calculations would lead to physically unacceptable solutions. Supposing that the fields \( \{v, y, P_v, P_y\} \) are sufficiently smooth as functions of \( r \) over the positive axis, the convergence can be ensured by imposing suitable boundary conditions on the fields, both at \( r = 0 \) and at infinity. We will discuss this point in section 3. Of course, such boundary conditions must hold at all times and, therefore, be preserved by the dynamical evolution. This issue will be analysed in section 4. In addition, it is clear from the form of the metric that, for real fields, the spacetime is Lorentzian with \( t \) being the time coordinate if and only if \( e^{2w} \) is positive for all values of \( r \). In fact, remembering that \( \dot{H} \) is a non-negative function on phase space and that \( D > 0 \), one can check from equation (2.2) that \( e^{2w} \) is strictly positive in \( r > 0 \) provided that \( e^{2w} \) is positive \([29]\). This last requirement amounts to the condition \( 2D > c_\theta^2 \Omega \). The only effect of this inequality when \( c_\theta \neq 0 \) is to restrict the admissible initial data for the fields \( \{v, y, P_v, P_y\} \). The reason is that \( e^{2w} \) is a constant of motion, because it does not depend explicitly on time (\( D \) is a constant) and commutes with the reduced Hamiltonian \( H_R = 1 - e^{-w} \) under Poisson brackets \([29]\). So \( e^{2w} \) remains positive in the evolution
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if it is originally positive. Hence, the inequality $2D > c_0^2 \Omega$ needs only be imposed at a given, initial time. Note also that $\Omega$ vanishes if so do all the fields. Consequently, for every value of $c_0$ and $D > 0$, there exists a region on phase space around the origin $v = y = P_v = P_y = 0$ where the considered inequality is satisfied.

On the other hand, accepting that the dynamical equations for our fields are valid not only in the region $r > 0$, but also when one approaches the axis, a consistency condition that must be verified is the constancy of the parameter $D$. Finally, if the considered spacetimes admit a reduced Hamiltonian formulation, it is necessary that the Hamiltonian be not only finite for physical solutions, but also differentiable on the corresponding phase space. All of these extra requirements will be analysed in section 5, where we will check that they are satisfied as a result of the boundary conditions imposed on the fields.

So, in summary, the situation is as follows. Although we have obtained a formal expression for the gauge-fixed metric and determined the equations of motion that satisfy the degrees of freedom, the existence of solutions with a rigorously defined metric is possible only if we can find suitable boundary conditions compatible with the dynamics and the constancy of $D$. Moreover, these solutions correspond to physically acceptable spacetimes only if the restriction $2D > c_0^2 \Omega$ is satisfied by the initial data. Finally, the reduced model possesses a well defined Hamiltonian formalism and a constant of motion that provides a valid notion of energy density only if the reduced Hamiltonian is differentiable. Hence, the importance of proving that the system admits a set of satisfactory boundary conditions and analysing their consequences.

3 Convergence of the integrals

In the rest of the paper, we will assume that the fields \( \{v, y, P_v, P_y\} \) are sufficiently smooth as functions of $r$ over the strictly positive axis. Then, any possible divergence in the integrals over $r$ that appear in the metric must arise either at infinity or around $r = 0$. We also assume that the inequality $2D > c_0^2 \Omega$ (necessary for $w$ to be real) is satisfied and that the value of $y$ when $r \to \infty$ has already been set equal to zero by a scaling of coordinates, fields and parameters [29]. However, we will not impose yet the vanishing of the limits $v_0$ and $v_\infty$ of the field $v$ on the axis and at infinity, respectively, although we will suppose that these limits are, at least, finite. So, for the moment, we will allow the presence in those regions of a dislocation in the $z$ direction.

In the rest of the paper, we will employ the following notation [33]. For any constant number $a$, the symbol $f = \bar{o}(r^a)$ when $r \to 0$ indicates that $f$ is much smaller than $r^a$ when one approaches the axis, so that the limit of $fr^{-a}$ vanishes at $r = 0$. Likewise, the notation $f = \bar{o}(r^a)$ when $r \to \infty$ means that the limit of $fr^{-a}$ vanishes at infinity. On the other hand, the symbol $f = o(g)$ when $r \to 0$ means that there exists a strictly positive number $\epsilon$ such that the limit of $r^{-\epsilon}f/g$ is zero on the axis. In addition, the symbol $f = o(g)$ when $r \to \infty$ implies that the limit of $r^\epsilon f/g$ at infinity vanishes for a certain $\epsilon > 0$. In those occasions in which it is clear from the context whether we are studying the behaviour of the solutions close to the axis or, in contrast, at spacelike infinity, we will employ the abbreviated notation $f = \bar{o}(r^a)$ and $f = o(g)$, obviating the appearance of the limit $r \to 0$ or $r \to \infty$. More importantly, denoting by $\xi$ any of the fields $\{v, y, P_v, P_y\}$, we will assume that the behaviour at infinity and around $r = 0$ is
smooth enough to guarantee that the condition $\xi = o(r^a)$ when $r \to \tilde{r}$ (with $\tilde{r} = 0$ or $\infty$) implies that $\partial_r \xi = o(r^{a-1})$ and $\partial^2_r \xi = o(r^{a-2})$ at $\tilde{r}$. Similarly, if $\xi = o(r^a)$ when $r \to \tilde{r}$, we assume that $\partial_r \xi = o(r^{a-1})$ and $\partial^2_r \xi = o(r^{a-2})$ in the considered limit.

We will also write $f = O(r^a)$ if there exist positive constants $M_1, M_2,$ and $M_3$ such that $|fr^{-a}| < M_1, |\partial_r fr^{-a}| < M_2,$ and $|\partial^2_r fr^{-a}| < M_3$ close to the axis $r = 0$. We will then say that the function $f$ is of the order or smaller than $r^a$ as $r \to 0$. In practice, as we have commented, we will only consider solutions whose fields are sufficiently smooth functions of $r$, including the limit in which one reaches the symmetry axis. For this kind of solutions, our definition of the symbol $f$ amounts to the existence of the limits of $fr^{-a}, \partial_r fr^{-a},$ and $\partial^2_r fr^{-a}$ when $r \to 0$.

With the assumptions introduced above, it is possible to check that our metric expressions are well defined for all values of $r > 0$ provided that $E_\infty$ is finite and that, if $c_0$ differs from zero, the integral $\Omega$ converges. The first of these conditions, $E_\infty < \infty$, implies that $H = o(r^{-1})$ on the axis and at infinity. Actually, a sufficient condition for $E_\infty$ to be finite is that $H = o(r^{-1})$ in the two considered regions. Although weaker conditions are possible, we will restrict our discussion to this quite general case from now on. Taking into account that $H$ is the sum of four non-negative factors, it is readily seen that our condition can be equivalently expressed in the form

\[
\begin{align*}
\partial_r v &= o(1), \\
P_v &= o(r^{-1}), \\
\partial_r y &= o(r^{-1}), \\
P_y &= o(1),
\end{align*}
\]

both when $r$ tends to zero and to infinity. For sufficiently smooth fields $v$ and $y$ on the axis, one then obtains that, around $r = 0$,

\[
v = v_0 + o(r), \quad y = y_0 + o(1).
\]

Here, $y_0$ is the limit of $y$ at $r = 0$, which we suppose finite. Similarly, far from the axis,

\[
v = v_\infty + o(1), \quad y = o(1).
\]

In the case that $c_0$ differs from zero, one still has to impose that $\Omega$ is finite. The convergence of the integral at infinity is already ensured by the finiteness of $E_\infty$. The convergence at $r = 0$, on the other hand, can be seen to require that $H = o(r)$ on the axis, and is satisfied, for instance, if the stronger condition $H = o(r)$ holds. Similar arguments to those presented above show that this last condition implies that, when $r \to 0$,

\[
v = v_0 + o(r^2), \quad y = y_0 + o(r), \quad P_v = o(1), \quad P_y = o(r).
\]

These conditions substitute for equations (3.2) and (3.3) on the axis when the spacetime contains a non-vanishing angular momentum.

