Stochastic description for open quantum systems

Esteban Calzetta
Departamento de Física,
Universidad de Buenos Aires, Ciudad Universitaria,
1428 Buenos Aires, Argentina

Albert Roura and Enric Verdaguer *
Departament de Física Fonamental,
Universitat de Barcelona, Av. Diagonal 647,
08028 Barcelona, Spain

A linear open quantum system consisting of a harmonic oscillator coupled linearly to an infinite set of independent harmonic oscillators is considered; these oscillators have a general spectral density function and are initially in thermal equilibrium. Using the influence functional formalism a formal Langevin equation can be introduced to describe the system’s fully quantum properties even beyond the semiclassical regime. It is shown that the reduced Wigner function for the system is exactly the formal distribution function resulting from averaging both over the initial conditions and the stochastic source of the formal Langevin equation. The master equation for the reduced density matrix is then obtained in the same way a Fokker-Planck equation can always be derived from a Langevin equation characterizing a stochastic process. We also show that the quantum correlation functions for the system can be deduced within the stochastic description provided by the Langevin equation. It is emphasized that when the system is not Markovian more information can be extracted from the Langevin equation than from the master equation. These results can be straightforwardly extended to the case of nonlinear coupling when treated perturbatively up to quadratic order.

I. INTRODUCTION

Open quantum systems are a recurrent topic in quantum physics. By an open quantum system we mean a subsystem, which is the system of interest, within a larger closed quantum system undergoing unitary evolution [1]. The system of interest is simply called the “system” and the remaining degrees of freedom are called the “environment”. In practice many quantum systems can be treated as open quantum systems provided a natural separation between the system and the environment can be unambiguously established. One of the main features of open quantum systems is that unlike closed systems they exhibit nonunitary evolution. These systems are of interest in condensed matter physics [2,3], quantum optics [4], quantum measurement theory [5], nonequilibrium field theory [6–9], quantum cosmology [10,11] and semiclassical gravity [12,13].

Among the most widely used examples of open quantum system is the quantum Brownian motion (QBM) model, which consists of a single massive particle in a potential (usually a quadratic potential) interacting with an infinite set of independent harmonic oscillators which are initially in thermal equilibrium [14]. The coupling may be linear both in the system and environment variables or may be nonlinear in some or all of these variables. The frequencies of the environment oscillators are distributed according to a prescribed spectral density function, the simplest case corresponding to the so-called ohmic environment. The linear coupling provides a good description of many open quantum systems in condensed matter physics [15,2], but in field theory [16], quantum cosmology [11] and semiclassical gravity [17–19,13] the coupling is usually nonlinear. Part of the interest of the linear systems is that they are in many cases exactly solvable and detailed studies of different aspects of open quantum systems can be performed. One of the issues that have received much attention in recent years is environment-induced decoherence as a mechanism to understand the transition from the quantum to the classical regime [20,21].

Concepts such as the Feynman and Vernon influence functional method [22], the closed time path (CTP) effective action [23–27], the reduced density matrix, the reduced Wigner function, the master equation, the Fokker-Planck equation and the Langevin equation are some of the key words associated to the study of open quantum systems. One of the purposes of this paper is to review the place of these concepts in the QBM model and to establish their often subtle interrelations. Thus, let us first review some of those concepts and recall their main features.

*Also at Institut de Física d’Altes Energies (IFAE), Barcelona, Spain.
The reduced density matrix is defined from the density matrix of the whole closed system by tracing out the environment. Its dynamical evolution may be given in terms of the Feynman and Vernon influence functional [22]. The influence functional is defined from a path integral involving the action of the system and the environment and an integration of the environment degrees of freedom. Its use in the QBM model is widespread especially since Caldeira and Leggett were able to compute in closed form the propagator for the reduced density matrix in the case of linear coupling with an ohmic environment [15].

The master equation is a differential equation describing the evolution of the reduced density matrix. The master equation for linear coupling and ohmic environment at high temperature was first deduced by Caldeira and Leggett [15], it was extended to arbitrary temperature by Unruh and Zurek [28], and it was finally obtained for a general environment (i.e., for an arbitrary spectral density function) by Hu, Paz and Zhang [29]. This result can be extended to the case of nonlinear coupling by treating the interaction perturbatively up to quadratic order [30].

Closely related to the reduced density matrix is the reduced Wigner function (in fact one goes from one to the other by an integral transform). The reduced Wigner function is similar in many aspects to a distribution function in phase-space, although it is not necessarily positive definite, and the dynamical evolution it satisfies is similar to the Fokker-Planck equation for classical statistical systems [31–33]. This equation is, of course, entirely equivalent to the master equation for the reduced density matrix and we will often also refer to it as the master equation. The reduced density matrix has been used to study decoherence induced by the environment [34–38,28–30]. The Wigner function has also been used in studies of emergence of classicality induced by an environment [39], especially in quantum cosmology [10,11].

The Langevin equation [40,41] is another relevant equation for open quantum systems. This equation has either been introduced phenomenologically [33] to describe the effect of the environment into a classical system (Brownian motion) or it has been derived as a classical or semiclassical limit. Thus, Gell-Mann and Hartle [42,43] in the framework of the consistent histories approach to quantum mechanics [44,45] considered the decoherence functional, which is closely related to the influence functional for open quantum systems, to measure the degree of classicality of the system. They were able to show that under certain conditions there exists a semiclassical limit in an open system which may be suitably described by a Langevin equation with the self-correlation of the stochastic source given by the noise kernel which appears in the decoherence functional.

Langevin type of equations as a suitable tool to study the semiclassical limit have been used recently in semiclassical gravity and cosmology [17–19,46,13]. In inflationary cosmology they have been used to describe the stochastic effect on the inflation field [47–53] or the stochastic behavior of large-scale gravitational perturbations [54], which is important for cosmological structure formation. So far the Langevin equation has been restricted to describe the classical or semiclassical limit. A closer look at the influence functional, however, reveals that a formal Langevin equation can be extracted from this functional (at least for quadratic influence actions) independently of the existence of a classical limit.

Our main purpose in this paper is to show that this equation contains relevant information on quantum properties of the system. In fact, it generally contains more information than the master equation. Our findings are summarized in the following two main results. The first result shows that for a system interacting bilinearly with a general environment the reduced Wigner function is exactly the formal distribution function in the system phase-space given in terms of an average both in the initial conditions and the stochastic source of the Langevin equation. A differential equation describing the dynamics for the reduced Wigner function can then be deduced in the same way a Fokker-Planck equation is usually derived in statistical mechanics from any Langevin equation characterizing a stochastic process [41]. This provides in passing an alternative derivation of the master equation. The second important result shows that a partial set of quantum correlation functions for the system variables can be obtained within the stochastic description provided by the Langevin equation. In fact, by slightly extending our derivation an explicit expression is also obtained for a generating functional (the CTP generating functional), from which all the relevant quantum correlation functions can be obtained.

We also point out that these correlation functions cannot be obtained, in general, from the master equation. Thus, the Langevin equation is a very useful tool to gain information on the quantum properties of the system even beyond the semiclassical regime, when it no longer describes the actual trajectories of the system.

The plan of the paper with a summary of the relevant results is the following. In Sec. II we briefly review the QBM model describing a harmonic oscillator coupled bilinearly to a bath of infinite harmonic oscillators in thermal equilibrium with an arbitrary spectral distribution (general environment). We summarize the main formulas that beginning with the influence functional lead to the reduced density matrix operator and to the master equation. We also give the equivalent evolution equation for the reduced Wigner function. These results are well known and the computational details, which can be found in the references provided, are omitted. We also show how the Langevin equation for the system is usually derived from the influence functional in a simple way by a formal trick. Finally, we refer to the consistent histories formalism of quantum mechanics for the interpretation and justification of the Langevin equation as a suitable stochastic description for the actual trajectories (histories) of the system in the
In Sec. III we derive one of the main results of this paper. As it has been mentioned before, we show that the reduced Wigner function can be written as a formal distribution function for the system variables defined as follows: We consider the ensemble of systems in phase-space which obey the formal Langevin equation for a given realization of the stochastic force and arbitrary initial conditions; the average both over the initial conditions (which involves the initial reduced Wigner function) and the different realizations of the stochastic source gives just the reduced Wigner function. The key technical point that makes this result relatively easy to derive is the computation of the path integral defining the reduced Wigner function in terms of variables which include the initial conditions and the stochastic force. For this we introduce a functional change which involves the formal Langevin equation. As far as we know this result is new. We are aware of a related result by Halliwell and Zoupas [55] that show a similar relation in the limit of large times; see also Ref. [56].

Having shown that the reduced Wigner function is a formal distribution function, it is clear that the master equation for the reduced Wigner function can be deduced using the techniques to derive the Fokker-Planck equation [41]. We recall that whereas a Langevin equation describes how an individual system evolves stochastically, a Fokker-Planck equation describes how the distribution for an ensemble of systems evolves deterministically in phase-space. The details of this derivation are left to Appendix A. This constitutes an alternative derivation of the master equation to those given by Hu, Paz and Zhang [29] (see also Ref. [57]) and Halliwell and Yu [58]. We also note that this process is not reversible in the sense that there may be many Langevin equations that give the same master equation. This just reflects the fact that when the dynamics is not Markovian (in the sense of Ref. [36]) more information can be extracted from the Langevin equation than from the master equation.

In Sec. IV we obtain the second main result of this paper. We show that quantum correlation functions for the system variables can be obtained within the stochastic description provided by the Langevin equation by explicitly computing the closed time path (CTP) generating functional for the system. It turns out that this generating functional can be written as an average over the initial conditions times a term that depends on the noise kernel, which contains the information on the fluctuations induced by the environment on the system.

We also show that quantum correlation functions cannot be obtained using the propagators for the reduced density matrix unless the system is Markovian, a fact which is discussed in Appendix B. Note that this is in contrast with a closed system where the unitary propagators, which are solutions of the Schrödinger equation, allow one to obtain all the information about the existing quantum correlations.

In Sec. V we briefly discuss the extension of our results to the case of system-environment nonlinear interaction when treated perturbatively up to quadratic order. Finally, in Sec. VI we summarize our results by giving a scheme of the interconnections between genuine quantum objects, such as the master equations both for the reduced density matrix and the reduced Wigner function, and the quantum correlation functions for the system, as well as those elements appearing in the alternative (and useful) stochastic description for the open quantum system such as the Langevin equation and the corresponding Fokker-Planck equation.

