Gravitational Higgs Mechanism

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Abstract

We discuss the gravitational Higgs mechanism in domain wall background solutions that arise in the theory of 5-dimensional Einstein-Hilbert gravity coupled to a scalar field with a non-trivial potential. The scalar fluctuations in such backgrounds can be completely gauged away, and so can be the graviphoton fluctuations. On the other hand, we show that the graviscalar fluctuations do not have normalizable modes. As to the 4-dimensional graviton fluctuations, in the case where the volume of the transverse dimension is finite the massive modes are plane-wave normalizable, while the zero mode is quadratically normalizable. We then discuss the coupling of domain wall gravity to localized 4-dimensional matter. In particular, we point out that this coupling is consistent only if the matter is conformal. This is different from the Randall-Sundrum case as there is a discontinuity in the δ-function-like limit of such a smooth domain wall - the latter breaks diffeomorphisms only spontaneously, while the Randall-Sundrum brane breaks diffeomorphisms explicitly. Finally, at the quantum level both the domain wall as well as the Randall-Sundrum setups suffer from inconsistencies in the coupling between gravity and localized matter, as well as the fact that gravity is generically expected to be delocalized in such backgrounds due to higher curvature terms.

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I. INTRODUCTION

In the Brane World scenario the Standard Model gauge and matter fields are assumed to be localized on branes (or an intersection thereof) embedded in a higher dimensional bulk [1–17]. The volume of dimensions transverse to the branes is automatically finite if these dimensions are compact. On the other hand, the volume of the transverse dimensions can be finite even if the latter are non-compact. In particular, this can be achieved by using [13] warped compactifications [18] which localize gravity on the brane [14].

A class of examples with localized gravity is given by domain wall solutions interpolating between two AdS vacua. Such backgrounds spontaneously break diffeomorphism invariance of the theory, which results in gravitational Higgs mechanism. One of the aims of this paper is to study the gravitational Higgs mechanism in some detail. In particular, we compute the spectrum of normalizable modes in domain wall background solutions that arise in the theory of $D$-dimensional Einstein-Hilbert gravity coupled to a scalar field with a non-trivial potential. The scalar fluctuations in such backgrounds can be completely gauged away, and so can be the graviphoton fluctuations. On the other hand, we show that the graviscalar fluctuations do not even have plane-wave normalizable modes. As to the $(D-1)$-dimensional graviton fluctuations, in the case where the domain wall interpolates between two AdS vacua (so that the volume of the transverse dimension is finite) the massive modes are plane-wave normalizable, while the zero mode is quadratically normalizable. We also discuss the case of domain walls interpolating between AdS and Minkowski vacua (in this case the volume of the transverse dimension is infinite) [19], where we have the same conclusions as in the previous case except that the $(D-1)$-dimensional graviton zero mode is no longer quadratically normalizable but plane-wave normalizable.

We then discuss the coupling of domain wall gravity to localized $(D-1)$-dimensional matter. In particular, we point out that this coupling is consistent only if the matter is conformal. This is different from what happens in the Randall-Sundrum model for the reason that there if a discontinuity between the $\delta$-function-like limit of a smooth domain wall and the Randall-Sundrum brane - the former breaks diffeomorphisms only spontaneously, while the latter breaks some of the diffeomorphisms explicitly. We also point out that, in the finite volume cases with localized gravity, at the quantum level there is an inconsistency in the coupling between the domain wall gravity and localized matter as the latter generically is no longer conformal. This is not unrelated to the fact that at the quantum level gravity is generically expected to be delocalized due to higher curvature terms [20–22]. As to the Randall-Sundrum case, where we also expect that gravity is generically expected to be delocalized at the quantum level, an inconsistency in the coupling between brane world gravity and brane matter is generically expected to arise due to the fact that in this case the graviscalar does not decouple in the ultra-violet [21].

Finally, we point out that the aforementioned difficulties do not arise in the recent proposal of [17], where we have completely localized gravity on a solitonic brane. In particular, in this case there are no propagating degrees of freedom in the bulk, and no inconsistency related to higher curvature terms or graviscalar coupling is expected to arise at the quantum level.
II. GRAVITY IN DOMAIN WALL BACKGROUNDS

Consider a single real scalar field $\phi$ coupled to gravity with the following action$^1$:

$$ S = M_P^{D-2} \int d^D x \sqrt{-G} \left[ R - \frac{4}{D-2} (\nabla \phi)^2 - V(\phi) \right] , $$

where $M_P$ is the $D$-dimensional (reduced) Planck scale, and $V(\phi)$ is the scalar potential for $\phi$. The equations of motion read:

$$ \frac{8}{D-2} \nabla^2 \phi = V_\phi , $$

$$ R_{MN} - \frac{1}{2} G_{MN} R = \frac{4}{D-2} \left[ \nabla_M \phi \nabla_N \phi - \frac{1}{2} G_{MN} (\nabla \phi)^2 \right] - \frac{1}{2} G_{MN} V \, . $$

The subscript $\phi$ in $V_\phi$ denotes derivative w.r.t. $\phi$.