Let us finally discuss the behaviour of the metric components (with our choice of coordinates) in the limit $r \to 0$. Although the axis is in principle excluded from our spacetime, the possible singularities at $r = 0$ are indeed quite weak, allowing the metric to be well defined there. In the absence of spin, $c_0 = 0$, it is straightforward to check that the metric components are finite on the axis when conditions (3.2) and (3.3) are satisfied. For spinning solutions, on the other hand, the condition $v = v_0 + o(r^2)$ when $r$ is small, together with the convergence of $\Omega$, can be seen to guarantee that $N^z - vN^\theta$ has a finite
limit when \( r \to 0 \). Then, the only metric component that can diverge on the axis is the diagonal \( t \) component. The potentially divergent terms in that component come from the contribution of \(-e^y[e^{2w}N^2 - r^2(N')^2]\). Given the expression of \( w \) and that \( \bar{H} = o(r) \), it turns out [29] that this expression has nevertheless a finite limit when \( r \to 0 \). Hence, the inclusion of spin does not destroy the finiteness of the metric at \( r = 0 \).

4 Dynamical Stability

Since the metric expressions must be rigorously defined at all instants of time, the boundary conditions on the fields must be satisfied at every single moment and, therefore, be compatible with the evolution dictated by equations (2.6)-(2.9). Otherwise, the system would not admit solutions that respect the conditions imposed at \( r = 0 \) and at infinity. In particular, this would imply that the metric expressions diverge on the solutions of the model, so that they would not lead to acceptable spacetimes. In this section, we will analyse whether the conditions introduced in section 3 are dynamically stable and, if the answer is in the negative, replace them with stronger conditions that are preserved in time.

Let us first study the asymptotic region far from the axis, \( r \to \infty \). Remembering that \( e^{2w_{\infty}} \) and \( \bar{E}_{\infty} \) are positive and finite, it is not difficult to check that equations (3.2) and (3.4), together with our equations of motion, imply that

\[
\dot{v} = o(1), \quad \dot{y} = o(r^{-1}), \quad \dot{P}_v = o(r^{-3}), \quad \dot{P}_y = o(r^{-1}).
\]

(4.1)

Then, our boundary conditions at infinity are stable, because the time derivatives provide subdominant contributions, compared to the leading terms in the fields. In particular, the behaviour \( \dot{v} = o(1) \) ensures that \( v_{\infty} \) is time independent. From now on, we will employ the notation \( v_{\infty}^c \) to remember the fact that this limit is constant.

The analysis of the stability around the symmetry axis \( r = 0 \) is much more involved. The solutions with and without spin must be studied separately, because the factor \( e^{-2w} \) that appears in the equations of motion behaves in a different way: it has a finite limit when \( r \to 0 \) if \( c_0 \) vanishes, but diverges like \( c_0^2/(4r^2) \) otherwise. In subsections 4.1 and 4.2 we will study, respectively, the families of spacetimes with zero and non-vanishing angular momentum.

Before doing this, nevertheless, let us point out that there exists at least an infinite family of solutions whose behaviour around the axis is the same in the presence or absence of spin. These solutions satisfy much stronger boundary conditions at \( r = 0 \) than those proposed in section 3. Namely, the fields \( v - v_0, y - y_0, P_v, P_y \), as well as their derivatives of any order with respect to \( r \), decrease at \( r = 0 \) faster than any homogeneous polynomial of \( r \). In other words, calling \( \{\psi\} \equiv \{v - v_0, y - y_0, P_v, P_y\} \), we have that the limit of \( r^{-a} \partial^m r \psi \) vanish when \( r \to 0 \) for all non-negative values of the integer numbers \( a \) and \( m \). Examining the equations of motion (2.6)-(2.9), one can see that these boundary conditions are indeed preserved by the evolution. The reason is that the right-hand side of those equations are given by terms in which we always find one of our fields, or one of their derivatives, multiplied by a factor that diverges, at most, like a negative power of \( r \). Therefore, all time derivatives turn out to vanish faster than any positive power of \( r \) on the axis. This proves the stability and implies that the limits of \( v \) and \( y \) at
$r = 0$ (i.e. $v_0$ and $y_0$) must be time independent on those solutions. Note also that the above set of spacetimes contains as a particular subfamily the solutions in which one can find a neighbourhood $r < r_0$ of the axis (with $r_0$ being a positive number) where the momenta $P_v$ and $P_y$ vanish and the fields $v$ and $y$ are constant. The existence of this special, infinite family of solutions was already noted in [29]. Finally, let us comment that the rapid decrease of the fields at $r = 0$ in these cylindrical spacetimes indicates a trivial interaction between the vacuum degrees of freedom and the axial singularity. Consequently, had we restricted our analysis to just this type of solutions, the physical interest of our discussion would be severely limited.

4.1 Vanishing angular momentum

We will now discuss the stability of the boundary conditions on the axis when the angular momentum vanishes. In this subsection, we will restrict our considerations to solutions that admit an expansion in powers of $r$ around the symmetry axis. Obviously, these do not include the solutions with rapid decrease of the fields at $r = 0$ commented above.

When $c_\theta$ vanish, the factor $e^{-2w\bar{E}[r]}$ that appears on the right-hand side of all the equations of motion reduces to the constant parameter $D$, greatly simplifying the calculations. In addition, accepting the existence of power series for the fields, the conditions (3.2) and (3.3) translate into

$$v = v_0 + O(r^2), \quad y = y_0 + O(r), \quad P_v = O(1) \quad \text{and} \quad P_y = O(r).$$

Employing this behaviour and the dynamical equation for $v$, one concludes that, in fact, $P_v = O(r)$ and the limit of $v$ must be constant. We will denote this constant value by $v_0^c$. Finally, equation (2.9) and our conditions restrict the field $y$ to have the form $y = y_0 + O(r^2)$. The limit $y_0$, on the other hand, does not need to be constant, but can vary in the evolution.

In conclusion, the dynamically stable boundary conditions on the axis are

$$v = v_0^c + O(r^2), \quad y = y_0 + O(r^2), \quad P_v = O(r), \quad P_y = O(r). \quad (4.2)$$

In particular, these conditions apply to cylindrical waves, case in which $D = 1$. Although the boundary conditions for these waves have already been discussed in the literature [4, 10], we have included their analysis for completeness. More importantly, it seems that the results of [10] for waves with general polarization contain some mistakes. Equations (3.2), (3.4) and (4.2) correct previous proposals and provide the behaviour that must be imposed on the fields in the asymptotic regions far and around the axis. For vanishing fields $v$ and $P_v$, on the other hand, these conditions can be seen to agree (modulo the supplementary assumption of expansions in powers of $r^{-1}$ at infinity) with those imposed by Ashtekar and Pierri for linearly polarized waves.

A more detailed analysis of the compatibility of the boundary conditions and the dynamics shows that the power series of the fields must be of the form

$$v = v_0^c + \sum_{m=0}^{\infty} V_m r^{2m+2}, \quad P_v = \sum_{m=0}^{\infty} P_m r^{2m+1},$$

$$y = y_0 + \sum_{m=0}^{\infty} Y_m r^{2m+2}, \quad P_y = \sum_{m=0}^{\infty} Q_m r^{2m+1}. \quad (4.3)$$

The coefficients of these series (except $v_0^c$) are functions of the rescaled time $\tau = F_\infty t$ (where we have used that $F_\infty$ is a constant of motion in the absence of spin). In fact,
let us suppose that we know, at all instants of $\tau$, the value of $y_0$ and the coefficients $\{V_m, Y_m, P_m, Q_m\}$ for all $m \leq n$, $n$ being an integer. It is not difficult to check that the power expansion of the equations of motion provides then all the information needed to determine the next-order approximation to the fields, i.e. the coefficients with subindex equal to $n + 1$. Moreover, the power series turn out to be determined just by the functions of time $y_0$ and $V_0$ and the constant $v_0^c$. Actually, once these coefficients are known, the lowest-order contributions in the dynamical equations for $v$ and $y$ allow one to obtain the values of $P_0$ and $Q_0$, respectively. In addition, with equation (2.9) and the knowledge of $y_0$, $V_0$ and $Q_0$, one can fix $V_0$. The iterative process outlined above leads then to the determination of all other coefficients.