II. INFLUENCE FUNCTIONAL FORMALISM FOR OPEN QUANTUM SYSTEMS. AN EXAMPLE

A. Linear QBM model

Here we review a QBM model as an example of linear open quantum system. Let us consider a harmonic oscillator of mass $M$, the “system”, coupled to a bath of independent harmonic oscillators of mass $m$, the “environment”. For simplicity, let us assume that the system and environment are linearly coupled. The action for the whole set of degrees of freedom is defined by:

$$S[x, \{q_j\}] = S[x] + S[\{q_j\}] + S_{int}[x, \{q_j\}],$$

where the terms on the right-hand side correspond to the action of the system, the environment and the interaction term respectively, which are given by:

$$S[x] = \int dt \left( \frac{1}{2} M \dot{x}^2 - \frac{1}{2} M \Omega^2 x^2 \right),$$

$$S[\{q_j\}] = \sum_j \int dt \left( \frac{1}{2} m \dot{q}_j^2 - \frac{1}{2} m \omega_j^2 q_j^2 \right),$$

where the terms on the right-hand side correspond to the action of the system, the environment and the interaction term respectively, which are given by:

$$S[x] = \int dt \left( \frac{1}{2} M \dot{x}^2 - \frac{1}{2} M \Omega^2 x^2 \right),$$

$$S[\{q_j\}] = \sum_j \int dt \left( \frac{1}{2} m \dot{q}_j^2 - \frac{1}{2} m \omega_j^2 q_j^2 \right),$$

3
\[
S_{\text{int}}[x, \{q_j\}] = \sum_j c_j \int dt x(t)q_j(t) = \int_0^\infty d\omega \frac{2m\omega}{\pi c(\omega)} I(\omega) \int dt x(t)q(t; \omega),
\]

where we introduced the spectral density \( I(\omega) = \sum_j \pi c_j^2 (2m\omega_j)^{-1} \delta(\omega - \omega_j) \) in the last equality, \( c(\omega) \) and \( q(t; \omega) \) are functions such that \( c(\omega_j) = c_j \) and \( q(t; \omega_j) = q_j(t) \), \( c_j \) being system-environment coupling parameters, and \( \Omega \) and \( \omega_j \) are, respectively, the system and environment oscillator frequencies.

The reduced density matrix for an open quantum system is defined from the density matrix \( \rho \) of the whole system by tracing out the environment degrees of freedom

\[
\rho_r(x_f, x'_f, t_f) = \int \prod_j dq_j \rho(x_f, \{q_j\}, x'_f, \{q_j\}, t_f).
\]

The evolution for the reduced density matrix, which is in general nonunitary and even non-Markovian, can be written as

\[
\rho_r(x_f, x'_f, t_f) = \int dx dx' J(x_f, x'_f, t_f; x_i, x'_i, t_i) \rho_r(x_i, x'_i, t_i),
\]

where the propagator \( J \) is defined in a path integral representation by

\[
J(x_f, x'_f, t_f; x_i, x'_i, t_i) = \int_{x(t_i)=x_i}^{x(t_f)=x_f} D\rho \int_{x'(t_i)=x'_i}^{x'(t_f)=x'_f} D\rho' e^{i\{S[x]-S[x'] + S_{IF}[x, x']\}/\hbar},
\]

where \( S_{IF}[x, x'] \) is the influence action introduced by Feynman and Vernon [22]. When the system and the environment are initially uncorrelated, i.e., when the initial density matrix factorizes (\( \tilde{\rho}(t_i) = \hat{\rho}_s(t_i) \otimes \hat{\rho}_e(t_i) \), where \( \hat{\rho}_s(t_i) \) and \( \hat{\rho}_e(t_i) \) mean, respectively, the density matrix operators of the system and the environment at the initial time) the influence functional, defined by \( F[x, x'] = \exp(iS_{IF}[x, x'])/\hbar \), can be expressed in the following way:

\[
F[x, x'] = \prod_j \int dq_{j(f)}^i dq_{j(i)} \int_{q_{i(f)}=q_{i(f)}^i}^{q_{j(f)}=q_{j(f)}^i} Dq_j \int_{q_{i(i)}=q_{i(i)}^i}^{q_{j(i)}=q_{j(i)}^i} Dq_j' \exp \left[ \frac{i}{\hbar} \left\{ S[\{q_j\}] - S[\{q_j'\}] + S[x, \{q_j\}] \right\} 
- S[x', \{q_j'\}] \right] \cdot \rho_e(\{q_j'\}, \{q_j(\cdot)\}, t_i).
\]

When the initial density matrix for the environment \( \rho_e(\{q_j'\}, \{q_j(\cdot)\}, t_i) \) is Gaussian, the path integrals can be exactly performed and one obtains [22,15]:

\[
S_{IF}[x, x'] = -2 \int_{t_i}^{t_f} ds \int_{t_i}^{t_f} ds' \Delta(s) D(s, s') X(s') + \frac{i}{2} \int_{t_i}^{t_f} ds \int_{t_i}^{t_f} ds' \Delta(s) N(s, s') \Delta(s'),
\]

where \( X(s) \equiv (x(s) + x'(s))/2 \) and \( \Delta(s) \equiv x'(s) - x(s) \). The kernels \( D(s, s') \) and \( N(s, s') \) are called the dissipation and noise kernel, respectively. When the bath is initially in thermal equilibrium these kernels are given by [22]:

\[
D(s, s') = \int_0^\infty d\omega \frac{I(\omega)}{\pi} \sin \omega(s - s'),
\]

\[
N(s, s') = \int_0^\infty d\omega \frac{I(\omega)}{\pi} \coth \left( \frac{\hbar \omega}{2kT} \right) \cos \omega(s - s').
\]

Both kernels are related by the usual fluctuation-dissipation relation [59]. When no special form is assumed for the spectral density \( I(\omega) \), this is usually referred to as a general environment. One of the most common particular cases is the so-called Ohmic environment, defined by \( I(\omega) \sim \omega \) (some high frequency cut-off may be sometimes naturally introduced).
From Eqs. (2.5) and (2.7) a differential equation for the system’s reduced density matrix known as the master equation can be derived. The expression for a general environment was first obtained by Hu, Paz and Zhang [29] (see [58] for an alternative derivation):

$$i\hbar \frac{\partial \rho_r}{\partial t} = -\frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \right) \rho_r + \frac{1}{2} M \Omega^2(x^2 - x'^2) \rho_r + \frac{1}{2} M \delta(t)(x^2 - x'^2) \rho_r - iMC(t)(x - x')^2 \rho_r,$$

(2.12)

where the functions $\delta \Omega^2(t)$, $A(t)$, $B(t)$ and $C(t)$ represent a frequency shift, a dissipation factor and two diffusive factors, respectively. For explicit expressions of these functions see Appendix A. An alternative representation for the system reduced density matrix is the reduced Wigner function $W_r(X, p, t)$ defined as

$$W_r(X, p, t) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} d\Delta \exp(\frac{i\hbar}{\Delta}) \rho_r(X - \Delta/2, X + \Delta/2, t).$$

(2.13)

It follows immediately that the master equation (2.12) can be written in the following equivalent form:

$$\frac{\partial W_r}{\partial t} = \{H_R, W_r\}_{PB} + 2A(t) \frac{\partial \rho_r}{\partial p} + \hbar B(t) \frac{\partial^2 W_r}{\partial q \partial p} + \hbar MC(t) \frac{\partial^2 W_r}{\partial p^2},$$

(2.14)

where $\{H_R, W_r\}_{PB} = -(p/M) \partial W_r / \partial q + M \Omega^2(t) q \partial W_r / \partial p$ with $\Omega^2(t) = \Omega^2 + \delta \Omega^2(t)$. This equation is formally similar to the Fokker-Planck equation for a distribution function.

**C. Langevin equation**

1. Derivation by a formal trick

From the influence functional a Langevin equation for the system dynamics may be derived by a formal trick. First, let us note that the equations of motion for the expectation value of the system position operator in the Heisenberg picture, $\hat{\xi}(t)$, can be obtained from the CTP effective action $\Gamma_{CTP}[x, x']$. This is an effective action obtained from a generating functional introduced by Schwinger [23], from which true expectation values instead of transition matrix elements can be naturally obtained (see Refs. [24-27] for details). In operator language the CTP generating functional is defined by

$$Z_{CTP}[J, J'] \equiv \text{Tr} \left[ T \exp(iJ \cdot \hat{\xi}) \hat{\rho}(t_i) \hat{T} \exp(-iJ' \cdot \hat{\xi}) \right],$$

(2.15)

where $\hat{\rho}(t_i)$ is the initial density matrix for the whole closed quantum system (system plus environment) and the trace is also taken over the whole system. $T$ and $\hat{T}$ denote time and anti-time ordering for the position operators of the system $\hat{\xi}(t)$ in Heisenberg picture and a simplified notation in which $\cdot$ denotes an integration in time $\int_{t_i}^{t_f} dt$ has also been introduced. The meaning of the CTP generating functional can be understood as follows. Let us assume an initial pure state $|\Psi(t_i)\rangle$ which evolves under a source $J(t)$ evolves in the interaction picture as $T \exp(iJ \cdot \hat{\xi}) |\Psi(t_i)\rangle$. We may define $Z_{CTP}$ as a transition amplitude for some initial state to evolve under $J(t)$ to some final state at time $t_f$ and then to evolve back in time under a source $J'(t)$ summing over all final states as $\sum_{\alpha} \langle \Psi(t_i) | \hat{T} \exp(-iJ' \cdot \hat{\xi}) | \alpha, t_f \rangle \langle \alpha, t_f | T \exp(-iJ \cdot \hat{\xi}) | \Psi(t_i) \rangle$, where $|\alpha, t_f\rangle$ is a complete basis of eigenstates of $\hat{\xi}$ at the final time $t_f$. This may be rewrittren as $\sum_{\alpha} \langle \alpha, t_f | T \exp(-iJ \cdot \hat{\xi}) | \Psi(t_i) \rangle \langle \Psi(t_i) | \hat{T} \exp(-iJ' \cdot \hat{\xi}) | \alpha, t_f \rangle$, which reduces to (2.15) when we have an arbitrary state defined by the density matrix $\hat{\rho}(t_i)$ instead of the pure state $|\Psi(t_i)\rangle$.