In the following we will be interested in solutions to the above equations of motion where the metric has the following warped $[18]$ form

$$ ds^2 = \exp(2A) \eta_{MN} dx^M dx^N \, , $$

where $\eta_{MN}$ is the flat $D$-dimensional Minkowski metric, and the warp factor $A$ and the scalar field $\phi$ are non-trivial functions of $z \equiv x^D$ but are independent of the other $(D-1)$ coordinates $x^\mu$. With this Ansatz we have the following equations of motion for $\phi$ and $A$ (prime denotes derivative w.r.t. $z$):

$$ \frac{8}{D-2} [\phi'' + (D-2) A' \phi'] - V_\phi \exp(2A) = 0 \, , $$

$$ (D-1)(D-2)(A')^2 - \frac{4}{D-2} (\phi')^2 + V \exp(2A) = 0 \, , $$

$$ (D-2) [A'' - (A')^2] + \frac{4}{D-2} (\phi')^2 = 0 \, . $$

We can rewrite these equations in terms of the following first order equations

$$ \phi' = \alpha W_\phi \exp(A) \, , $$

$$ A' = \beta W \exp(A) \, , $$

where

$$ \alpha \equiv \sigma \frac{\sqrt{D-2}}{2} \, , $$

$$ \beta \equiv -\sigma \frac{2}{(D-2)^{3/2}} \, . $$

$^1$Here we focus on the case with one scalar field for the sake of simplicity. In particular, in this case we can absorb a (non-singular) metric $Z(\phi)$ in the $(\nabla \phi)^2$ term by a non-linear field redefinition. This cannot generically be done in the case of multiple scalar fields $\phi^i$, where one must therefore also consider the metric $Z_{ij}(\phi)$. 

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and \( \sigma = \pm 1 \). Moreover, the scalar potential \( V \) is related to the function \( W = W(\phi) \) via

\[
V = W_\phi^2 - \gamma^2 W^2 ,
\]

where

\[
\gamma^2 \equiv \frac{4(D - 1)}{(D - 2)^2} .
\]

In the supersymmetric context the function \( W(\phi) \) is interpreted as the superpotential, while the equations (8) and (9) are the BPS equations, which imply that the domain wall breaks 1/2 of the original supersymmetries.

**A Simple Example**

Let us give a simple example of a domain wall solution of the above type. Thus, let

\[
W = \xi \left[ \zeta \phi - \frac{1}{3} \zeta^3 \phi^3 \right] ,
\]

where \( \xi \) and \( \zeta \) are parameters. The domain wall solution is then given by:

\[
\phi(y) = \frac{1}{\zeta} \tanh \left[ \alpha \zeta^2 (y - y_0) \right] ,
\]

\[
A(y) = \frac{2\beta}{3\alpha \zeta^2} \left\{ \ln \left( \cosh \left[ \alpha \zeta^2 (y - y_0) \right] \right) - \frac{1}{4 \cosh^2 \left[ \alpha \zeta^2 (y - y_0) \right]} \right\} + A_0 ,
\]

where \( y_0 \) and \( A_0 \) are integration constants, which we will set to zero in the following. Then the point \( y = 0 \) corresponds to the “center” of the domain wall, and at this point the warp factor vanishes: \( A(0) = 0 \). For convenience reasons here instead of the \((x^\mu, z)\) coordinate system we are using the \((x^\mu, y)\) coordinate system, where

\[
dy = \exp(A) dz .
\]

In the following we will set the integration constant in the solution \( y = y(z) \) of (17) so that the \( z = 0 \) point corresponds to the \( y = 0 \) point.

Note that in the above solution the volume of the \( y \) direction, which is given by

\[
v = \int dy \exp[(D - 1)A] ,
\]

is finite. This implies that gravity is localized on the domain wall. In the following we will be interested in precisely such domain walls.

**A. Normalizable Modes**

Let us now study gravity in the above type of smooth domain wall backgrounds. In particular, here we would like to compute the spectrum of normalizable modes. Thus, let us consider small fluctuations around the domain wall solution (15) and (16):
where for convenience reasons we have chosen to work with $\tilde{h}_{MN}$ instead of metric fluctuations $h_{MN} = \exp(2A)\tilde{h}_{MN}$. Also, let $\varphi$ be the fluctuation of the scalar field around the background $\phi = \phi(z)$.

In terms of $\tilde{h}_{MN}$ the full $D$-dimensional diffeomorphisms (corresponding to $x^M \to x^M - \xi^M$)

$$\delta h_{MN} = \nabla_M \xi_N + \nabla_N \xi_M$$

(20)

are given by the following gauge transformations (here we use $\xi_M \equiv \exp(2A)\tilde{\xi}_M$):

$$\delta \tilde{h}_{MN} = \partial_M \tilde{\xi}_N + \partial_N \tilde{\xi}_M + 2A' \eta_{MN} \omega,$$

(21)

where $\omega \equiv \tilde{\xi}_D$. As to the scalar field $\varphi$, we have:

$$\delta \varphi = \phi' \omega.$$  

(22)

Since the domain wall solution does not break diffeomorphisms explicitly but spontaneously, the linearized equations of motion are invariant under the full $D$-dimensional diffeomorphisms.