Finally, let us note that, instead of the functions of the rescaled time $y_0$ and $V_0$, one can choose as degrees of freedom in the power series the values of $y_0$ and of all the sets of coefficients $\{V_m, Y_m, P_m, Q_m\}$ at a fixed, initial time $\tau_0$. These values, together with $v_0^c$, completely determine the fields at $\tau_0$. Given such initial data, the equations of motion, which are first-order differential equations, fix the evolution of the fields.

### 4.2 Case with spin

Let us now search for stable boundary conditions at $r = 0$ when $c_\theta$ does not vanish. Like in the previous subsection, we will only consider solutions whose fields $\{v, y, P_v, P_y\}$ admit a power expansion in $r$ around the symmetry axis. Again, this assumption does not hold in the family of spacetimes with a rapid decrease of the fields discussed at the start of section 4. In addition, we will restrict our discussion to solutions in which the field $v$ has a fixed constant limit at $r = 0$, which we will call $v_0^c$. In contradistinction with the situation found for vanishing angular momentum, where the stability of the boundary conditions requires that $v_0$ should be constant, a heuristic analysis of the equations of motion seems to indicate that now $\dot{v}_0$ may actually differ from zero. However, we will concentrate our attention exclusively on solutions whose spacelike helical structure in the vicinity of the axis corresponds, at most, to a constant dislocation [28]. This includes, in particular, the case of a spinning cosmic string [14, 15], in which the dislocation is absent. The possibility of $v_0$ being allowed to vary in time will be discussed elsewhere.

With our hypotheses, conditions (3.5) for the convergence of the integrals appearing in the metric expressions become

$$v = v_0^c + O(r^3), \quad y = y_0 + O(r^2), \quad P_v = O(r), \quad P_y = O(r^2). \quad (4.4)$$

On the other hand, according to equation (2.2), the factor $e^{-2w}E[r]$ that is present in all of the dynamical equations diverges at $r = 0$ like the inverse square of $r$ when $c_\theta \neq 0$. Therefore, the condition that $P_v$ should be at most of order $r$ is compatible with the equation of motion for this momentum only if $v = v_0^c + O(r^4)$. With $v_0^c$ being a constant, the equation for $\dot{v}$ then requires that $P_v$ must satisfy the stronger requirement $P_v = O(r^5)$. Concerning the conditions on $y$ and $P_y$, we have, in fact, two possibilities. For a generally time-dependent value of $y_0$, equation (2.7) imposes that $P_y = O(r^3)$. Then, it turns out that the subdominant correction to $y_0$ must be at least of order $r^6$. The reason is that, owing to the time independence of the parameter $D$, the system must satisfy the consistency condition $P_v \partial_v v + P_y \partial_y y = o(r^4)$, as we will see in section 5. This requirement, together with the stability of the condition on $P_y$, leads to the result $y = y_0 + O(r^6)$.
The second possibility is that the value of $y_0$ is a fixed constant, $y_0^c$. One would thus have $y = y_0^c + O(r^2)$. In this case the equation of motion for $y$ imposes the condition $P_y = O(r^2)$.

We may express these two sets of stable boundary conditions in the symbolic form

$$v = v_0^c + O(r^4), \quad P_v = O(r^6), \quad y = y_0^{(\kappa)} + O(r^{4+2\kappa}), \quad P_y = O(r^{4-\kappa}),$$

(4.5)

where the parameter $\kappa$ can be equal to either plus or minus unity, and $y_0^{(1)}$ generally depends on time, whereas $y_0^{(-1)}$ is an alternative notation for $y_0^c$. Note that the possibility $\kappa = -1$ was not included in the discussions of [29].

With the above boundary conditions, one can see that, on solutions to the equations of motion, the power expansions of the fields must be of the following type

$$v = v_0^c + \sum_{m=0}^{\infty} V_m r^{2m+4}, \quad P_v = \sum_{m=0}^{\infty} P_m r^{2m+5},$$

$$y = y_0^{(\kappa)} + \sum_{m=0}^{\infty} Y_m r^{2m+4+2\kappa}, \quad P_y = \sum_{m=0}^{\infty} Q_m r^{2m+4-\kappa}.$$

(4.6)

Except $v_0^c$ and $y_0^{(-1)} = y_0^c$, which are constants, all the coefficients in these series depend, in principle, on the modified time coordinate $\tau = \int_0^t \frac{1}{F_\infty(\tilde{t})} d\tilde{t}$. This redefinition of time absorbs the common factor $F_\infty$ that appears in the dynamical equations.

Actually, all the coefficients in these series can be found if one knows, at all instants of $\tau$, the values of $v_0^c$, $V_0$, and $y_0^{(\kappa)}$, as well as the value of $Y_0$ when $\kappa = -1$. The proof of this statement is sketched in appendix A. The main difference with respect to the situation described when the angular momentum vanishes is that now the coefficients of the power expansion are not all functionally independent; instead, they satisfy relations which do not involve time derivatives. This issue is also discussed in appendix A. As a result of such functional relations, one can, in fact, prove by induction that, at a generic initial time $\tau_0$, the series (4.6) can be completely determined if one knows at that moment just the values of $v_0^c$, $y_0^{(\kappa)}$ and all the sets of the form $\{V_n, Q_n, P_n, Y_n\}$, where $n$ is any non-negative integer. Note that, since the dynamics is dictated by first-order differential equations, the commented collection of coefficients, evaluated at $\tau_0$, provide then all the information needed to fix the power expansion of the fields around $r = 0$ at all instants of time.

From these comments, it is also clear that a possible procedure to construct admissible solutions is the following. At a certain initial time, determine the initial values of the fields in a region around $r = 0$ by fixing the collection of independent coefficients given above. Try then to analytically continue such initial values to the whole positive semiaxis. If the continuation is possible and satisfies the boundary conditions (3.2) and (3.4), use such initial data, together with the equations of motion, to arrive at a physical solution. Otherwise, employ as initial data the result of a smooth matching between the initial values obtained around $r = 0$ and any initial fields in the region far from the axis that satisfy the boundary conditions at infinity.

5 Consistency of the formalism

In the previous section, we have proved that there exist boundary conditions that ensure that the metric expressions, which had been obtained by means of formal integrations, are
meaningful at all instants of time on the solutions of the system. In order to prove that such solutions lead, in fact, to physically acceptable spacetimes with a rigorously defined energy density, we want to show now that the introduced boundary conditions guarantee also that the dynamics is fully consistent and that the system possesses a well defined (reduced) Hamiltonian formalism. Let us remind that, in the asymptotic region \( r \to \infty \), the boundary conditions are given by equations (3.2) and (3.4), where the limit of \( v \) is a constant (i.e. \( v_\infty = v_c^\infty \)). On the other hand, close to the axis \( r = 0 \), the behaviour of the fields is dictated by equation (4.2) if the angular momentum vanishes, and by equation (4.5) if \( c_\theta \) differs from zero. Remember also that, in this last case, the constant \( \kappa \) can adopt the values \( \pm 1 \). In addition, it is worth noting that, although this behaviour on the axis was deduced assuming that the basic fields admit power expansions in \( r \), the conditions apply as well to the set of solutions with rapid decrease of the fields at \( r = 0 \) discussed in section 4. So, for all of the considered solutions, it will suffice to prove the consistency of the Hamiltonian dynamics when requirements (4.2) or (4.5) are satisfied.