The effective action $\Gamma_{CTP}$ can then be deduced in the usual way [25-27] using a Legendre transformation. The equations of motion obtained from the CTP effective action for the expectation values are clearly seen to be real and causal [25,26]. They read:

$$\frac{\delta}{\delta x(t)} \Gamma_{CTP}[x, x'] \right|_{x' = x = \hat{\xi}} = 0.$$

(2.16)
To lowest order in \( \hbar \) (at tree level in diagrammatic language), which is denoted by the superscript \(^0\), the CTP effective action for an open quantum system corresponds to

\[
\Gamma^{(0)}_{\text{CTP}}[x, x'] = S[x] - S[x'] + S_{IF}[x, x'].
\]  

(2.17)

In fact, for a linear open quantum system \( \Gamma^{(0)}_{\text{CTP}}[x, x'] \) coincides with the exact expression for the CTP effective action \( \Gamma_{\text{CTP}}[x, x'] \). Note that the noise term, which is the imaginary part of the influence action (2.9), does not contribute to the equation of motion (2.16). Furthermore, this equation provides information just about the “averaged” dynamics of the system. An “improved” version for the equations of motion which is supposed to take into account the fluctuations of the system induced by the environment can be obtained by performing the following formal “trick”. Consider the following mathematical identity for Gaussian functional integrals:

\[
e^{-\frac{1}{2}\Delta \cdot N \cdot \Delta / \hbar} = \int \mathcal{D}\xi P[\xi] e^{i\xi \Delta / \hbar},
\]

where

\[
P[\xi] = \det(2\pi \hbar N)^{-\frac{1}{2}} e^{-\frac{1}{2} \xi (\hbar N)^{-1} \cdot \xi}.
\]

(2.18)

(2.19)

\( P[\xi] \) can be interpreted as the functional probability distribution for a Gaussian stochastic source \( \xi(t) \) with \( \langle \xi(t) \rangle = 0 \) and \( \langle \xi(t) \xi(t') \rangle = \hbar N(t, t') \). This interpretation is possible whenever \( N(t, t') \) is positive semidefinite, which can be shown to hold for any reasonable case. The trick consists in defining an “improved” semiclassical effective action as

\[
S_{\text{eff}}[x, x'; \xi] = S[x] - S[x'] + \Re S_{IF}[x, x'] + \xi \cdot (x' - x),
\]

(2.20)

which satisfies \( e^{iS_{\text{eff}}[x, x'; \xi] / \hbar} = e^{i\Gamma^{(0)}_{\text{CTP}}[x, x'] / \hbar} \) and leads to equations of motion with a stochastic force which endows a stochastic character to the dynamics of the system:

\[
\frac{\delta}{\delta x(t)} S_{\text{eff}}[x, x'; \xi] \bigg|_{x' = x} = 0,
\]

(2.21)

or, equivalently,

\[
\frac{\delta}{\delta x(t)} \Gamma^{(0)}_{\text{CTP}}[x, x'] \bigg|_{x' = x} = \xi(t).
\]

(2.22)

Since the stochastic source has zero mean, by averaging the stochastic equation (2.22) one exactly recovers the averaged equation of motion (2.16).

2. Derivation via the decoherent histories formalism

A deeper justification of the Langevin equation (2.22) as the system dynamical equation in a semiclassical limit was given by Gell-Mann and Hartle \([42,43]\) in the framework of the consistent histories approach to quantum mechanics pioneered by Griffiths and Omnès \([44,45]\). Let us briefly summarize the main steps of this derivation.

Histories are essentially chains of projectors\(^{1}\) on quantum properties of a closed quantum system at different instants of time \( C_\alpha = P_{\alpha_1}(t_1) \ldots P_{\alpha_n}(t_n) \), where every \( P_{\alpha}(t_i) \) is the corresponding projector at time \( t_i \) in the Heisenberg picture and \( \alpha \) denotes the \( n \)-plet \((\alpha_1, \ldots, \alpha_n)\). All the possible projectors at each instant of time must be both exhaustive and exclusive, i.e., \( \sum \alpha \ P_{\alpha}(t_i) = I \) and \( P_{\alpha_i}(t_i) P_{\beta_i}(t_i) = 0 \) for \( \alpha_i \neq \beta_i \), respectively. Two histories \( C_\alpha \) and \( C_{\alpha'} \) are said to be disjoint whenever there exists at least an instant of time for which \( P_{\alpha_i}(t_i) P_{\alpha'_i}(t_i) = 0 \). It is important to stress that histories should always be considered as elements belonging to a given family or set of histories\(^{2}\). Another fundamental object within this framework is the so-called decoherence functional, which is defined as follows:

\(^{1}\)In fact, the concept of history is extended to allow for the definition of union and intersection of histories as well as that of complementary history \([42,43,60]\). They can also be generalized to include branch-dependence \([42,36] \) and continuous time projections \([61]\).

\(^{2}\)Strictly speaking, any admissible family of histories should satisfy a number of axioms. Namely, the union, intersection and complementary histories of any elements of a family should also belong to that family.
where we made use of the previously introduced notation in which the additivity axiom for disjoint histories is satisfied, which is not the case when histories interfere. This lack of interference for a set of histories, known as the decoherence condition, is characterized via the decoherence functional by the fact that \( D(\alpha, \alpha') = 0 \) for any pair of disjoint histories \( \alpha \) and \( \alpha' \) belonging to that set. The probability assigned to a given history is then \( p(\alpha) = D(\alpha, \alpha) \). In fact, the weaker condition \( \Re D(\alpha, \alpha') = 0 \) is sufficient to guarantee that probabilities can be consistently assigned to any element of a given family of histories.

Within the decoherent histories framework there are two requirements for the “emergence of classicality”. The first one is decoherence, which allows for classical probabilities to be consistently assigned to histories describing the evolution of the system. The second requirement is that these probabilities should be peaked near histories corresponding to solutions of classical equations of motion. For open quantum systems the entanglement arising between system and environment may provide enough induced decoherence for those sets of histories describing properties of the system. In addition, the back reaction of the environment on the system dynamics leads to dissipation and noise. Therefore, under certain conditions the system exhibits a classical but stochastic dynamics which can be properly described in terms of a Langevin equation. This has been explicitly shown by Gell-Mann and Hartle [42,43] for some particular cases of the QBM models considered in this paper. There are two main steps in the deduction of the Langevin equation as a suitable description of the system histories. First, the environment degrees of freedom are integrated out. This step implies the appearance of the influence functional in the path integral representation of the decoherence functional for the open quantum system. Second, to get a set of decoherent histories, further coarse graining in the system trajectories is required. The expression for the decoherence functional is then:

\[
D(\alpha, \alpha') = \int_\alpha \mathcal{D}x \int_{\alpha'} \mathcal{D}x' e^{i(S[x] - S[x']) + S_{IF}[x,x']/\hbar},
\]

where \( \alpha \) and \( \alpha' \) denote the restrictions on the system trajectories (chains of projectors at different instants of time for the system position within those cells defining the coarse graining) when performing the path integrals, corresponding to the different coarse-grained histories which constitute a decoherent set. Decoherence results from the interplay between the effect of the environment through the imaginary part of the influence action (the noise kernel, see Eq. (2.9)), the size of the cells defining the coarse graining, and the time separation between every projection (restriction on the system trajectory). Altogether results in the exponential suppression of the decoherence functional for any pair of disjoint histories.

III. STOCHASTIC FORMAL DESCRIPTION OF THE SYSTEM'S FULLY QUANTUM DYNAMICS

In this Section we show that the reduced Wigner function can be written as a formal distribution function for some stochastic process, and using this result we deduce the corresponding Fokker-Planck equation.

A. Reduced density matrix and Wigner function

In order to find an explicit expression for the reduced density matrix (2.5) at a time \( t_f \), we need to compute the path integrals appearing in Eq. (2.7) for the reduced density matrix propagator. Let us begin by rewriting the influence action (2.9) as

\[
S_{IF}[x, x'] = \int_{t_i}^{t_f} ds \int_{t_i}^{t_f} ds' \Delta(s) H_{bare}(s, s') X(s') + \frac{i}{2} \int_{t_i}^{t_f} ds \int_{t_i}^{t_f} ds' \Delta(s)N(s, s') \Delta(s') \equiv X \cdot H_{bare} \cdot \Delta + \frac{i}{2} \Delta \cdot N \cdot \Delta,
\]

where we made use of the previously introduced notation in which \( \cdot \) denotes an integration in time \( \int_{t_i}^{t_f} dt \) and we have defined \( H_{bare}(s, s') \) as formally equivalent to \( -2D(s, s')\theta(s - s') \). Being the product of two distributions the latter expression is not well defined in general and suitable regularization and renormalization may be required; see [62] for details. The local divergences present in \( H_{bare}(s, s') = H(s, s') + H_{div} \delta(s - s') \) can be canceled by suitable counterterms \( \Omega_{\text{div}} \) in the bare frequency of the system \( \Omega = \Omega_{\text{ren}} + \Omega_{\text{div}} \). From now on we will consider that this infinite renormalization, if necessary, has already been performed so that both \( \Omega_{\text{ren}} \) and \( H(s, s') \) are free of divergences. Now we perform three main steps.

\[
D(\alpha, \alpha') = \text{Tr}(C_{\alpha}(t_i)C_{\alpha'}). \tag{2.23}
\]
First, we integrate the system action by parts,

\[ S[x] - S[x'] = -M \int_{t_i}^{t_f} dt \dot{X}(t) \dot{\Delta}(t) - \Omega_{ren}^2 X(t) \Delta(t)) = -M \dot{X} \Delta \Big|_{t_i}^{t_f} + M \int_{t_i}^{t_f} dt \Delta(t) \left( \frac{d^2}{dt^2} + \Omega_{ren}^2 \right) X(t). \]  

(3.2)