In the following we will use the following notations for the component fields:

$$H_{\mu\nu} \equiv \tilde{h}_{\mu\nu}, \quad A_{\mu} \equiv \tilde{h}_{\mu D}, \quad \rho \equiv \tilde{h}_{DD}.$$ 

(23)

In terms of the component fields $H_{\mu\nu}$, $A_{\mu}$ and $\rho$, the full $D$-dimensional diffeomorphisms read:

$$\delta H_{\mu\nu} = \partial_{\mu} \tilde{\xi}_{\nu} + \partial_{\nu} \tilde{\xi}_{\mu} + 2\eta_{\mu\nu} A' \omega,$$

$$\delta A_{\mu} = \tilde{\xi}'_{\mu} + \partial_{\mu} \omega,$$

$$\delta \rho = 2\omega' + 2A' \omega,$$

$$\delta \varphi = \phi' \omega.$$ 

(24-27)

In the following we will also use the notation $H \equiv H_{\mu}^\mu$.

The linearized equations of motion read:

$$\partial_{\sigma} \partial^\sigma H_{\mu\nu} + \partial_{\mu} \partial_{\nu} H - \partial_{\mu} \partial_{\nu} H_{\sigma\nu} - \partial_{\nu} \partial_{\mu} H_{\sigma\mu} - \eta_{\mu\nu} \left[ \partial_{\sigma} \partial^\sigma H - \partial^\sigma \partial^\sigma H_{\sigma\rho} \right] + H''_{\mu\nu} - \eta_{\mu\nu} H'' + (D - 2) A' \left[ \tilde{H}'_{\mu\nu} - \eta_{\mu\nu} H' \right] -$$

$$\left\{ \partial_{\mu} A_{\nu} + \partial_{\nu} A_{\mu} - 2\eta_{\mu\nu} \partial^\sigma A_{\sigma} + (D - 2) A' \left[ \partial_{\mu} A_{\nu} + \partial_{\nu} A_{\mu} - 2\eta_{\mu\nu} \partial^\sigma A_{\sigma} \right] \right\} +$$

$$\partial_{\mu} \partial_{\rho} \rho - \eta_{\mu\nu} \partial_{\sigma} \partial^\rho \rho + \eta_{\mu\nu} [(D - 2) A' \rho' - V \exp(2A) \rho] =$$

$$\frac{8}{D - 2} \eta_{\mu\nu} \varphi' \varphi' + \eta_{\mu\nu} \varphi V_{\phi} \exp(2A),$$

(28)

$$\left[ \partial^\mu H_{\mu\nu} - \partial \varphi H_{\mu\nu} \right] - \partial^\mu F_{\mu\nu} + (D - 2) A' \partial_{\rho} \rho =$$

$$\frac{8}{D - 2} \varphi' \partial_{\nu} \varphi,$$

(29)

$$- \left[ \partial^\mu \partial^\nu H_{\mu\nu} - \partial^\mu \partial \varphi H \right] + (D - 2) A' \left[ H' - 2\partial^\sigma A_{\sigma} \right] + V \exp(2A) \rho =$$

$$\frac{8}{D - 2} \varphi' \varphi' - \varphi V_{\phi} \exp(2A),$$

(30)

$$\partial_{\mu} \partial^\mu \varphi + \varphi'' + (D - 2) A' \varphi' - \frac{D - 2}{8} \varphi V_{\phi} \exp(2A) -$$

$$\frac{1}{2} \varphi' \left[ 2 \partial^\mu A_{\mu} + \rho' - H' \right] - \frac{D - 2}{8} \rho V_{\phi} \exp(2A) = 0,$$

(31)
where $F_{\mu \nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the $U(1)$ field strength for the graviphoton.

Next, let us note that the field $\varphi$ can be completely eliminated from the above equations of motion [20,19]. Indeed, this is achieved via diffeomorphisms with

$$\omega = -\varphi/\phi'.$$

That is, $\varphi$ is not a propagating degree of freedom in this gauge [20,19]. This is an important point, which implies that not only the $\varphi$ zero mode but the entire field $\varphi$ is “eaten” in the gravitational Higgs mechanism.