We will first prove that the dynamics is compatible with the time independence of the parameter \( D \), assuming that the equations of motion remain valid in the limit \( r \to 0 \). Note that, if this compatibility could not be reached, one would be forced either to admit that the system does not possess physically acceptable solutions or to try and introduce rather artificial sources on the symmetry axis that could account for the constancy of \( D \).

The constant \( D \) determines the subleading term in \( e^{2w} \) around the axis \( r = 0 \) in the presence of spin, and the leading contribution when \( c_\theta \) vanishes. In the case with non-zero angular momentum, using that \( \bar{H} = o \left( r^{-1} \right) \) for \( r \to 0 \), one can check

\[
e^{2w} = \frac{4}{c_\theta^2} r^2 - \frac{16D}{c_\theta^4} r^4 + o(r^4). \tag{5.1}
\]

Similarly, when there is no angular momentum, the condition \( \bar{H} = o(r^{-1}) \) ensures that \( e^{2w} = 1/D + o(1) \). Therefore, the hypothesis that \( D \) is constant is compatible with the dynamics if and only if \( \partial_t(e^{2w}) = o(r^4) \) when \( c_\theta \neq 0 \), and \( \partial_t(e^{2w}) = o(1) \) when \( c_\theta \) vanishes. On the other hand, the equations of motion for our system, together with the expression of \( e^{2w} \), lead to [29]

\[
\partial_t(e^{2w}) = 2F_\infty \bar{E}[r] \left( P_\phi \partial_r v + P_\theta \partial_r y \right). \tag{5.2}
\]

Since \( \bar{E}[0] = 1 \), we then conclude that the term \( P_\phi \partial_r v + P_\theta \partial_r y \) must be of the form \( o(r^4) \) around the axis when \( c_\theta \) differs from zero, and of the form \( o(1) \) in the absence of spin. It is then easy to check that the respective boundary conditions (4.5) and (4.2) guarantee this requirement.

In order to present a well defined Hamiltonian dynamics, our family of cylindrical spacetimes must not only possess a real and finite reduced Hamiltonian \( H_R \), but this Hamiltonian must also be differentiable on phase space. Otherwise, \( H_R \) would not generate a true canonical transformation and, therefore, a valid time evolution [30]. Note that, if that happened to be the case, the system would be missing an acceptable notion of energy density. We have already commented that, once the integrals \( \bar{E}_\infty \) and \( \Omega \) that appear in equation (2.10) are known to converge, the reality and finiteness of \( H_R \) amount just to the condition \( 2D > c_\theta^2 \Omega \). This inequality is preserved in the evolution, and can hence be regarded as a mere restriction on the initial values of the fields [29]. Moreover, there always exists a non-empty region of phase space where this inequality is satisfied. On the
other hand, it is proved in appendix B that our boundary conditions guarantee also that
the Hamiltonian \( H_R \) is, in fact, differentiable. Hence, our conditions, together with the
restriction \( 2D > c_\theta^2 \Omega \) on the initial values, ensure that the Hamiltonian dynamics is well
defined and that there exists a meaningful constant of motion that provides the energy
density.

6 Metric at spacelike infinity

Once we have determined the boundary conditions that must be satisfied by the basic
fields of the system, we can discuss the behaviour of the metric close to the axis \( r = 0 \)
and at spacelike infinity, \( r \gg 1, |t| \). The knowledge of this behaviour will be essential to
determine the geometrical properties of our solutions and reach a well founded physical
interpretation of the vacuum spacetimes under consideration. Let us first analyse the
region at spacelike infinity.

Taking into account that when \( r \to \infty \) the behaviour of the fields is governed, for
all values of the spin parameter \( c_\theta \), by conditions (3.2) and (3.4), with \( v_\infty = v_\infty^c \) being a
constant, a trivial calculation shows

\[
E[r] = E_\infty[1 + o(1)], \quad e^{2w} = e^{2w_\infty} + o(1),
\]

\[
N^\theta = \frac{c_\theta}{2r^2} e^{w_\infty} + o(r^{-2}), \quad N^z - vN^\theta = \bar{o}(r^{-2}).
\]  

(6.1)

The cylindrical metric adopts then the asymptotic form

\[
ds^2 = [1 + o(1)] \left[-(dt - e^{w_\infty} \frac{c_\theta}{2} d\theta)^2 + r^2 d\theta^2 + e^{2w_\infty} dr^2 \right] + [1 + o(1)] \left\{dz + \bar{o}(r^{-2}) dt - [v^c_\infty + \bar{o}(1)] d\theta \right\}^2.
\]  

(6.2)

Disregarding the possible existence of a constant dislocation in the \( z \) direction (i.e. taking
\( v^c_\infty = 0 \)) and neglecting contributions of the form \( o(1) \) in the metric, as well as terms of the
type \( o(r^2) \) and \( \bar{o}(1) \) in the diagonal \( \theta \) component and the \( z\theta \) component, respectively, we obtain precisely the metric that would be created by a spinning cosmic string. Moreover,
the density of angular momentum of the studied cylindrical solution, \( c_\theta/2 \), coincides with
the spin (per unit length) of the string that would produce the approximate metric. In
addition, the reduced Hamiltonian of our spacetime equals the energy density of the string,
namely, \( 1 - e^{-w_\infty} \). Obviously, the deficit angle at spacelike infinity is simply this energy
density multiplied by a factor of \( 2\pi \).

The resemblance in the analysed asymptotic region to the exterior metric of a spinning
string is considerably enhanced when one restricts one’s attention to solutions with

\[
v = \bar{o}(r^{-2}), \quad y = \bar{o}(r^{-2}), \quad P_v = \bar{o}(r^{-3}), \quad P_y = \bar{o}(r^{-1}).
\]  

(6.3)

Note that this behaviour guarantees that conditions (3.2) and (3.4) are satisfied. A
particular subfamily of spacetimes in which these asymptotic restrictions hold is that
formed by all solutions with fields of compact support, namely, (smooth) solutions whose
fields \( v, y, P_v \) and \( P_y \) vanish in the region \( r \geq r_1 \), with \( r_1 \) being a positive number.
From the equations of motion of our system, it is not difficult to check that the compact
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support of these fields is respected in the evolution, so that the noted kind of solutions exist. Employing conditions (6.3), a detailed calculation leads to the following metric at spacelike infinity:

\[ ds^2 = \left[ 1 + \bar{o}(\rho^{-2}) \right] \left[ -\left( dt - e^{w} \frac{C_0}{2} d\theta \right)^2 + \rho^2 d\theta^2 + e^{2w} d\rho^2 + dz^2 \right] + \bar{o}(\rho^{-2}) dz d\theta + \bar{o}(\rho^{-4}) dt dz. \] (6.4)

Here \( \rho^2 = r^2 + e^{2w} c_0^2 / 4 \) is a new radial coordinate. So we see that the approximation in the asymptotic region to the exterior metric of a (possibly) spinning string has been improved, with respect to the general case, to the level \( \bar{o}(1) \) in the diagonal \( \theta \) component and at least to the level \( \bar{o}(r^{-2}) = \bar{o}(r^{-2}) \) in the rest of the metric components.