Second, we perform the Gaussian path integral for \( \Delta(t) \). The Jacobian determinant for the change of integration variables \( \int_{x_i}^{x_f} dx \int_{x'_i}^{x'_f} dx' \rightarrow \int_{X_i}^{X_f} DX \int_{\Delta_i}^{\Delta_f} D\Delta \) is the unity. From now on we consider \( \hbar = 1 \) and perform the Gaussian path integral for \( \Delta(t) \) with \( \Delta_i \) and \( \Delta_f \) fixed:

\[ \int_{X_i}^{X_f} DX \int_{\Delta_i}^{\Delta_f} D\Delta e^{iLX} e^{-\frac{1}{2} \Delta \cdot N \Delta} = \left( \det \frac{N}{2\pi} \right)^{-\frac{1}{2}} \int_{X_i}^{X_f} DX e^{-\frac{1}{2}(LX) \cdot N^{-1} \cdot (LX)}, \]

(3.3)

where \( L(t,t') \equiv M \left( \frac{d^2}{dt^2} + \Omega_{ren}^2 \right) \delta(t-t') + H(t,t') \). The integration over \( \Delta_i \) gives,

\[ \rho_r(X_f - \Delta_f/2, X_f + \Delta_f/2,t_f) = \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} d\Delta \int_{-\infty}^{\infty} dX \int_{X_i}^{X_f} DX e^{-\frac{1}{2}(LX) \cdot N^{-1} \cdot (LX)} \delta(X_f - \Delta_f/2, X_i + \Delta_i/2,t_i) \]

\[ = 2\pi \left( \frac{N}{2\pi} \right)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dX \int_{X_i}^{X_f} DX e^{-\frac{1}{2}(LX) \cdot N^{-1} \cdot (LX)} e^{-iMX_i(\Delta_f/2)} W_r(X_i, MX_i, t_i), \]

(3.4)

where in the last step we used Eq. (2.13), which defines the reduced Wigner function.

Third, we carry out the following functional change:

\[ X(t) \rightarrow \begin{cases} X_i = X(t_i), & p_i = M \dot{X}_i = M \dot{X}(t_i), \xi(t) = (L \cdot X)(t) \end{cases}. \]

(3.5)

Note that with this change the function \( X(t) \) gets substituted by the initial conditions \( (X_i, p_i) \) and the function \( \xi(t) \) in the path integration. It is important to note that at this point the function \( \xi(t) \) is not a stochastic process but just a function over which a path integral is performed. The functional change (3.5) is invertible as can be explicitly seen:

\[ \{X_i, p_i, \xi(t)\} \rightarrow X(t) = X_o(t) + \int_{t_i}^{t_f} dt' G_{rel}(t,t') \xi(t'), \]

(3.6)

where \( G_{rel}(t',t'') \) is the retarded (i.e., \( G_{rel}(t',t'') = 0 \) for \( t' \leq t'' \)) Green function for the linear integro-differential operator associated to the kernel \( L(t,t') \), and \( X_o(t) = \int_{t_i}^{t_f} dt' G_{rel}(t,t') \xi(t') \) is a solution of the inhomogeneous equation \( (L \cdot X_o)(t) = \xi(t) \) with initial conditions \( X_o(t_i) = 0 \) and \( \partial X_o(t')/\partial t' \big|_{t'=t_i} = 0 \). On the other hand, \( X_o(t) \) is a solution of the homogeneous equation \( (L \cdot X_o)(t) = 0 \), with initial conditions \( X_o(t_i) = X_i \) and \( \dot{X}(t_i) = p_i/M \). Since the change is linear, the Jacobian functional determinant will be a constant (this can be clearly seen by skeletonizing the path integral). After performing the functional change, we obtain

\[ \rho_r(X_f - \Delta_f/2, X_f + \Delta_f/2,t_f) = K \int_{-\infty}^{\infty} dX_f \int_{-\infty}^{\infty} dp_i \int DX \delta(X(t_f) - X_f) e^{-\frac{1}{2} \xi \cdot N^{-1} \cdot \xi} e^{-iMX_i(\Delta_f/2)} W_r(X_i, p_i, t_i), \]

(3.7)

where the delta function \( \delta(X(t_f) - X_f) \) was introduced to restrict the functional integral \( \int D\xi \) with free ends, in order to take into account the restriction on the final points of the allowed paths for the integral \( \int_{X_i}^{X_f} DX \) appearing in Eq. (3.4). The contribution from the Jacobian has been included in the constant \( K \). In order to determine this constant, we demand the reduced density matrix to remain normalized, i.e., that \( Tr \rho_r(t_f) = 1 \) if \( Tr \rho_r(t_i) = 1 \):

\[ 1 = \int_{-\infty}^{\infty} dX_f \rho_r(X_f, X_f, t_f) = K \int dX_f \int DX \delta(X(t_f) - X_f) e^{-\frac{1}{2} \xi \cdot N^{-1} \cdot \xi} \int_{-\infty}^{\infty} dX_i \int_{-\infty}^{\infty} dp_i W_r(X_i, p_i, t_i) \]

\[ = K \int DX e^{-\frac{1}{2} \xi \cdot N^{-1} \cdot \xi} \int_{-\infty}^{\infty} dX_i \int_{-\infty}^{\infty} dp_i W_r(X_i, p_i, t_i). \]

(3.8)

Now, from Eq. (2.13) it can be checked that \( Tr \rho_r(t_i) = 1 \) implies \( \int_{-\infty}^{\infty} dX_i \int_{-\infty}^{\infty} dp_i W_r(X_i, p_i, t_i) = 1 \). The constant \( K \) is thus determined to be
Finally, using the definition (2.13) for the Wigner function and the fact that
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} d\Delta f e^{i\Delta t \cdot \frac{1}{2} \epsilon} e^{-i \Delta t \cdot \frac{1}{2} \epsilon} = \delta(M \dot{X}(t_f) - p_f),
\]
we get an expression for the reduced Wigner function
\[
W_r(X_f, p_f, t_f) = K \int_{-\infty}^{\infty} dX_i \int_{-\infty}^{\infty} dp_i \int D\xi \delta(X(t_f) - X_f) \delta(M \dot{X}(t_f) - p_f) e^{-\frac{1}{2} \epsilon} W_r(X_i, p_i, t_i),
\]
which can be written in the following suggestive way:
\[
W_r(X_f, p_f, t_f) = \left\langle \left\langle \delta(X(t_f) - X_f) \delta(M \dot{X}(t_f) - p_f) \right\rangle_{\xi} \right\rangle_{X_i, p_i},
\]
where
\[
\left\langle \ldots \right\rangle_{\xi} \equiv [\text{det}(2\pi N)]^{-\frac{1}{2}} \int D\xi \ldots e^{-\frac{1}{2} \epsilon}.
\]
Thus the reduced Wigner function can be interpreted as an average over a Gaussian stochastic process \( \xi(t) \) with \( \langle \xi(t) \rangle_{\xi} = 0 \) and \( \langle \xi(t) \xi(t') \rangle_{\xi} = N(t, t') \) as well as an average over the initial conditions characterized by a distribution function \( W_r(X_i, p_i, t_i) \). It is only after formally interpreting \( \xi(t) \) as a stochastic process characterized by Eq. (3.13) that the equation defining \( \xi(t) \) in the functional change (3.5)
\[
(L \cdot X)(t) = \xi(t),
\]
can be regarded as a Langevin equation. We insist that, in general, Eq. (3.15) is not meant to describe the actual trajectories of the system, but it should rather be regarded as a formal tool. We should also remark that \( X(t_f) \) and \( \dot{X}(t_f) \) in Eq. (3.12) correspond to solutions of the Langevin equation (3.15) for a given realization of \( \xi(t) \), and that \( X_f \) and \( p_f \) are coordinates of a point in phase space.

Note, in addition, that although \( W_r(X_i, p_i, t_i) \) is real, which follows from the hermiticity of the density matrix, and properly normalized, in general it is not positive everywhere and, thus, cannot be considered as a probability distribution. The fact that the Wigner function cannot be interpreted as a phase space probability density is crucial since most of the nonclassical features of the quantum state are tightly related to the Wigner function having negative values. For instance, a coherent superposition state is typically characterized by the Wigner function presenting strong oscillations with negative values in the minima [39, 38], which are closely connected to interference terms.

Equation (3.12) is the main result of this section and shows that the reduced Wigner function can be interpreted as a formal distribution in phase space. This result will now be used to derive the corresponding Fokker-Planck equation.

### B. From Langevin to Fokker-Planck: recovery of the master equation

As mentioned above there is a simple one-to-one correspondence between any density matrix and the associated Wigner function introduced in (2.13). Taking this correspondence into account, the equation satisfied by the reduced Wigner function is equivalent to the master equation satisfied by the reduced density matrix. By deriving Eq. (3.12) with respect to time and using the Langevin-type equation in (3.15), one can obtain a Fokker-Planck differential equation describing the time evolution of the system’s reduced Wigner function. The details of the calculation can be found in Appendix A. This is yet another alternative way to derive the master equation (2.14). Our result is, of course, in agreement with that obtained in the previous derivations of Refs. [58] and [29] (see also [57]). It is given by (A16)
\[
\frac{\partial W_r}{\partial t} = \{H_R, W_r\}_{PB} + 2A(t) \frac{\partial(pW_r)}{\partial p} + B(t) \frac{\partial^2 W_r}{\partial X \partial p} + MC(t) \frac{\partial^2 W_r}{\partial p^2},
\]
(3.16)
where \( A(t), B(t) \) and \( C(t) \) are explicitly given by (A10), (A17) and (A18).