Note that setting $\varphi$ to zero uses up some diffeomorphisms, but the residual diffeomorphisms are sufficient to also gauge $A_\mu$ away. Indeed, after we remove $\varphi$ from the equations of motion, we can use the diffeomorphisms with $\omega \equiv 0$ but non-trivial $\tilde{\xi}_\mu$ to set $A_\mu$ to zero without otherwise changing the form of the equations of motion. We, therefore, obtain:

$$\partial_\sigma \partial^\sigma H_{\mu \nu} + \partial_\mu \partial_\nu H - \partial_\mu \partial^\sigma H_{\sigma \nu} - \partial_\nu \partial^\sigma H_{\sigma \mu} - \eta_{\mu \nu} \left[ \partial_\sigma \partial^\sigma H - \partial^\rho \partial_\rho H_{\sigma \mu} \right] + H''_{\mu \nu} - \eta_{\mu \nu} H'' + (D - 2) A' \left[ H_{\mu \nu} - \eta_{\mu \nu} H \right] +$$

$$\partial_\mu \partial_\nu \rho - \eta_{\mu \nu} \partial_\sigma \partial^\sigma \rho + \eta_{\mu \nu} \left[ (D - 2) A' \rho' - V \exp(2A) \rho \right] = 0 ,$$

$$\left[ \partial^\mu H_{\mu \nu} - \partial_\nu H \right] + (D - 2) A' \partial_\nu \rho = 0 ,$$

$$- \left[ \partial^\mu \partial^\nu H_{\mu \nu} - \partial^\rho \partial_\rho H \right] + (D - 2) A' H' + V \exp(2A) \rho = 0 ,$$

$$\phi' \left[ \rho' - H \right] + \frac{D - 2}{4} \rho \exp(2A) = 0 .$$

Here we note that the graviscalar component cannot be gauged away after we perform the above gauge fixing.

Here we note that not all of the above equations are independent. Thus, differentiating (33) with $\partial^\mu$, we obtain an equation which is identically satisfied once we take into account (34) as well as the on-shell expressions for $A$ and $\phi$. Also, if we take the trace of (33), then we obtain an equation which is identically satisfied once we take into account (34), (35) and (36) as well as the on-shell expressions for $A$ and $\phi$. This, as usual, is a consequence of Bianchi identities.

Now we are ready to discuss normalizable modes in the above domain wall background. Let us first consider the normalizable modes for the graviscalar $\rho$. Thus, we can eliminate $H_{\mu \nu}$ from (34), (35) and (36), which gives us the following second order equation for $\rho$:

$$\rho'' + \psi A' \rho' + \partial^\mu \partial_\mu \rho + F \rho = 0 ,$$

where

$$\psi(z) \equiv D + \frac{2\alpha}{\beta} \frac{W_{\phi \phi}}{W},$$

and

$$F(z) \equiv 2 \exp(2A) \left[ (D - 1) \beta^2 W^2 - W_{\phi}^2 + 2\alpha \beta W W_{\phi \phi} + \alpha^2 W_{\phi} W_{\phi \phi} \right] .$$

Let us now assume that $\rho$ satisfies the $(D - 1)$-dimensional Klein-Gordon equation

$$\partial^\mu \partial_\mu \rho = m^2 \rho .$$
In the following we will assume that $m^2 \geq 0$. As to the $m^2 < 0$ modes, they cannot be normalizable - indeed, the domain wall is a kink-like object, and is therefore stable, so no tachyonic modes are normalizable.

We need to understand the asymptotic behavior of $\rho$ at large $z$. To do this, let us first note that at large $z$ the function $\psi(z)$ goes to a constant asymptotic value:

$$\psi(z \to \pm \infty) \equiv \psi_0 .$$

Here for simplicity we are assuming that $W(-\phi) = -W(\phi)$, so that the asymptotic values of $\psi(z)$ at $z \to \pm \infty$ are the same. Also, note that

$$\psi_0 > D .$$

Thus, for instance, in the example given by (15) and (16) we have

$$\psi_0 = D + \frac{3}{2}(D-2)\zeta^2 .$$

In the following, since we are interested in the asymptotic behavior of $\rho$ at large $z$, we will replace $\psi(z)$ in (37) by $\psi_0$.

To proceed further, it is convenient to rescale $\rho$ as follows:

$$\rho \equiv \bar{\rho} \exp \left[ -\frac{1}{2} \psi_0 A \right] .$$

At large $z$ the equation (37) then reads:

$$\bar{\rho}'' + \left[ m^2 + F - \frac{1}{2} \psi_0 A'' - \frac{1}{4} \psi_0^2 (A')^2 \right] \bar{\rho} = 0 .$$

Note that at large $z$ the functions $F$, $A''$ and $(A')^2$ go to zero as $\sim 1/z^2$. We therefore have the following leading behavior for $\bar{\rho}$ at large $z$:

$$\bar{\rho}(z) = C_1 \cos(mz) + C_2 \sin(mz) ,$$

where $C_1, C_2$ are some constant coefficients.