7 Metric near the axis

7.1 Vanishing angular momentum

Since the boundary conditions at \( r = 0 \) are different for solutions with and without angular momentum, the corresponding behaviour of the metric must be studied separately. Let us first analyse the case in which the axis is not endowed with spin. The conditions (4.2) on the fields imply that \( \bar{E}[r] = 1 + O(r^2) \), and it is then straightforward to see that

\[ ds^2 = \left[ 1 + O(r^2) \right] e^{y_0} \left[ -dt^2 + r^2 d\theta^2 + \frac{dr^2}{D} + e^{-2y_0} dz^2 \right] + O(r^2) dz d\theta, \] (7.1)

where \( \bar{z} = z - v_0^c \theta \) and \( \bar{t} = t/\sqrt{\bar{E}_\infty} \) is a rescaled time. In defining this coordinate, we have employed that \( \bar{E}_\infty = D e^{2w} \) is a constant of motion when \( c_\theta \) vanishes. Neglecting contributions or order \( r^4 \) to the diagonal \( \theta \) component and of order \( r^2 \) to the rest of the metric elements, we see that, near the axis, the metric describes the exterior of a straight cosmic string, except for the possible periodic structure in the \( \bar{z} \) coordinate and the allowed time dependence of the factor \( e^{y_0} \), which remains constant for a static string.

Several comments are in order at this point. First, let us remark that the time coordinates used in our approximations near the axis and at spacelike infinity differ by a scaling that depends on the considered solution. It is not difficult to see that this scaling reduces to the identity only for the flat solution with vanishing fields \( v - v_0^c \), \( y \), \( P_y \) and \( P_v \). This solution is possible only if \( v_0^c = v_\infty^c \) and \( y_0 = 0 \).

On the other hand, the periodicity of \( \theta \) leads to the identification of points \( (\theta, \bar{z}) \) and \( (\theta + 2\pi, \bar{z} - 2\pi v_0^c) \). To avoid this spacelike helical structure, we will assume in the following that \( v_0^c \) vanishes. Let us also note that there exist solutions in which \( y_0 \) is actually constant. For instance, this occurs in the solutions with fields that decrease rapidly at \( r = 0 \), as discussed in section 4. Furthermore, we proved in subsection 4.1 that the time-dependent coefficients \( y_0 \) and \( V_0 \) could be considered as the true degrees of freedom contained in the expansions (4.3) around the axis. Therefore, the possibility that \( y_0 \) is set equal to a constant is indeed available.

In addition, it is known that (at least when the angular momentum vanishes) cylindrical gravity in a vacuum can be reduced to three dimensions by employing the existence
of a translational Killing vector \([1, 30]\). This reduction leads to three-dimensional gravity with axial symmetry coupled to two scalar fields, namely \(v\) and \(y\) \([34]\). From this point of view, the metric in three dimensions associated with our spacetimes in the vicinity of the axis is \(ds^2 = -d\bar{t}^2 + r^2d\theta^2 + dr^2/D\), modulo relative corrections of the order of \(r^2\) or smaller. This is exactly the 3-metric created by a point particle of mass \(m = 1 - \sqrt{D}\) \([14]\). With this perspective, the time variation displayed by \(y_0\) in equation (7.1) can be understood just as a time-dependent lift of the metric from three to four dimensions. This time dependence is generally necessary to compensate the radial variation of the fields around the axis, so as to finally arrive at a vacuum solution for \(r > 0\).

We also note that the metric (7.1) presents a deficit angle around the axis, a property that is characteristic of stringy defects. Moreover, this deficit angle is independent of time, namely \(2\pi(1 - \sqrt{D})\), as it would correspond to a static string with linear energy density equal to \(m\), i.e. the mass of the point particle obtained in the three-dimensional version of the system. Thus, we see that, like in the case of a straight string, the effect of the singularity on the axis is to introduce a constant deficit when \(D \neq 1\). Note that, in order for this deficit to be positive, one must restrict the parameter \(D\) to be smaller than unity. For \(D = 1\), on the other hand, the spacetime is completely regular \([29]\) and the considered solutions are purely gravitational waves. Finally, it is worth commenting that the approximations of the metric given in equations (6.2) and (7.1) can be checked to describe the behaviour of the solutions with quasi-regular axis analysed in Refs. \([24, 25]\), as well as those contained in the soliton spacetimes of class A2 and B2 constructed by Garriga and Verdaguer \([26]\). In agreement with the above discussion, such solutions have been interpreted as representing the interaction of cylindrical waves with a cosmic string.

### 7.2 Case with spin

When \(c_\theta \neq 0\), there exist two acceptable sets of boundary conditions (assuming a constant value for \(v\) at \(r = 0\)), each of them leading to a different metric behaviour. We will first consider the case in which the field \(y\) may have a time-dependent limit on the axis. The corresponding boundary conditions are given by equation (4.5) with \(\kappa = 1\). Remember that these conditions apply as well to solutions with fields that decrease rapidly at \(r = 0\), even though they were originally deduced for fields that admit an expansion in powers of \(r\). One can then easily see that

\[
e^{-2w} = \frac{c_\theta^2}{4r^2} \left[1 + \frac{r^2}{A^2} + O(r^6)\right], \quad \bar{E}[r] = 1 + O(r^6),
\]

\[
N^\theta = \frac{c_\theta}{2r^2} F_\infty [1 + 2\Omega r^2 + O(r^6)], \quad N^z - v N^\theta = N^z_0 + O(r^2). \tag{7.2}
\]

Here, \(\Omega\) is the convergent integral defined in equation (2.10) and

\[
N^z_0 = c_\theta F_\infty \int_0^\infty \frac{dr}{r^3} (v - v_0^c) \bar{E}[r], \quad A = \frac{c_\theta}{2\sqrt{D}}. \tag{7.3}
\]

Note that \(N^z_0\) is finite, given our boundary conditions. Introducing then the coordinates

\[
\bar{t} = \sqrt{D} \int_0^\tilde{t} d\tilde{t} F_\infty(\tilde{t}), \quad \bar{\theta} = \theta + c_\theta \int_0^\tilde{t} d\tilde{t} \Omega(\tilde{t}) F_\infty(\tilde{t}),
\]

\[
\rho^2 = r^2 + A^2, \quad \bar{z} = z - v_0^c \theta + \int_0^\tilde{t} d\tilde{t} N^z_0(\tilde{t}), \tag{7.4}
\]
the metric around the axis \( r = 0 \) can be expressed in the form

\[
\begin{align*}
\mathcal{A} = 1 + O(\rho^6) e^{y_0(1)} & \left[ - \left( d\bar{\theta} - A d\bar{\rho} \right)^2 + \rho^2 d\bar{\rho}^2 + \frac{d\rho^2}{D} + e^{-2y_0(1)} d\bar{z}^2 \right] \\
+ O(\rho^2 d\bar{t} d\bar{z} + O(\rho^4) d\bar{z} d\bar{\theta} + O(\rho^4)(d\bar{t})^2).
\end{align*}
\] (7.5)

In the definition of our new coordinates, we have taken into account that the quantities \( F_\infty, N_0^0 \) and \( \Omega \) may implicitly depend on time when \( c_\theta \neq 0 \). Like in the absence of angular momentum, the time and angular coordinates of the expansions around the axis and at spacelike infinity [see equation (6.2)] differ for all but the flat solution, which has vanishing momenta \( \rho \) and \( y \) and constant fields \( v = v_0^c \) and \( y = 0 \). Concerning the axial coordinate \( \bar{z} \), we see that the periodicity of \( \bar{\theta} \) introduces again non-trivial identifications of points in the sections of constant time unless \( v_0^c \) vanishes. We will thus restrict our discussion to the case \( v_0^c = 0 \) from now on. The coordinates \( z \) and \( \bar{z} \) can then be seen to coincide on the flat solution; otherwise they generally differ. Finally, note that the new coordinate \( \rho \) is defined only over the semiaxis \((|A|, \infty)\), because, in our spacetime, \( r \) must be positive. In principle, however, one could try and make an analytic extension to values of \( \rho \) smaller than \(|A|\). This would correspond to imaginary values of \( \bar{r} \). It is important to remark that, in any case, such an extension would give rise to the appearance of CTCs because, from equation (7.5) and at least in the region with \( \rho \) smaller but close to \(|A|\), the diagonal \( \bar{\theta} \) component of the metric would then be negative, indicating the presence of a timelike vector field with closed orbits.