It should be pointed out that whereas one can derive the Fokker-Planck equation from the Langevin equation, the opposite is not possible in general. One can always consider Langevin equations with stochastic sources characterized by different noise kernels which, nevertheless, lead to the same Fokker-Planck equation and, thus, the same master equation. This can be argued from the expressions obtained in the derivation of the Fokker-Planck equation. Let us consider, for simplicity, the situation corresponding to local dissipation. A local contribution to the noise gives no contribution to \( B(t) \), but it does contribute to \( C(t) \) as can be seen from Eqs. (A17) and (A18) taking into account that \( G_{ret}(t, t) = 0 \) and \( \partial G_{ret}(t', t)/\partial t'|_{t'\rightarrow t} = M^{-1} \). Thus, one can always choose any noise kernel that gives the desired \( B(t) \) and then add the appropriate local contribution to the noise kernel to get the desired \( C(t) \) keeping \( B(t) \) fixed. Note that changing the noise kernel does not change \( A(t) \). To illustrate the fact that there exist different noise kernels giving the same \( B(t) \), as was stated above, one may consider the particular case corresponding to the weak dissipation limit so that \( G_{ret}(t, t') \sim (M\Omega)^{-1} \sin \Omega(t - t') \theta(t - t') \). To see that a different \( \tilde{N}(t, t') \) giving the same \( B(t) \) as \( N(t, t') \) exists reduces then to show that there is at least one nontrivial function \( \nu(s, t) = \tilde{N}(t, t') - N(t, t') \) (with \( s = t - t' \)) such that for any \( t \)

\[
\int_0^t ds \sin(\Omega \nu(s, t)) = 0, \tag{3.17}
\]

which can be shown to be the case.

The fact that different Langevin equations lead to the same master equation\(^3\) reflects that the former contains more information than the latter. This fact can be qualitatively understood in the following way. In the influence functional it is only the evolution of the environment degrees of freedom that is traced out. Of course, having integrated over all the possible quantum histories for the environment, no correlations in the environment can be obtained. Nevertheless, since the system is interacting with the environment, non-Markovian correlations for the system at different times may in general persist. On the other hand, when considering either the reduced density matrix or its propagator, also the system evolution, except for the final state, is integrated out. Consequently, information on non-Markovian correlations for the system is no longer available. Thus, only when the system’s reduced dynamics is Markovian, i.e. the influence functional is local in time, we expect that the Langevin equation and the master equation contain the same information. In particular, for a Gaussian stochastic source, as in our case, the Langevin equation contains the information about the system correlations at different times which the Fokker-Planck equation cannot in general account for. Only in the case in which the dynamics generated by the Langevin equation is Markovian one can compute the correlation functions just from the solutions of the Fokker-Planck equation or, equivalently, the master equation for the propagator \( J(x_2, x'_2, t_2; x_1, x'_1, t_1) \); see Eq.(2.7). The key point is the fact that the propagator for the reduced density matrix only factorizes when the influence action is local. In Appendix B we give a detailed argument on this point.

It is important to note that for a closed quantum system the evolution determined by the time evolution operators \( U(t_2, t_1) \) obtained from the Schrödinger equation is always unitary and, thus, also Markovian. That is why the Schrödinger equation suffices to get the correlation functions for a closed quantum system. On the contrary, for an open quantum system the evolution is nonunitary and, provided the influence action is nonlocal, not even Markovian.

### IV. CORRELATION FUNCTIONS

We have seen that the reduced Wigner function or equivalently the reduced density matrix and the master equation governing these functions can be obtained from a formal stochastic description provided by the Langevin equation (3.15). In this Section we show that also entirely quantum correlation functions for the system can be obtained by means of the stochastic description developed in the previous Section.

\(^3\)In fact, what we showed was that a Langevin equation contains in general more information that the corresponding Fokker-Planck equation. To extend this assertion to the master equation, one should make sure that different Langevin equations leading to the same Fokker-Planck equation can be obtained from an influence functional. Indeed this can be shown to be the case provided that one considers general Gaussian initial states for the environment.
Let us go back to the CTP generating functional introduced in Eq. (2.15) in operator language. As will be seen below, all the relevant quantum correlation functions for the system can be obtained from this functional. In terms of path integrals the generating functional is expressed, after integrating out the environment, as:

\[ Z_{CTP}[J,J'] = \int dx f \int dx d'x' \int dx \int dx' e^{iJ \cdot x - iJ' \cdot x'} e^{i[S[x] - S[x'] + S_{IF}[x,x']]} \rho_r(x, x', t), \]

where use was made of the influence action introduced in Eq. (2.7). Equivalently, we may rewrite the previous equation changing to semism and difference variables with \( J_\Sigma = (J(t) + J'(t))/2 \) and \( J_\Delta = J'(t) - J(t) \), integrating the system action by parts and using (3.2)-(3.3):

\[ Z_{CTP}[J_\Sigma, J_\Delta] = \int dX f \int dxX \int_{X_i}^{X_f} DX \int_{\Delta_i}^{\Delta_f} D\Delta e^{-iJ_\Sigma \cdot X \cdot \Delta \cdot L \cdot X} e^{-\frac{i}{2} L \cdot N \cdot \Delta} \]

\[ e^{iM X \cdot \Delta} \rho_r(X_i - \Delta_i/2, X_i + \Delta_i/2, t_i). \]

We can proceed analogously as we did in (3.3)-(3.4); perform the Gaussian path integral in \( \Delta(t) \) with \( \Delta_i \) and \( \Delta_f \) fixed (with \( \Delta_f = 0 \)) and then integrate over \( \Delta_i \) to get

\[ Z_{CTP}[J_\Sigma, J_\Delta] = 2\pi (\det \frac{N}{2\pi})^{-\frac{1}{4}} \int dX f \int dxX \int_{X_i}^{X_f} DX e^{-iJ_\Delta \cdot X \cdot \Delta \cdot L \cdot X} e^{-\frac{1}{2} L \cdot N \cdot \Delta} \cdot \Delta \cdot L \cdot X \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \]

\[ \cdot \rho_r(X_i - \Delta_i/2, X_i + \Delta_i/2, t_i), \]

where we used the definition (2.13) of the reduced Wigner function. Performing the functional change specified in (3.5) and following similar steps to those in (3.6)-(3.9) we get:

\[ Z_{CTP}[J_\Sigma, J_\Delta] = K \int_{-\infty}^{\infty} dX_i \int_{-\infty}^{\infty} dp_i \int \mathcal{D} \xi e^{-iJ_\Delta \cdot X \cdot \Delta \cdot L \cdot X} \]

\[ e^{-iJ_\Delta \cdot X \cdot \Delta \cdot L \cdot X} e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \Delta \cdot L \cdot X \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \rho_r(X_i, p_i, t_i) \]

\[ = \langle e^{-iJ_\Delta \cdot X \cdot \Delta} \rangle_{X_i, p_i} e^{-iJ_\Delta \cdot G_{ret} + iJ_\Sigma \cdot N^{-1}} \cdot \xi \cdot \xi \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \Delta \cdot L \cdot X \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \rho_r(X_i, p_i, t_i) \]

\[ = \langle e^{-iJ_\Delta \cdot X \cdot \Delta} \rangle_{X_i, p_i} e^{-iJ_\Delta \cdot G_{ret} + iJ_\Sigma \cdot N^{-1}} \cdot \xi \cdot \xi \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \Delta \cdot L \cdot X \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \rho_r(X_i, p_i, t_i) \]

\[ = \langle e^{-iJ_\Delta \cdot X \cdot \Delta} \rangle_{X_i, p_i} e^{-iJ_\Delta \cdot G_{ret} + iJ_\Sigma \cdot N^{-1}} \cdot \xi \cdot \xi \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \Delta \cdot L \cdot X \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \rho_r(X_i, p_i, t_i) \]

where \( X_o(t) \) was the solution to the homogeneous equation \( (L \cdot X_o)(t) = 0 \) with initial conditions \( X_i, p_i \) and the value of the normalization constant \( K \) was fixed in (3.9). Here we used the definitions (3.13) and (3.14) in the second equality, separated the factors which just depend on the initial conditions from those which just depend on \( \xi \) in the third equality, and computed the average \( \langle \ldots \rangle_\xi \) which amounts to a Gaussian functional integral, in the fourth equality. We also introduced the notation \( (A)^T(t, t') = A(t', t) \). The final result for the CTP generating functional follows trivially from the last expression in Eq. (4.4):

\[ Z_{CTP}[J_\Sigma, J_\Delta] = \langle e^{-iJ_\Delta \cdot X \cdot \Delta} \rangle_{X_i, p_i} e^{-iJ_\Delta \cdot G_{ret} + iJ_\Sigma \cdot N^{-1}} \cdot \xi \cdot \xi \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \Delta \cdot L \cdot X \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \rho_r(X_i, p_i, t_i) \]

\[ = \langle e^{-iJ_\Delta \cdot X \cdot \Delta} \rangle_{X_i, p_i} e^{-iJ_\Delta \cdot G_{ret} + iJ_\Sigma \cdot N^{-1}} \cdot \xi \cdot \xi \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \Delta \cdot L \cdot X \cdot e^{-\frac{i}{2} L \cdot N \cdot \Delta} \cdot \rho_r(X_i, p_i, t_i) \]

It is interesting to note that the first factor contains all the information about the initial conditions of the system, whereas the information about the fluctuations induced on the system by the environment is essentially contained in the second factor through the noise kernel. This is the key result of this section, which will allow to relate the quantum with the stochastic correlation functions.

By performing the appropriate Legendre transformation of \( W[J, J'] = -i \ln Z_{CTP}[J, J'] \) \[25-27\], the CTP effective action for the system can be exactly obtained to be

\[ \Gamma_{CTP}[x, x'] = S[x] - S[x'] + S_{IF}[x, x']. \]
Since one can always write $J$ and $J'$ in terms of $J_\Sigma$ and $J_\Delta$ as $J_\Sigma = (J(t) + J'(t))/2$ and $J_\Delta = J'(t) - J(t)$, the right-hand side of Eq. (4.7) can be expressed as a linear combination of terms of the type
\[
i^{r+s} \frac{\delta}{\delta J_\Sigma} \left( \frac{\delta}{\delta J_\Delta} \right)^s Z_{CTP}[J_\Sigma, J_\Delta]\bigg|_{J_\Sigma, J_\Delta = 0},
\]
with $0 \leq r \leq n + m$, $0 \leq s \leq n + m$ and $r + s = n + m$. To obtain an explicit expression one must evaluate Eq. (4.8) with the final result for the CTP generating functional (4.5). We will give some examples in the following subsection.