Next, note that the norm for the graviscalar is given by

$$||\rho||^2 \propto \int dz \, \exp(DA)\rho^2 ,$$

where the measure $\exp(DA)$ comes from $\sqrt{-G}$. In terms of $\bar{\rho}$ we have

$$||\rho||^2 \propto \int dz \, \exp[(D - \psi_0)A] \bar{\rho}^2 .$$

Since $A$ goes to $-\infty$ at large $z$, we conclude that, due to (42), none of the $m^2 > 0$ modes are even plane-wave normalizable. Moreover, since the function $F$ in (37) is non-trivial, we do not have a quadratically normalizable zero mode either. Thus, we conclude that $\rho$ is not a propagating degree of freedom in the above background, and should be set to zero.
Next, let us turn to the normalizable modes for the graviton $H_{\mu\nu}$. From (34), (35) and (36) it follows that, since $\rho \equiv 0$, we have
\[ \partial^\mu H'_{\mu\nu} = H' = 0 . \] (49)
This then implies that we can use the residual $(D - 1)$-dimensional diffeomorphisms (for which $\omega \equiv 0$, and $\xi_\mu$ are independent of $z$) to bring $H_{\mu\nu}$ into the transverse-traceless form:
\[ \partial^\mu H_{\mu\nu} = H = 0 . \] (50)
It then follows from (33) that for the modes of the form
\[ H_{\mu\nu} = \xi_{\mu\nu}(x^\rho)\Sigma(z) , \] (51)
where
\[ \partial^\sigma \partial_\sigma \xi_{\mu\nu} = m^2 \xi_{\mu\nu} , \] (52)
the $z$-dependent part of $H_{\mu\nu}$ satisfies the following equation:
\[ \Sigma'' + (D - 2)A'\Sigma' + m^2 \Sigma = 0 , \] (53)
Let us rescale $\Sigma$ as follows:
\[ \Sigma \equiv \tilde{\Sigma} \exp \left[ -\frac{1}{2}(D - 2)A \right] . \] (54)
The equation (53) now reads:
\[ \tilde{\Sigma}'' + \left[ m^2 - \frac{1}{2}(D - 2)A'' - \frac{1}{4}(D - 2)^2(A')^2 \right] \tilde{\Sigma} = 0 . \] (55)
At large $z$ we therefore have:
\[ \tilde{\Sigma}(z) = D_1 \cos (mz) + D_2 \sin (mz) , \] (56)
where $D_1, D_2$ are some constant coefficients.

Next, note that the norm for the graviton is given by
\[ ||H_{\mu\nu}||^2 \propto \int dz \exp[(D - 2)A]\Sigma^2 , \] (57)
where, unlike the graviscalar case, the measure $\exp[(D - 2)A]$ comes from $\sqrt{-G}R$. In terms of $\tilde{\Sigma}$ we have
\[ ||H_{\mu\nu}||^2 \propto \int dz \tilde{\Sigma}^2 . \] (58)
Thus, we see that the $m^2 > 0$ modes of $H_{\mu\nu}$ are plane-wave normalizable. Moreover, we also have a quadratically normalizable zero mode for $H_{\mu\nu}$. This zero mode is given by $\Sigma' = 0$.

Thus, as we see, in smooth domain wall backgrounds of the aforementioned type only the $H_{\mu\nu}$ components correspond to propagating degrees of freedom, while others either can be gauged away or do not have normalizable modes. This is a result of the gravitational Higgs mechanism.
Infinite Volume Cases

We would like to end this subsection with the following remark. Above we considered domain walls interpolating between two AdS vacua. Here we note that we can also have domain walls interpolating between AdS and Minkowski vacua [19]. Let us consider a simple example of such a domain wall. Thus, let

\[
W = \xi \left[ \zeta \phi - \frac{1}{3} \zeta^3 \phi^3 - \frac{2}{3} \right], \tag{59}
\]

where \(\xi\) and \(\zeta\) are parameters. The domain wall solution is then given by:

\[
\phi(y) = \frac{1}{\zeta} \tanh \left[ \alpha \zeta^2(y - y_0) \right], \tag{60}
\]

\[
A(y) = \frac{2\beta}{3\alpha\zeta^2} \left\{ \ln \left( \cosh \left[ \alpha \zeta^2(y - y_0) \right] \right) - \frac{1}{4} \cosh^2 \left[ \alpha \zeta^2(y - y_0) \right] \right\} - \frac{2}{3} \beta \xi (y - y_0) + A_0, \tag{61}
\]

where, as before, \(y_0\) and \(A_0\) are integration constants.

Note that in the above solution the volume of the \(y\) direction, which is given by (18), is infinite. This implies that gravity is not localized on the domain wall. The above computations for the spectrum of normalizable modes can be straightforwardly applied to this case as well. In particular, it is not difficult to show that, as before, the only normalizable modes are those corresponding to the \((D - 1)\)-dimensional graviton \(H_{\mu \nu}\). The difference, however, is that all of the modes \(m^2 \geq 0\) are only plane-wave normalizable, that is, we do not have a quadratically normalizable zero mode in this case.