In terms of \( \rho \), the expression \( O(\rho \nu) \) can be rewritten in the form \( O(|\rho - |A||^a) \) for any real number \( a \). On the other hand, disregarding the small \( \bar{t} \bar{z} \) and \( \bar{z} \bar{\theta} \) components, the contributions of the type \( O(\rho^4) \) to the diagonal \( \bar{t} \) component, and all relative corrections to the metric of the order of \( \rho^6 \), we arrive at a metric near the section \( \rho = |A| \) that describes the gravitational field created by a spinning string, except for the fact that now \( y_0 \) may depend on time. Of course, this time dependence does not show up in the solutions with rapidly decreasing fields, since \( y_0 \) is then a constant. In addition, from our discussion in subsection 4.2, where we showed that the function \( y_0(t) \) could be considered as one of the degrees of freedom contained in the series (4.6), it follows that there exist solutions with power series at \( r = 0 \) in which \( y_0 \) remains constant in time.

In the general case in which \( y_0 \) depends on time, a line of reasoning similar to that presented for spacetimes with vanishing momenta allows us to regard the approximation of the metric around \( \rho = |A| \) as a time-dependent lift to four dimensions of the three-dimensional metric produced by a rotating point particle with mass \([14]\). In addition, it is worth noting that the approximated metric again presents a deficit angle that is constant. In this spirit, the arguments given in subsection 7.1 support the interpretation of the metric in the analysed region as that associated with a spinning string interacting with an environment of gravitational waves. This spinning string is characterized by two parameters, namely, the corresponding density of angular momentum \( c_\theta / 2 \) and the deficit angle \( 2\pi(1 - \sqrt{D}) \). Actually, this value of the deficit is not modified by the presence of spin, since it takes the same expression when \( c_\theta \) vanishes.

To end this section, we will discuss the behaviour of the metric around the axis \( r = 0 \) when the limit of \( y \) is a fixed constant \( y_0^c \), so that the boundary conditions are given by
equation (4.5) with \( \kappa = -1 \). These imply

\[
e^{-2w} = \frac{c_\theta^2}{4r^2} \left[ 1 + \frac{r^2}{A^2} + O(r^4) \right], \quad \bar{E}[r] = 1 + O(r^4),
\]

\[
N^\theta = \frac{c_\theta}{2r^2} F_\infty [1 + 2\Omega r^2 + O(r^4)], \quad N^z - vN^\theta = N^z_0 + O(r^2).
\]

Then, employing equation (7.4), we can write the metric in the form

\[
ds^2 = \left[ 1 + O(r^4) \right] e^y \left[ -\left( d\bar{t} - A d\bar{\theta} \right)^2 + \rho^2 d\theta^2 + \frac{d\rho^2}{D} \right] + e^{-y} dz^2 + O(r^2) d\bar{t} d\bar{z} + O(r^4) d\bar{z} d\bar{\theta}.
\]

Here, \( e^y = e^{y_0} + O(r^2) \). Note that, around \( \rho = |A| \), the metric presents, in fact, a constant deficit angle. The corrections to this deficit are at most of the order of \( r^4 \), even if the subdominant contributions to \( y_0^c \) can be of the type \( O(r^2) \). Like in the case \( \kappa = 1 \), we will restrict our attention to the possibility that \( v_0^c \) vanishes. Since \( y_0^c \) is a constant, it is then clear that the metric in the region \( 0 < \rho - |A| \ll 1 \) can be interpreted as that originated by a string with spin equal to \( c_\theta/2 \) and a conical deficit given by \( 2\pi(1 - \sqrt{D}) \).

## 8 Discussion and conclusions

We have considered the most general cylindrical solution to the Einstein equations in vacuo that, in principle, does not contain the symmetry axis \( r = 0 \). This axis, which may be singular, has been allowed to possess spin. However, to arrive at metric expressions that are well defined in the whole region \( r > 0 \) [29], we have supposed that the linear momentum in the axis direction vanishes. These spacetimes were analysed in a recent work [29], where a gauge-fixing procedure that removes all of the gravitational constraints was introduced. The resulting reduced model can be described by four fields on phase space, \( \{v, y, P_v, P_y\} \), whose dynamics is generated, at least formally, by a reduced Hamiltonian \( H_R \). The singularity on the axis can be characterized by two parameters: a real constant \( c_\theta \) that provides (twice) the density of angular momentum and a positive number \( D \) that determines the behaviour of the purely radial component of the metric around \( r = 0 \). The latter of these parameters should be a constant in order for the dynamics to be consistent and the system to possess a conserved energy density [29], given by the value of \( H_R \).

We have first searched for boundary conditions which ensure that the reduced system obtained after completing the gauge fixing is well defined. These conditions describe the behaviour of the basic fields \( \{v, y, P_v, P_y\} \) at spacelike infinity, \( r \gg 1, |\bar{t}| \), and near the axis \( r = 0 \). More explicitly, the consistency of the system implies the following requirements. First, all the metric expressions which have been found by means of formal integrations must be meaningful. Second, in order for the reduced Hamiltonian to generate a well-defined dynamics and provide the energy density of the system, \( H_R \) must be real, finite and differentiable on phase space. Third, these requirements must be satisfied at all instants of time and, therefore, the boundary conditions must be dynamically stable. Finally, assuming that the equation of motion for the metric function \( w \) (that appears in the diagonal radial component) remains valid in the limit \( r \to 0 \), one has to check that the
value of the parameter $D$ is, in fact, preserved in the evolution. The existence of boundary conditions that satisfy the above requirements is most fundamental; otherwise, the reduced dynamics would not be consistent and there would not exist physically acceptable (non-trivial) solutions in cylindrical vacuum gravity endowed with a non-vanishing angular momentum.

In our discussion, we have supposed that our basic fields are smooth over the whole semi-axis $r > 0$ for all possible values of the time coordinate $t$. In addition, denoting by $\xi$ any of these fields, we have assumed that both the requirements $\xi = o(r^a)$ and $\xi = o(r^a)$, either at $r = 0$ or at infinity, automatically imply that the derivatives $\partial_r \xi$ and $\partial_r^2 \xi$ display a behaviour similar to that of the field $\xi$, but with the exponent $a$ replaced with $a - 1$ and $a - 2$, respectively. With this assumption, the boundary conditions at spacelike infinity turn out to be given by equations (3.2) and (3.4). In the latter of these equations, $v_\infty$ (i.e. the asymptotic limit of $v$) is a fixed number that reflects the possible existence of a constant dislocation in the $z$ direction. To avoid the appearance of an asymptotic spacelike helical structure [28], one only has to make $v_\infty = 0$.

The boundary conditions that must be introduced on the axis $r = 0$, on the other hand, differ for solutions with or without angular momentum. When $c_\theta$ vanishes, the conditions are given by equation (4.2). Again, to prevent the existence of a screw dislocation in the axis direction [28], one must demand that $v$ vanishes at $r = 0$, i.e. $v_0 = 0$. For non-zero spin, there exists more than one set of admissible boundary conditions. We have only considered in detail the case in which the limit of $v$ on the axis $r = 0$ is constant, thus corresponding to a fixed dislocation. Then, the behaviour of the fields depend on whether the limit of $y$ is also fixed and time independent or, in contrast, is allowed to vary in time. In both situations, the boundary conditions can be written in the symbolic form (4.5), where $\kappa = -1$ for the first of the considered possibilities ($y_0^{-1}$ is constant) and $\kappa = 1$ otherwise.