B. Particular cases: the two-point quantum correlation functions

Let us write the explicit expressions for all the possible two-point quantum correlation functions of the system position operators. The following results for the second functional derivatives of the CTP generating functional using the final result (4.5) will be useful:

\[
\begin{align*}
\frac{\delta}{\delta J_\Sigma(t_2)} & \frac{\delta}{\delta J_\Sigma(t_1)} Z_{CTP}[J_\Sigma, J_\Delta]\bigg|_{J_\Sigma, J_\Delta = 0} = 0, \\ 
\frac{\delta}{\delta J_\Sigma(t_2)} & \frac{\delta}{\delta J_\Delta(t_1)} Z_{CTP}[J_\Sigma, J_\Delta]\bigg|_{J_\Sigma, J_\Delta = 0} = -i G_{ret}(t_1, t_2) = -i (G_{ret})^T(t_2, t_1), \\ 
\frac{\delta}{\delta J_\Delta(t_2)} & \frac{\delta}{\delta J_\Sigma(t_1)} Z_{CTP}[J_\Sigma, J_\Delta]\bigg|_{J_\Sigma, J_\Delta = 0} = -i G_{ret}(t_2, t_1), \\ 
\frac{\delta}{\delta J_\Delta(t_2)} & \frac{\delta}{\delta J_\Delta(t_1)} Z_{CTP}[J_\Sigma, J_\Delta]\bigg|_{J_\Sigma, J_\Delta = 0} = -\langle X_o(t_1) X_o(t_2) \rangle_{X_i, p_i} - (G_{ret} \cdot N \cdot (G_{ret})^T)(t_2, t_1). 
\end{align*}
\]

We begin with the Wightman function $G^+(t_1, t_2)$:

\[
G^+(t_1, t_2) \equiv \text{Tr} \left[ \hat{x}(t_2) \hat{\rho}(t_1) \hat{x}(t_1) \right] = \frac{\delta}{\delta J_\Sigma(t_1)} \frac{\delta}{\delta J_\Sigma(t_2)} Z_{CTP}[J_\Sigma, J_\Delta]\bigg|_{J_\Sigma, J_\Delta = 0} = \frac{1}{4} \frac{\delta}{\delta J_\Sigma(t_1)} \frac{\delta}{\delta J_\Sigma(t_2)} - \frac{1}{2} \frac{\delta}{\delta J_\Sigma(t_1)} \frac{\delta}{\delta J_\Delta(t_2)} + \frac{1}{2} \frac{\delta}{\delta J_\Delta(t_1)} \frac{\delta}{\delta J_\Sigma(t_2)} - \frac{\delta}{\delta J_\Delta(t_1)} \frac{\delta}{\delta J_\Delta(t_2)} Z_{CTP}[J_\Sigma, J_\Delta]\bigg|_{J_\Sigma, J_\Delta = 0}.
\]

Substituting (4.9a)-(4.9d) into Eq. (4.10), we get

\[
G^+(t_1, t_2) = \frac{i}{2} \left[ G_{ret}(t_2, t_1) - G_{ret}(t_1, t_2) \right] + \langle X_o(t_1) X_o(t_2) \rangle_{X_i, p_i} + (G_{ret} \cdot N \cdot (G_{ret})^T)(t_1, t_2). 
\]

Similarly, for the Hadamard and Jordan functions, denoted by $G^{(1)}(t_1, t_2)$ and $G(t_1, t_2)$ respectively, as well as the Feynman propagator $G^F(t_1, t_2)$ we have

\[
\begin{align*}
G^{(1)}(t_1, t_2) & \equiv \text{Tr} \left[ [\hat{x}(t_1), \hat{x}(t_2)] \hat{\rho}(t_1) \right] = \frac{1}{2} \left[ \frac{\delta}{\delta J_\Sigma(t_1)} \frac{\delta}{\delta J_\Sigma(t_2)} - 2 \frac{\delta}{\delta J_\Sigma(t_2)} \frac{\delta}{\delta J_\Delta(t_1)} \right] Z_{CTP}[J_\Sigma, J_\Delta]\bigg|_{J_\Sigma, J_\Delta = 0} = 2 \left[ \langle X_o(t_1) X_o(t_2) \rangle_{X_i, p_i} + (G_{ret} \cdot N \cdot (G_{ret})^T)(t_1, t_2) \right], \\
G(t_1, t_2) & \equiv \text{Tr} \left[ [\hat{x}(t_1), \hat{x}(t_2)] \frac{\delta}{\delta J_\Sigma(t_1)} \hat{\rho}(t_1) \right] = \left[ \frac{\delta}{\delta J_\Sigma(t_1)} \frac{\delta}{\delta J_\Sigma(t_2)} - \frac{\delta}{\delta J_\Delta(t_2)} \frac{\delta}{\delta J_\Sigma(t_1)} \right] Z_{CTP}[J_\Sigma, J_\Delta]\bigg|_{J_\Sigma, J_\Delta = 0} = i (G_{ret}(t_2, t_1) - G_{ret}(t_1, t_2)), \\
\end{align*}
\]
\[ G^F(t_1, t_2) \equiv Tr [ T (\hat{x}(t_1) \hat{x}(t_2)) \rho(t_1)] = \left[ 1 - \frac{1}{2} \frac{\delta}{\delta J_\Sigma(t_2)} \frac{\delta}{\delta J_\Sigma(t_1)} \right] Z_{CTP}[J_\Sigma, J_\Delta] \bigg|_{J_\Sigma, J_\Delta = 0} = -i (G_{ret}(t_1, t_2) + G_{ret}(t_2, t_1)) + \frac{1}{2} G^{(1)}(t_1, t_2) \]

Combining the previous results, the following alternative expressions, which may look more familiar, are obtained for the retarded propagator and the Wightman function:

\[ G_{ret}(t_1, t_2) = iG(t_1, t_2)\theta(t_1 - t_2), \]
\[ G^+(t_1, t_2) = \frac{1}{2} \left[ G(t_1, t_2) + G^{(1)}(t_1, t_2) \right]. \]

It should be stressed that all the previous results for the quantum correlation functions, as well as expression (4.5) for the CTP generating functional are valid for a general initial state of the system, i.e., an arbitrary initial reduced density matrix. On the other hand, as implied by Wick's theorem, for the particular case of a Gaussian initial state, any \( n \)-point correlation function can be obtained as a linear combination of products of the two-point correlation functions obtained in this subsection.

C. Stochastic and quantum correlation functions. An illustrative example

From expression (4.8) for the case \( r = 0 \), a connection can be established between the correlation functions for the Gaussian stochastic process associated to \( \xi(t) \) via the Langevin-type equation in (3.15) with \( W(t, X, p, t_0) \) as the distribution function for the initial conditions, and some quantum correlation functions corresponding to quantum expectation values of products of Heisenberg operators at different instants of time. Any correlation function for the former stochastic process can be obtained from its characteristic functional in the usual way:

\[ \langle \langle X(t_1) ... X(t_n) \rangle \rangle_{X_i, p_i} = i^s \left( \frac{\delta}{\delta K} \right)^s \langle \langle e^{-iK \cdot X} \rangle \rangle_{X_i, p_i} \bigg|_{K = 0}. \]

The generating functional for the aforementioned stochastic process is, in turn, related to the full CTP generating functional previously introduced as follows:

\[ \langle \langle e^{-iK \cdot X} \rangle \rangle_{X_i, p_i} = Z_{CTP}[J_\Sigma = 0, J_\Delta = K]. \]

Substituting Eq. (4.16) into Eq. (4.15), rewriting \( J_\Delta \) in terms of \( J \) and \( J' \), and using expression (4.7), we can express the correlation functions for the stochastic process in terms of quantum correlation functions for the system observables. In particular, for \( s = 2 \) we have

\[ \langle \langle X(t_1) X(t_2) \rangle \rangle_{X_i, p_i} = \frac{1}{4} \left( \langle \langle T \hat{x}(t_1) \hat{x}(t_2) \rangle \rangle + \langle \langle \hat{x}(t_1) \hat{x}(t_2) \rangle \rangle + \langle \langle \hat{x}(t_2) \hat{x}(t_1) \rangle \rangle + \langle \langle \hat{T} \hat{x}(t_1) \hat{x}(t_2) \rangle \rangle \right) = \frac{1}{2} \langle \langle \hat{x}(t_1), \hat{x}(t_2) \rangle \rangle, \]

where, as usual, we used \( \langle \langle \rangle \rangle \) to denote the quantum expectation value \( Tr[...\hat{\rho}(t_i)] \).

On the other hand, concentrating on the stochastic description provided by the left-hand side of Eq. (4.17) and elaborating a little bit on it by using Eq. (3.6) and taking into account that \( \xi(t) \) is a Gaussian stochastic process characterized by \( \langle \xi(t) \rangle = 0 \) and \( \langle \xi(t_1) \xi(t_2) \rangle = N(t_1, t_2) \), we can write

\[ \langle \langle X(t_1) X(t_2) \rangle \rangle_{X_i, p_i} = \langle \langle [X_0(t_1) + (G_{ret} \cdot \xi)(t_1)] [X_0(t_2) + (G_{ret} \cdot \xi)(t_2)] \rangle \rangle_{X_i, p_i} = \langle X_0(t_1) X_0(t_2) \rangle_{X_i, p_i} + (G_{ret} \cdot N \cdot (G_{ret})^T)(t_1, t_2). \]

Hence, the final result is

\[ \frac{1}{2} \langle \langle \hat{x}(t_1), \hat{x}(t_2) \rangle \rangle = \langle X_0(t_1) X_0(t_2) \rangle_{X_i, p_i} + (G_{ret} \cdot N \cdot (G_{ret})^T)(t_1, t_2). \]
in agreement with Eq. (4.12a). The left-hand side of Eq. (4.19) is the quantum correlation function, which can therefore be described within the stochastic scheme in terms of two separate contributions: the first term on the right-hand side corresponds entirely to the dispersion in the initial conditions, whereas the second term is due to the fluctuations induced by the stochastic source appearing in the Langevin-type equation (3.15). It should be strongly remarked that, as discussed in Appendix B, for the general case of a nonlocal influence action no quantum correlation functions (except for the trivial case of $n = 1$) can be expressed in terms of the propagators for the reduced density matrix, which are obtained from the master equation.