B. Comparison with the Global Case

For comparative purposes we would like to end this section by briefly discussing what happens if we turn off gravity. In this case the relevant action is

\[
S = \int d^D x \left[ -\frac{4}{D-2} (\partial \phi)^2 - V(\phi) \right], \tag{62}
\]

where the potential \(V\) is given by

\[
V = W^2_\phi. \tag{63}
\]

Let us focus on backgrounds where \(\phi\) depends non-trivially on \(x^D \equiv y\) but is independent of the other \((D - 1)\) coordinates \(x^\mu\). The equation of motion for \(\phi\) can then be written as

\[
\phi_y = \alpha W_\phi. \tag{64}
\]

As before, let

\[
W = \xi \left[ \zeta \phi - \frac{1}{3} \zeta^3 \phi^3 \right], \tag{65}
\]
where $\xi$ and $\zeta$ are parameters. The domain wall solution is then given by:

$$\phi(y) = \frac{1}{\zeta} \tanh \left[ \alpha \xi \zeta^2 (y - y_0) \right], \quad (66)$$

where $y_0$ corresponds to the “center” of the domain wall, which, in this case, is a kink. In the following we will set $y_0 = 0$.

Let us now discuss the spectrum of normalizable modes. The equation of motion for small fluctuations $\varphi$ around the background is given by

$$\varphi_{yy} + \partial^\mu \partial_\mu \varphi - \alpha^2 \left[ W_{\phi\phi}^2 + W_\phi W_{\phi\phi\phi} \right] \varphi = 0. \quad (67)$$

Let us assume that $\varphi$ satisfies the $(D - 1)$-dimensional Klein-Gordon equation

$$\partial^\mu \partial_\mu \varphi = m^2 \varphi. \quad (68)$$

In the following we will assume that $m^2 \geq 0$. As to the $m^2 < 0$ modes, they are not normalizable as the above kink background is stable.

In the background (66) we have:

$$\varphi_{yy} + \left[ m^2 - 4m_*^2 \left( 1 - \frac{3}{2 \cosh^2 (m_* y)} \right) \right] \varphi = 0, \quad (69)$$

where

$$m_* \equiv |\alpha \xi | \zeta^2. \quad (70)$$

The spectrum of normalizable modes is then as follows [1]. There is a quadratically normalizable zero mode given by (note that in this setup the measure in the norm of the $\varphi$ field is trivial)

$$\varphi(x^\mu, y) = \frac{\chi(x^\mu)}{\cosh^2 (m_* y)}, \quad (71)$$

where $\chi(x^\mu)$ satisfies the massless $(D - 1)$-dimensional Klein-Gordon equation. The modes with masses $0 < m \leq 2m_*$ are not normalizable except for an isolated mode with $m^2 = 3m_*^2$, which is quadratically normalizable. Finally, the modes with masses $m > 2m_*$ are plane-wave normalizable. Thus, we have a mass gap in this model. The zero mode is the Goldstone mode of the broken translational invariance in the $y$ direction. Upon gauging the diffeomorphisms, that is, once we include gravity, we expect that this mode is eaten in the corresponding gravitational Higgs mechanism. As we saw in the previous subsection, this is indeed the case. Note, however, that in the gravitational Higgs mechanism not only the zero mode but all the other $\varphi$ modes are eliminated as well including the aforementioned isolated massive quadratically normalizable mode with $m^2 = 3m_*^2$. 

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III. COUPLING TO LOCALIZED MATTER

In this section we would like to study how gravity in the domain wall background couples to localized matter. To do this, let us introduce a probe $\delta$-function-like codimension one brane with matter localized on it. We will refer to the hypersurface corresponding to this brane as $\Sigma$, and we will denote its location in the $z$ direction via $z_0$. For this probe brane not to affect the domain wall background, it must be tensionless, and its coupling to the scalar field $\phi$ must vanish as well.

The term describing the interaction of brane matter with bulk fields is given by

$$S_{\text{int}} = \int_{\Sigma} d^{D-1}x \left[ \frac{1}{2} T_{\mu\nu} H^{\mu\nu} + \frac{8}{D-2} \Theta \phi \right],$$

where $T_{\mu\nu}$ is the (properly normalized) energy-momentum tensor, and $\Theta$ is the coupling of $\phi$ to the brane matter. The invariance under the diffeomorphisms (24) and (27) implies that the energy-momentum tensor is conserved

$$\partial^\mu T_{\mu\nu} = 0,$$

while the trace of the energy momentum tensor and the coupling to the scalar field must satisfy the following condition (note that $\phi'$ does not vanish anywhere):

$$\Theta = -\frac{D-2}{8} \frac{A'(z_0)}{\phi'(z_0)} T,$$

where $T \equiv T_\mu^\mu$.

To proceed further, we need equations of motion for small perturbations in the presence of the above matter sources. As before, in these equations we can gauge $A_\mu$ and $\phi$ away. Then the resulting equations of motion read:

$$\partial_{\sigma} \partial^{\sigma} H_{\mu\nu} + \partial_{\sigma} \partial^{\sigma} H - \partial_{\mu} \partial^{\sigma} H_{\sigma\nu} - \partial_{\nu} \partial^{\sigma} H_{\sigma\mu} - \eta_{\mu\nu} \left[ \partial_{\sigma} \partial^{\sigma} H - \partial^{\sigma} \partial^\sigma H_{\sigma\rho} \right] +$$