By means of a careful analysis of the equations of motion and the Hamiltonian, we have proved that these boundary conditions are stable and that, together with the requirement $2D > c_\theta^2 \Omega$ (which restricts the admissible initial data for the basic fields), guarantee that the metric and Hamiltonian dynamics are rigorously defined. We have also checked that the evolution is compatible with the constancy of $D$. So all consistency conditions are satisfied. In addition, employing the boundary conditions, we have been able to calculate the approximate form of the metric in the region close to the axis $r = 0$ and at spacelike infinity. The physical picture that arises from this analysis is the following.

In the region near $r = 0$ (and assuming that $v_0 = 0$), the metric describes a ‘rotating’ conical geometry, with constant deficit angle and a generally non-vanishing angular momentum. This approximate metric corresponds to the lift from three to four dimensions of the metric produced by a point particle with constant mass and spin given by $1 - \sqrt{D}$ and $c_\theta/2$. The lift is in general time dependent, to account for the possible variation of the fields around $r = 0$ and yield a solution to the Einstein equations at all points of the spacetime. The 4-metric can thus be regarded as that caused, in the analysed region, by a spinning cosmic string with linear density of energy and angular momentum determined by the mass and spin parameters of the analogous particle in three dimensions. Note that, in order for this energy density to be non-negative, one must restrict the parameter $D$ to be equal or smaller than unity. It is also worth remarking that the time and cylindrical coordinates that are naturally associated with this spinning string do not coincide with
those selected in our gauge fixing, which are specially adapted to the asymptotic region far from the axis. The relation between both sets of coordinates is given by equation (7.4) (with $v_0^c = 0$). This relation continues to be valid even if $c_0$ vanishes [see equation (6.2)].

Of particular importance is the change of radial coordinate in the presence of spin. This change is precisely that which would remove the region with CTCs from the exterior of a spinning cosmic string, mapping the resulting spacetime to the sector $r > 0$. This opens the possibility of analytically continuing our spacetime to the region of imaginary values of $r$ (namely, to $\rho < |A|$) at the price of introducing CTCs. Note that, in our original spacetime, the existence of CTCs is actually precluded. An infinite family of solutions in which the continuation to imaginary values of $r$ can be straightforwardly performed is that with fields of rapid decrease at $r = 0$, considered in the beginning of section 4. It is clear that the metric of any of these spacetimes can be matched smoothly at $r = 0$ with the metric of a spinning cosmic string describing the region $\rho \leq |A|$, providing in this way an extended solution that contains timelike orbits generated by $\partial_\theta$.

Since our spacetimes are the most general solution to the Einstein equations in a vacuum for $r > 0$, the only field content outside the axis $r = 0$ is that corresponding to gravitational waves. Consequently, the studied spacetimes represent an ensemble of gravitational waves surrounding a singular axis that is characterized by its spin density and by the constant deficit angle detected in its vicinity. These are precisely the effects that a spinning string would produce. In this sense, the analysed spacetimes can be regarded as describing the most general interaction that is permitted in general relativity between cylindrical waves and strings with constant density of energy and spin.

This interpretation is in accordance with the asymptotic form of the metric (6.2) at spatial infinity. It is known that, in the region $r \gg 1, |t|$, a cylindrical wave causes a deficit angle that is proportional to the total energy density contained in the wave; namely, the deficit is $2\pi (1 - \sqrt{\bar{E}_\infty})$ [10]. In addition, the wave does not carry angular momentum. Therefore, in the case that the gravitational wave surrounds a cosmic string, the deficit angle at spatial infinity should be produced by the combined effect of both phenomena, and the angular momentum should be originated in the string. So, for a cylindrical wave interacting with a spinning string, one would expect the asymptotic metric at spacelike infinity to describe a conical geometry with a spin parameter equal to that of the string, but with a different constant deficit angle. This is, in fact, the result that we have obtained.

Let us analyse in more detail the relation between the deficit angles encountered at $r = 0$ and at infinity: $2\pi (1 - \sqrt{D})$ and $2\pi (1 - e^{-w_\infty})$, respectively. In the absence of spin, the relation is relatively simple; at spacelike infinity, one only has to divide the parameter $D$ by $\bar{E}_\infty$, a constant of motion that determines the energy of the wave. This quotient can be interpreted in terms of the associated $C$-energy [8] as an addition of energies. In the spinning case, on the other hand, the relation is much more complicated. The quantity $\hat{E}_\infty$ is not a constant of motion anymore, and the gravitational wave does not possess a preserved energy density by its own (since the $C$-energy is not conserved). Moreover, the existence of angular momentum introduces corrections to the deficit angle that depend on the gravitational fields not just through the functional $\hat{E}_\infty$, but also via $\Omega$. The spin decreases the effective value of $D$ by an amount of $c_0^2 \Omega/2$. The resulting deficit is, apart from the usual factor of $2\pi$, equal to the total energy density of the system, given in equation (2.10) and which is again conserved. Furthermore, as we commented in section
2, the positivity of \( \bar{H} \) on phase space implies that \( H_R \geq 1 - \sqrt{D} \). Consequently, in absolutely all of the spacetimes that we have considered, the deficit angle at infinity is greater than or equal to that around the axis \( r = 0 \). This is a general prediction that, in principle, could be verified experimentally, had we the possibility of making measurements in cylindrical gravitational systems containing a stationary stringy defect or, alternatively, in systems that could mimic the dynamics of the gravitational field in this situation.

Our result can also be rephrased by saying that, although the energy of the composite system is not the sum of the energies corresponding to the spinning string and the cylindrical wave, the presence of a gravitational wave always results in an increase of the total energy. It is not difficult to check that the only situation in which the two studied deficit angles coincide is when the fields \( v \) and \( y \) are constant and the momenta \( P_v \) and \( P_y \) vanish all over the spacetime, i.e. for the flat solution which (modulo a constant dislocation, given by the fixed value of \( v \)) describes the vacuum region, free of CTCs, in the exterior of a string with mass and spin densities equal to \( 1 - \sqrt{D} \) and \( c_\theta /2 \).

It is worth commenting that, although the degrees of freedom of our family of spacetimes are those corresponding to purely gravitational cylindrical waves (namely, the fields \( v \) and \( y \) and their canonical momenta), the equations of motion that dictate the evolution in configuration space (i.e. in terms of \( v \) and \( y \)) are not truly hyperbolic partial differential equations, except in the absence of spin. The reason is the appearance of the factor \( e^{-2w E[r]} \) in all of the dynamical equations (2.6)-(2.9). This factor has a local dependence on the fields if and only if \( c_\theta \) vanishes, in which case it reduces to the constant \( D \). In other words, the existence of angular momentum introduces, via the process of gauge fixing \([29]\), a high non-locality in the dynamics of cylindrical gravity.

Finally, in order to confirm the interpretation that we have put forward for our family of spacetimes as cylindrical waves interacting with spinning strings, it would be interesting to analyse the Riemann tensor of the solutions, paying a particular attention to the contributions that, in the form of distributions concentrated on line sources, could account for the appearance of the axial singularity \([14, 35]\). In addition, one might also study the behaviour of the Riemann tensor at null infinity, determining the radiative content of the gravitational field in this region (as it was done, e.g., in Refs. \([25, 26]\)). These issues will be the subject of future research.