From Eqs. (4.15) and (4.16) it is clearly seen that only those quantum correlation functions which are obtained by functionally differentiating the CTP generating functional with respect to $J_\Delta$ (but not $J_X$) an arbitrary number of times can be related to the stochastic correlation functions (4.15). Let us, therefore, see what is the general expression for all the quantum correlation functions that can be directly obtained from the stochastic description. We begin with the classical correlation functions (4.15) for the stochastic processes $X(t)$ which are solutions of the Langevin-type equation with stochastic source $\xi(t)$ and initial conditions averaged over the initial reduced Wigner function. Then we write these correlation functions in terms of path integrals and use the results of Secs. III and IV to relate them to a subclass of quantum correlation functions for the system:

$$
\langle (X(t_1) \ldots X(t_n))_X \rangle_{X_1, \ldots, X_n} = \frac{\det(2\pi iN)}{2\pi} \int dX \int \lim_{\Delta \to 0} \frac{dX^\prime}{\pi} \int dp \int D\xi e^{-\frac{1}{2} \xi (\hbar N)^{-1} \xi} \delta(X(t_1) - X(t'_1))X(t_1) \ldots X(t_n) W_r(X_1, \ldots, X_n) = Tr^* \left[ X(t_1) \ldots X(t_n) \hat{\rho}(t) \right],
$$

(4.20)

where $\hat{X}(t_i) = (\hat{x}(t_i) + \hat{x}'(t_i))/2$ and

$$
Tr^* \left[ \hat{x}(t_1) \ldots \hat{x}'(t_r) \ldots \hat{x}(t_s) \ldots \hat{x}'(t_u) \hat{\rho}(t) \right] = Tr \left[ T \hat{x}(t_1) \ldots \hat{x}(t_{r-1}) \hat{x}(t_r) \ldots \hat{x}(t_{u-1}) \hat{x}(t_u) \ldots \right] \hat{\rho}(t).
$$

(4.21)

where both the initial density matrix and the trace correspond to the whole closed quantum system (i.e., system plus environment) and $T$ and $\tilde{T}$ denote time and anti-time ordering, respectively. It is then straightforward to show that

$$
\langle (X(t_1) \ldots X(t_n))_{X_1, \ldots, X_n} = 2^{-n} \sum_{m=0}^{n} \frac{1}{m!(n-m)!} \sum_{\sigma \in S_n} Tr \left[ \left\{ T \prod_{i=\sigma(1)}^{\sigma(m)} \hat{x}(t_i) \right\} \hat{\rho}(t) \right] \left\{ \tilde{T} \prod_{j=\sigma(m+1)}^{\sigma(n)} \hat{x}(t_j) \right\}.
$$

(4.22)

where $\sigma \in S_n$ are all the possible permutations for a set consisting of $n$ elements.

V. PERTURBATIVE TREATMENT FOR NONLINEAR COUPLING

Throughout the paper we have considered the case in which the system and the environment are both linear and that they are linearly coupled. However, the situation can be slightly generalized to the case of nonlinear interaction between system and environment provided the nonlinear part of the coupling can be treated perturbatively.

Let us begin by considering the case in which the nonlinearity in the system-environment comes from the environment. Suppose $S_{int}[x, \{q_j\}] = \lambda \sum_j c_j q_j^k$ with $k = 2, 3, \ldots$, where $\lambda$ is a perturbative parameter. One can compute $S_{IF}[x, x']$ perturbatively in $\lambda$ [30,62]. In fact one can use a set of Feynman rules [30] completely analogous to those used in the context of the CTP formalism. Alternatively, one may consider the case in which the nonlinearity comes entirely from the system: $S_{int}[x, \{q_j\}] = \sum_q c_j q_j f(x_q + \lambda x)$, where $f(z)$ is in general a nonlinear function and $\lambda$ is a perturbative parameter that controls the expansion around a given fixed background function $x_q(t)$, usually taken to be a solution of the system equations of motion without considering the coupling to the environment. When computing the influence action, we will get analogous results to the linear case but with the following sum and difference coordinates

$$
\hat{X} = \frac{1}{2} (f(x_b + \lambda x) + f(x_b + \lambda x'))
$$

$$
\hat{\Delta} = f(x_b + \lambda x') - f(x_b + \lambda x),
$$

instead of $X$ and $\Delta$. Expanding perturbatively in $\lambda$, $\hat{X}$ and $\hat{\Delta}$ can be expressed in terms of $X$ and $\Delta$:

14
FIG. 1. Diagram showing the interconnections between different quantum properties of an open quantum system on the one hand, and between the elements of the stochastic description on the other hand, as well as the connection between both levels of description. Labels Q and S stand for quantum objects and stochastic ones, respectively.

\[ \dot{X} = f(x_b) + \lambda f'(x_b)X + \frac{\lambda^2}{2} f''(x_b) \left[ X^2 + \left( \Delta/2 \right)^2 \right] + O(\lambda^3) \]

\[ \dot{\Delta} = \lambda f'(x_b)\Delta + \lambda^2 f''(x_b)X\Delta + O(\lambda^3). \]

In both cases we recover the usual structure for the influence action (2.9) if we keep up to quadratic order in \( \lambda \): the term including \( X\Delta \) is a dissipation term, whereas the term including \( \Delta\Delta \) is a noise term (being quadratic in \( \Delta \), it will give rise to a Gaussian stochastic source). Of course, both kinds of nonlinearities can be simultaneously considered when treated perturbatively.

VI. SUMMARY AND DISCUSSION

In this paper we have considered the stochastic description of a linear open quantum system. We have shown that the reduced Wigner function can be written as a formal distribution function for a stochastic process given by a Langevin-type equation. The master equation has then been deduced as the corresponding Fokker-Planck equation for the stochastic process. We have also shown that the stochastic correlation functions for the system variables can be written in terms of quantum correlation functions. To summarize our results we give a diagram (see Fig. 1) with all the interconnections between the influence functional, the Langevin equation, the Fokker-Planck equation, the master equation and the correlation functions.

Quantum objects (label Q) are placed at the top of the diagram whereas those objects associated to the stochastic description (label S) are placed at the bottom. The equivalence between the reduced Wigner function with a distribution function in the framework of the stochastic description provided by the Langevin equation reflects the connection formally established between both levels of description. One can take advantage of this connection by working within the stochastic framework to obtain relevant information about quantum properties. In the stochastic context, one can always obtain a Fokker-Planck equation for the distribution function associated to the stochastic process resulting from a given Langevin equation. From the connection previously stated, the Fokker-Planck equation can be translated into the master equation for the reduced Wigner function or, equivalently, for the reduced density matrix. Similarly, correlation functions for the stochastic processes which are solutions of the Langevin equation are closely related to quantum correlation functions for system observables.

Finally, we should insist on the fact that, although we have exploited the formal description of open quantum systems in terms of stochastic processes, a classical statistical interpretation is not always possible. Thus, although the Wigner function is a real and properly normalized function providing a distribution for the initial conditions of our formal stochastic processes, it is not a true probability distribution function in the sense that it is not positive definite.
The derivation of the Fokker-Planck equation from the Langevin equation with local dissipation is well understood (see Ref. [41]). However, in our case the existence of nonlocal dissipation makes it convenient to review the main steps. Let us begin by computing $\partial W_r/\partial t$ from expression (3.12),

$$\frac{\partial W_r(X,p,t)}{\partial t} = \left\langle \left( \dot{X}(t)\delta'(X(t) - X)\delta(M\dot{X}(t) - p) \right) \right\rangle_{X_i,p_i} + \left\langle \left( \delta(X(t) - X)M\dot{X}(t)\delta'(M\dot{X}(t) - p) \right) \right\rangle_{X_i,p_i},$$

where the fact that $\dot{X}(t)$, $\partial/\partial X(t)$ and $\partial/\partial \dot{X}(t)$ may be replaced by $p/M$, $-\partial/\partial X$ and $-\partial/\partial p$ respectively, since they are multiplying the delta functions, was used in the second equality. Let us now concentrate on the expectation value appearing in the last term and recall the expectation values defined in (3.13)-(3.14). We will consider the Langevin-type equation

$$(L \cdot X)(t') = \xi(t'),$$

(A2)

corresponding to the functional change (3.5) and substitute the corresponding expression for $M\ddot{X}(t)$ so that the last expectation value in (A1) can be written as

$$-M\Omega_{ren}^2 X W_r(X,p,t) + \left\langle \left( - \int_{t_i}^t dt H(t,t')X(t') + \xi(t) \right) \delta(X(t) - X)\delta(M\dot{X}(t) - p) \right\rangle_{X_i,p_i}.$$

(A3)

Any solution of Eq. (A2) can be written as

$$X(t') = X_h(t') + \int_{t_i}^t dt'' \tilde{G}_{adv}(t',t'')\xi(t''),$$

(A4)

where $X_h(t')$ is a solution of the homogeneous equation $(L \cdot X)(t') = 0$ such that $X_h(t) = X$, $\dot{X}_h(t) = p/M$ and $\tilde{G}_{adv}(t',t'')$ is the advanced (i.e., $\tilde{G}_{adv}(t',t'') = 0$ for $t' \geq t''$) Green function for the linear integro-differential operator associated to the kernel $L(t,t')$. The particular solution of the inhomogeneous Eq. (A2) $\dot{X}_i(t') = \int_{t_i}^t dt'' \tilde{G}_{adv}(t',t'')\xi(t'')$ has boundary conditions $\dot{X}_i(t) = 0$, $\dot{X}_i(t')/\partial t'$ $|_{t'=t} = 0$. Both $X_h(t')$ and $\tilde{G}_{adv}(t',t'')$ can be expressed in terms of the homogeneous solutions $u_1(t')$ and $u_2(t')$, which satisfy $u_1(t_i) = 1$, $u_1(t) = 0$ and $u_2(t_i) = 0$, $u_2(t) = 1$ respectively;
We use the advanced propagator so that there is no dependence on the initial conditions at time \( t' = t_i \) coming from the homogeneous solution but just on the final conditions at time \( t' = t, \) i.e., on those the Fokker-Planck equation is written in terms of. Using expression (A4) the first term within the expectation value appearing in Eq. (A3) can be reexpressed as

\[
\int_{t_i}^{t} dt' H(t, t') \left\langle X(t') \delta(X(t) - X) \delta(M \dot{X}(t) - p) \right\rangle_{X, p, i} = \int_{t_i}^{t} dt' H(t, t') X_h(t') W_r(X, p, t) + \int_{t_i}^{t} dt' \int_{t_i}^{t} dt'' H(t, t') \tilde{G}_{adv}(t', t'') \left\langle \xi(t'') \delta(X(t) - X) \delta(M \dot{X}(t) - p) \right\rangle_{X, i, p, i} .
\]  