$$H''_{\mu\nu} = \Theta \left[ H_{\mu\nu} - \eta_{\mu\nu} H' \right] +$$

$$\partial_\sigma \partial_\rho \eta_{\mu\nu} \partial_\rho H - \eta_{\mu\nu} \partial_\sigma \partial_\rho H + \eta_{\mu\nu} \left[ (D-2) A' \rho - V \exp(2A) \rho \right] = -M_P^{D-2} T_{\mu\nu} \delta(z - z_0),$$

$$[\partial^{\rho} H_{\mu\nu} - \partial_\nu H_{\rho\mu}] + (D-2) A' \partial_\nu \rho = 0,$$  

$$- [\partial^{\rho} \partial^\rho H_{\mu\nu} - \partial_\mu \partial_\nu H] + (D-2) A' H' + V \exp(2A) \rho = 0,$$  

$$\phi' [\rho' - H'] + \frac{D-2}{4} \rho V_\phi \exp(2A) = 2 M_P^{2-D} \Theta \delta(z - z_0).$$

Note that the l.h.s. of the last equation contains no terms with the second derivative w.r.t. $z$, while for non-vanishing scalar coupling $\Theta$ the r.h.s. contains a $\delta$-function source term. This implies that this equation does not have a consistent solution unless we require that

$$\Theta = 0.$$  

This together with (74) implies that, unless $A'(z_0) = 0$, we must have $T = 0$. To avoid this restriction on the trace of the energy-momentum tensor, we must place the brane at the “center” of the domain wall (that is, we must set $z_0 = 0$), where we have $A'(0) = 0$. (Note
that if the domain wall interpolates between AdS and Minkowski vacua $A'\neq 0$ is non-vanishing everywhere, and the consistency requires that the brane matter be conformal [19].

Note that, with $\Theta = 0$, as before (76), (77) and (78) imply that $\rho$ must be set to zero. Moreover, we still have (49). It is then not difficult to see that (75) can be satisfied if and only if

$$T = 0,$$  \hspace{1cm} (80)

that is, the localized matter must be \textit{conformal}. If this condition is satisfied, then the solution for the graviton field $H_{\mu \nu}$ is given by

$$H_{\mu \nu}(p, z) = M_P^{2-D} \Omega(p, z) T_{\mu \nu},$$  \hspace{1cm} (81)

where $\Omega(p, z)$ is the solution to the following equation

$$\Omega''(p, z) + (D - 2)A'\Omega'(p, z) - p^2 \Omega(p, z) = -\delta(z)$$  \hspace{1cm} (82)

subject to the boundary conditions (for $p^2 \equiv p_\mu p^\mu > 0$)

$$\Omega(p, z \to \pm \infty) = 0.$$  \hspace{1cm} (83)

Here we have Fourier transformed the $(D - 1)$ coordinates $x^\mu$ ($p^\mu$ are the corresponding momenta), and Wick rotated to the Euclidean space (where the propagator is unique). The above solution describes a gravitational field of conformal matter localized on the brane.

### A. The $\delta$-function-like Limit and a Discontinuity

The above result, that localized matter cannot be consistently coupled to domain wall gravity unless the former is conformal, might at first appear surprising. In particular, naively it might seem that in the thin wall limit, where the domain wall becomes $\delta$-function like, one should reproduce the setup of [14], where it appears that non-conformal matter can, at least at the classical level, be coupled to bulk gravity. This, however, is not so for a simple reason which we would like to discuss next.

To begin with, let us note that if we take the limit $\zeta \to \infty$ in the domain wall solution (15) and (16), we obtain a $\delta$-function-like brane solution with vanishing scalar field and the warp factor

$$A(y) = -\frac{|y|}{\Delta},$$  \hspace{1cm} (84)

where

$$\Delta \equiv \frac{3}{2 \beta \xi}.$$  \hspace{1cm} (85)

This warp factor is of the same form as in the Randall-Sundrum model [14], where we have a codimension one brane with tension $f > 0$ embedded in the bulk with constant vacuum energy density $\Lambda < 0$.
\[ S = -f \int_{\text{brane}} d^{D-1}x \sqrt{-\hat{G}} + M_P^{D-2} \int d^Dx \sqrt{-G} [R - \Lambda], \quad (86) \]

where
\[ \hat{G}_{\mu\nu} \equiv \delta^M_\mu \delta^N_\nu G_{MN} \bigg|_{z=0}. \quad (87) \]

Here for definiteness we are assuming that the brane is located at \( z = 0 \). With the appropriately fine-tuned brane tension \( f \) and bulk vacuum energy density \( \Lambda \) in this model we then have a solution with precisely the warp factor of the form (84) and \((D - 1)\)-dimensional Poincaré invariance on the brane.

There is, however, a crucial difference between the above two setups. The smooth domain wall solution, even in the aforementioned limit, breaks diffeomorphisms only spontaneously, while the \( \delta \)-function-like brane source in (86) breaks diffeomorphisms \textit{explicitly}. We therefore have a discontinuity between the two setups. That is, for any arbitrarily large but finite value of the parameter \( \zeta \) gravity in the above domain wall background is qualitatively different from that in the Randall-Sundrum model. In particular, in the former case we always have an extra equation (78), which ensures that \( \rho \) vanishes everywhere, and this, in turn, leads to the requirement that the localized matter be conformal.