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**Appendix A**

In this appendix, we will discuss the freedom available in the choice of coefficients in the series (4.6). We will first show that all the coefficients in these series can be determined from the knowledge, at all instants of time \( \tau \), of the values of \( v_0^0 \), \( V_0 \), and \( y_0^{(k)} \), as well as the value of \( Y_0 \) when \( \kappa = -1 \). In order to demonstrate this statement, let us call \( \Gamma_n \) the set formed by \( v_0 \), \( y_0^{(k)} \) and all the coefficients \( \{V_m, Y_m, P_m, Q_m\} \) with \( m \leq n \), where \( n \) is an integer. By expanding the dynamical equations (2.6)-(2.9) in powers of \( r \), one
can check that the coefficients with subindex equal to \( n + 1 \) can be determined from the knowledge of \( \Gamma_n \) at all values of \( \tau \). Consequently, all the information needed to fix the series is contained in \( \Gamma_1 \). Let us now analyse the freedom available in the choice of this set of coefficients. One can check that the lowest-order contributions to \( \partial_\tau v \) and \( \partial_\tau y \) in equation (2.6) and (2.7), respectively, determine \( P_0 \) and \( Q_0 \) in terms of the constant \( y_0^{(1)} \) and the derivatives of \( V_0 \) and \( Y_0 \) with respect to \( \tau \) when \( \kappa = -1 \), or as functions of \( y_0^{(1)} \) and the derivatives of this coefficient and \( V_0 \) if \( \kappa = 1 \). In this last case, in addition, the contributions of order \( r^3 \) to \( \partial_\tau P_y \) turn out to fix \( Y_0 \) in terms of \( V_0 \) and \( y_0^{(1)} \), once \( Q_0 \) has been found. This concludes the proof of our assertion.

Let us consider now the collection of coefficients that appear in the series (4.6) at a certain initial time \( \tau_0 \), rather than as functions of time, and discuss the freedom that exists in the choice of such initial data. It is not difficult to check that the coefficients of these power expansions satisfy functional relations which do not involve time derivatives, so that they are not all independent at \( \tau_0 \). In particular, for \( \kappa = 1 \), the lack of contributions of order \( r \) and \( r^3 \) in \( \partial_\tau P_v \) implies that \( V_1 \) and \( V_2 \) are proportional to \( V_0 \). In addition, the requirement that the subdominant correction to \( \partial_\tau y_0^{(1)} \) should be of order \( r^6 \) turns out to fix the coefficients \( Q_1 \) and \( Q_2 \) as linear homogeneous functions of \( Q_0 \). Similarly, for \( \kappa = -1 \), the vanishing of the derivative of \( P_y \) with respect to \( \tau \) at orders \( r \) and \( r^3 \) leads to a functional dependence of the coefficients \( Y_1 \) and \( Y_2 \) on \( y_0^3 \), \( y_0^6 \), and \( V_0 \). Taking into account this dependence and demanding that \( \partial_\tau P_v \) does not include contributions of order \( r \) and \( r^3 \), one can also determine the coefficients \( V_1 \) and \( V_2 \) as functions of \( Y_0 \) and \( V_0 \). In general, more complicated relations appear when higher-order coefficients are considered. As a consequence of these relations, one can prove by induction that, in order to determine the series (4.6) at \( \tau_0 \), it suffices to know at that moment the values of \( v_0^3 \), \( y_0^{(\kappa)} \) and all the sets of coefficients \( \{V_{3n}, Q_{3n}, P_{3n}, Y_{3n}\} \) with \( n \geq 0 \).

**Appendix B**

In this appendix, we will prove that the reduced Hamiltonian \( H_R \) is differentiable on phase space once one adopts the boundary conditions (4.5) [or (4.2) if the angular momentum vanishes], (3.2), and (3.4) [where \( v_\infty = v_\infty^3 \)]. The variation of the reduced Hamiltonian is given by

\[
\delta H_R = \frac{e^{-w_\infty}}{4} \int_0^\infty dr \delta \bar{H} \left( 2 + c_2^2 \bar{E}_\infty F_\infty^2 \int_r^\infty ds \frac{ds}{s^3}\bar{E}[s] \right), \quad (B.1)
\]

where \( \bar{H} \) is the function on phase space defined in equation (2.4). The integrals that appear in the term in parentheses are all convergent for \( r > 0 \), owing to our boundary conditions. In addition, from a variation of the canonical momenta \( P_v \) and \( P_y \), one obtains

\[
\delta \bar{H} = 4 \left( \frac{\delta H}{r} \right) \left( P_v r^2 e^{2y} \delta P_v + P_y \delta P_y \right). \quad (B.2)
\]

Therefore, the Hamiltonian is differentiable with respect to these momenta if and only if the integral over \( r \) in the expression of \( \delta H_R \) converges (both at \( r = 0 \) and at infinity) for all possible variations of \( P_v \) and \( P_y \). Given the asymptotic behaviour (3.2) and the conditions on the axis, equation (4.2) or equation (4.5), the admissible variations of \( P_v \) and \( P_y \) turn out, in general, to be of the same order as the momenta themselves. It is
then a simply exercise to check that the studied integration over \( r \) leads, in fact, to a well defined variation of the reduced Hamiltonian.

The analysis of the variations of \( v \) and \( y \) is more complicated. In this case, one gets

\[
\delta \bar{H} = \frac{1}{r} \left[ 4P_v^2 r^2 e^{2y} \delta y - (\partial_r v)^2 e^{-2y} \delta y + r^2 \partial_r y \partial_r (\delta y) + \partial_r ve^{-2y} \partial_r (\delta v) \right].
\] (B.3)

Note that this variation contains the radial derivatives of \( \delta v \) and \( \delta y \). To get rid of these derivatives, one must perform an integration by parts. Consequently, the variation of the reduced Hamiltonian splits into a surface term and an integral expression. The surface term is

\[
e^{-w_\infty} \left( \frac{4}{r} \right) \left( 2 + c_\theta^2 \bar{E}_\infty F_\infty^2 \int_r^{\infty} \frac{ds}{s^3} \bar{E}[s] \right) \bigg|_0^\infty,
\] (B.4)

where we have employed the notation \( f^b_a = f(b) - f(a) \). The integral contribution to \( \delta \bar{H}_R \), on the other hand, can be obtained by replacing \( \delta \bar{H} \) on the right-hand side of equation (B.1) with

\[
4P_v^2 re^{2y} \delta y - (\partial_r v)^2 e^{-2y} \delta y - \partial_r (r \partial_r y) \delta y - \partial_r \left( \partial_r v e^{-2y} \right) \delta v,
\] (B.5)

and adding to the result the factor

\[
\frac{e^{-w_\infty}}{4} \int_0^{\infty} dr \frac{c_\theta^2}{r^4} \bar{E}_\infty F_\infty^2 \bar{E}[r] \left[ r^2 \partial_r y \delta y + \partial_r ve^{-2y} \delta v \right].
\] (B.6)

The differentiability of the reduced Hamiltonian then requires that the surface term (B.4) vanishes and that the integrals over \( r \) that determine the variation of \( H_R \) are convergent for all possible values of \( \delta v \) and \( \delta y \). Actually, according to the boundary conditions (3.4) (with \( v_\infty = v_{c\infty} \)), one has that \( \delta v = o(1) \) and \( \delta y = o(1) \) when \( r \to \infty \). Here, we have imposed that the variations of \( v \) preserve the value of \( v_{c\infty} \), since this is a given constant. Similarly, from equation (4.2) one concludes that, when \( c_\theta \) vanishes, the acceptable variations of our fields display the behaviour \( \delta v = O(r^2) \) and \( \delta y = O(1) \) around the axis. Finally, in the presence of spin, one obtains from equation (4.5) that \( \delta v = O(r^4) \) at \( r = 0 \), whereas the variation of \( y \) admits two different types of behaviour, depending on whether \( y_0^{(\kappa)} \) is a fixed constant or not. In the first case (\( \kappa = -1 \)), one arrives at \( \delta y = O(r^2) \); in the second case (\( \kappa = 1 \)), one gets \( \delta y = O(1) \), because now it is possible to vary the limit of \( y \) on the axis. With this information about the variations of the fields and our boundary conditions, it is not difficult to check that the reduced Hamiltonian is indeed differentiable with respect to \( v \) and \( y \). We therefore conclude that our boundary conditions guarantee that \( H_R \) is differentiable on phase space, as we wanted to show.

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