(A7)

The first term on the right-hand side can in turn be written as

\[
-(M \delta \Omega(t) X + 2 A(t) p) W_r(X, p, t),
\]  

(A8)

where

\[
\delta \Omega(t) = \frac{1}{M} \int_{t_i}^{t} dt' H(t, t') [u_2(t') - (\dot{u}_2(t)/\dot{u}_1(t)) u_1(t')],
\]  

(A9)

\[
A(t) = \frac{1}{2} (M \dot{u}_1(t))^{-1} \int_{t_i}^{t} dt' H(t, t') u_1(t').
\]  

(A10)

In order to find an expression for \( \left\langle \xi(t') \delta(X(t) - X) \delta(M \dot{X}(t) - p) \right\rangle_{X} \) we use Novikov’s formula for Gaussian stochastic processes [63], which corresponds essentially to use (3.13) and functionally integrate by parts with respect to \( \xi(t) \),

\[
\left\langle \xi(t') F(t; \xi) \right\rangle_{\xi} = \int_{t_i}^{t} dt'' N(t', t'') \left\langle \delta F(t; \xi) / \delta \xi(t'') \right\rangle_{\xi} .
\]  

(A11)

We then obtain the following expression:

\[
\left\langle \xi(t') \delta(X(t) - X) \delta(M \dot{X}(t) - p) \right\rangle_{X} = \int_{t_i}^{t} dt'' \int_{t_i}^{t} dt''' N(t', t'') \left\langle \frac{\delta X(t''')}{\delta \xi(t''')} \frac{\delta}{\delta X(t''')} \delta(X(t) - X) \delta(M \dot{X}(t) - p) \right\rangle_{X} + \frac{\delta \dot{X}(t''')}{\delta \xi(t''')} \delta \dot{X}(t) \delta(X(t) - X) \delta(M \dot{X}(t) - p) \right\rangle_{X} = \int_{t_i}^{t} dt'' \int_{t_i}^{t} dt''' N(t', t'') \delta(t''' - t) \left\langle - \left( \frac{\delta X(t''')}{\delta \xi(t''')} \partial X + M \frac{\delta \dot{X}(t''')}{\delta \xi(t''')} \partial p \right) \delta(X(t) - X) \delta(M \dot{X}(t) - p) \right\rangle_{X} = \int_{t_i}^{t} dt'' N(t', t'') \left\langle \left( \frac{\delta X(t)}{\delta \xi(t'')} \partial X + M \frac{\delta \dot{X}(t)}{\delta \xi(t'')} \partial p \right) \delta(X(t) - X) \delta(M \dot{X}(t) - p) \right\rangle_{X} ,
\]  

(A12)

where we used again the presence of the delta functions to substitute the functional derivatives \( \delta / \delta X(t'') \) and \( \delta / \delta \dot{X}(t'') \) by \( \delta(t''' - t) \cdot \partial / \partial X \) and \( \delta(t''' - t) \cdot M \cdot \partial / \partial p \), respectively, in the second equality. Functionally differentiating with respect to \( \xi(t'') \) expression (3.6) for \( X(t) \) and analogously for \( \dot{X}(t) \) we get

\[
\frac{\delta X(t')}{\delta \xi(t'')} = G_{rel}(t', t''),
\]  

(A13a)

\[
\frac{\delta \dot{X}(t')}{\delta \xi(t'')} = \frac{\partial}{\partial t'} G_{rel}(t', t''),
\]  

(A13b)
which after substitution into (A12) leads to

\[
\left\langle \xi(t') \delta(X(t) - X) \delta(M\dot{X}(t) - p) \right\rangle = -\int_{t_i}^{t} dt'' N(t', t'') \left( G_{\text{rel}}(t, t'') \frac{\partial}{\partial X} + M \frac{\partial G_{\text{rel}}(t, t'')}{\partial p} \right) W_r(X, p, t). \tag{A14}
\]

The retarded Green function can also be expressed in terms of the solutions of the homogeneous equation \( u_1(t) \) and \( u_2(t) \), which were previously introduced, as

\[
G_{\text{rel}}(t', t'') = \frac{1}{M} \frac{u_1(t') u_2(t'') - u_2(t') u_1(t'')}{u_1(t'') u_2(t') - u_2(t'') u_1(t')} \delta(t' - t''). \tag{A15}
\]

Note that it is important to use now the expression in terms of the retarded propagator \( G_{\text{rel}} \) and the initial conditions \( X_i \) and \( p_i \) (at time \( t' = t_i \)), since the “final” conditions \( X(t) \) and \( M\dot{X}(t) \) depend on \( \xi(t'') \) (for \( t'' < t \)). Putting all the terms together, i.e., (A3), (A7) and (A14), we reach the final expression for (A1):

\[
\frac{\partial W_r}{\partial \theta(t)} = \{H_R, W_r\}_{PB} + 2A(t) \frac{\partial(pW_r)}{\partial p} + B(t) \frac{\partial^2 W_r}{\partial X \partial p} + MC(t) \frac{\partial^2 W_r}{\partial p^2}, \tag{A16}
\]

where \( \delta \Omega(t) \) and \( A(t) \) are given by Eqs. (A9) and (A10), and

\[
B(t) = \int_{t_i}^{t} dt'' N(t, t'') G_{\text{rel}}(t, t''), \quad C(t) = \int_{t_i}^{t} dt'' \frac{\partial G_{\text{rel}}(t, t'')}{\partial t} - \int_{t_i}^{t} \int_{t_i}^{t'} dt' H(t, t') \int_{t_i}^{t'} dt'' G_{\text{adv}}(t', t'') \int_{t_i}^{t'} dt'' N(t'', t''') \frac{\partial G_{\text{rel}}(t, t''')}{\partial t''}. \tag{A17}
\]

The last two expressions were obtained by combining the second term within the expectation value appearing in (A3) and the second term on the right-hand side of Eq. (A7). It should be taken into account that if we put back the \( \hat{b} \)'s, there appears one with every noise kernel in Eqs. (A17) and (A18).

The expressions (A9), (A10), (A17) and (A18) for \( \delta \Omega(t) \), \( A(t) \), \( B(t) \) and \( C(t) \), respectively, coincide exactly with those of Ref. [58], which are in turn equivalent to those obtained in Ref. [29]. Thus, this derivation of the master equation based on a stochastic description is an alternative to those given previously.

**APPENDIX B: CORRELATION FUNCTIONS AND NONLOCAL INFLUENCE ACTION**

Let us see how the fact that the influence action is nonlocal implies that the propagator for the reduced density matrix does not factorize in time and, thus, the system evolution is non-Markovian. In this Appendix we will denote the integrand of the real part of the influence action by \( \mathcal{H} \equiv \Delta(t) \mathcal{H}(t, t') X(t') \) and the integrand of the imaginary part by \( \mathcal{N} \equiv \Delta(t) \mathcal{N}(t, t') \Delta(t')/2 \).

When the influence action is local \( \mathcal{H}(t, t') = \tilde{H}(t) \delta(t - t') \), \( \mathcal{N}(t, t') = \tilde{N}(t) \delta(t - t') \) and we have

\[
S_{IF}[x, x'; t_f, t_i] = \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \tilde{H} + i \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \tilde{N} = \int_{t_i}^{t_f} dt \tilde{H} + i \int_{t_i}^{t_f} dt \tilde{N}, \tag{B1}
\]

where we introduced the notation \( S_{IF}[x, x'; t_f, t_i] \), which is a functional of \( x(t) \) and \( x'(t) \) and also depends on the variables \( t_i \) and \( t_f \), to explicitly state the initial and final times defining the dependence domain considered for the functions \( x(t) \) and \( x'(t) \), which will play an important role in the subsequent discussion. Expression (B1) can then be decomposed as follows

\[
S_{IF}[x, x'; t_f, t_i] = \left( \int_{t_i}^{t_f} dt \tilde{H} + i \int_{t_i}^{t_f} dt \tilde{N} \right) + \left( \int_{t_i}^{t_f} dt \tilde{H} + i \int_{t_i}^{t_f} dt \tilde{N} \right) = S_{IF}[x, x'; t_f, t_1] + S_{IF}[x, x'; t_1, t_i], \tag{B2}
\]

so that the influence functional factorizes

\[
F_{IF}[x, x'; t_f, t_i] = e^{S_{IF}[x, x'; t_f, t_i]} = F_{IF}[x, x'; t_f, t_1] F_{IF}[x, x'; t_1, t_i], \tag{B3}
\]

and so does the reduced density matrix propagator, as can be straightforwardly seen from its path integral representation.
The cross terms like functions can be obtained from the reduced density matrix propagators as was done in Eq. (B5). It is, thus, clear that use was made both of the fact that the system action is local and (B3) applied to definition (2.7) for the reduced influence action is local.

The path integrals in the intermediate steps were decomposed in a way completely analogous to that used in (B4). Hence, the information on the correlation functions can be essentially obtained from the master equation when the influence action is local.

On the other hand, when the influence action is nonlocal,

\[
S_{IF}[x,x';t_f,t_i] = \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \mathcal{H} + i \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \mathcal{N} = \left( \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \mathcal{H} + \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \mathcal{H} + \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \mathcal{H} + \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \mathcal{H} \right) + i \left( \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \mathcal{N} + \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \mathcal{N} + \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \mathcal{N} + \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \mathcal{N} \right). \tag{B6}
\]

The cross terms like \( \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \mathcal{N} \) do not allow the influence action to be separated into terms that depend either on the “history” of the system just for times smaller than \( t_1 \) or just for times greater than \( t_1 \) (as happened in Eq. (B2)). This fact makes it impossible to factorize the influence functional as was done in Eq. (B3) and consequently implies that neither the reduced density matrix propagators factorize in the sense of Eq. (B4) nor the quantum correlation functions can be obtained from the reduced density matrix propagators as was done in Eq. (B5). It is, thus, clear
how the nonlocality of the influence action leads to a non-Markovian evolution for the system and the impossibility to obtain the correlation functions from the propagators for the reduced density matrix.