On the other hand, in the Randall-Sundrum model there is no analog of (78), and \( \rho \) need not vanish. As explained in [21], it is precisely this fact that allows a consistent (classical) coupling between brane matter and bulk gravity in this setup. This is precisely due to the fact that the brane in the Randall-Sundrum model breaks diffeomorphisms explicitly. There is, however, a price one has to pay for this. In particular, even though the graviscalar decouples from the brane matter in the infra-red, its coupling to the trace of the corresponding conserved energy-momentum tensor is non-vanishing in the \textit{ultra-violet} [21]. As explained in [21], at the quantum level this then generically leads to an inconsistency in the coupling between brane matter and bulk gravity in the Randall-Sundrum model.

\[ \text{B. Quantum Instability} \]

As we reiterated in the previous subsection, at the quantum level we expect an inconsistency in the coupling between bulk gravity and brane matter in the Randall-Sundrum model. Here we would like to point out that a similar conclusion holds for gravity in the above type of smooth domain wall backgrounds as well.

Thus, since gravity is localized, generically the localized matter will not remain conformal at the quantum level. This then implies that generically we have an inconsistency in the coupling between localized matter and bulk gravity at the quantum level. This gives us a hint that localization of gravity itself might not be stable against quantum corrections. In fact, this is indeed expected to be the case [20–22]. In particular, generic higher curvature terms actually delocalize gravity. Thus, inclusion of higher derivative terms of, say, the form
\[ \lambda \int d^Dx \sqrt{-G} R^k \quad (88) \]

into the bulk action would produce terms of the form [20–22] (the hatted quantities are \((D - 1)\)-dimensional)
\[ \lambda \int d^{D-1}x \, dy \exp[(D-2k-1)A] \sqrt{-\hat{\mathcal{G}}} \hat{R}^k. \]  

(89)

Assuming that \( A \) goes to \(-\infty\) at \( y \to \pm \infty \), for large enough \( k \) the factor \( \exp[(D-2k-1)A] \) diverges, so that at the end of the day gravity is no longer localized. In fact, for \( D = 5 \) delocalization of gravity takes place already at the four-derivative level once we include the \( R^2 \), \( R_{MN}^2 \) and \( R_{MNRS}^2 \) terms with generic coefficients with the only exception being the Gauss-Bonnet combination [20–22,17].

A possible way around this difficulty might be that all the higher curvature terms should come in “topological” combinations (corresponding to Euler invariants such as the Gauss-Bonnet term) so that their presence does not modify the \((D-1)\)-dimensional propagator for the bulk graviton modes [20–22,17]. That is, even though such terms are multiplied by diverging powers of the warp factor, they are still harmless. One could attempt to justify the fact that higher curvature bulk terms must arise only in such combinations by the fact that otherwise the bulk theory would be inconsistent to begin with due to the presence of ghosts. However, it is not completely obvious whether it is necessary to have only such combinations to preserve unitarity. Thus, in a non-local theory such as string theory unitarity might be preserved, even though at each higher derivative order there are non-unitary terms, due to a non-trivial cancellation between an infinite tower of such terms.

Recently, however, a novel approach to this problem has been proposed in [17]. The setup of [17] is the Einstein-Hilbert-Gauss-Bonnet gravity with negative cosmological term. As was shown in [17], at the special value of the Gauss-Bonnet coupling this theory has a codimension one solitonic brane solution. In this solution the brane is \( \delta \)-function-like, and gravity is completely localized on the brane. That is, there are no propagating degrees of freedom in the bulk, while on the brane we have purely \((D-1)\)-dimensional Einstein gravity. Thus, albeit the classical background is \( D \)-dimensional, the quantum theory is \((D-1)\)-dimensional. The aforementioned troubles with delocalization of gravity as well as consistency of the coupling between brane matter and bulk gravity are therefore absent in the model of [17]. In particular, the brane matter in this model need not be conformal, and the entire setup is stable against quantum corrections

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We would like to end our discussion by pointing out that the aforementioned difficulty with higher curvature terms does not arise in theories with infinite-volume non-compact extra dimensions [23–26,15,27,19,16]. However, in such scenarios consistency of the coupling between bulk gravity and brane matter might give rise to additional constraints. Thus, in some cases the brane world-volume theory must be conformal [27].

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2The only possible instability is related to \((D-1)\)-dimensional physics, namely, the cosmological constant on the brane. This solution, however, can be embedded in supergravity [17], where the solitonic brane is a BPS solution preserving 1/2 of the original supersymmetries, and the brane cosmological constant vanishes.

3In this case gravity is not localized as the volume of the transverse space is infinite, so the requirement that the brane matter is conformal need not be violated at the quantum level. It would be interesting to understand if there is any relation between such setups and [28]. Some speculations on this question were recently given in [17].
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