Note on Four $Dp$-Branes at Angles

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Abstract

In this note we analyse the dynamical potential of a system of four $Dp$-branes at arbitrary angles. The equilibrium configurations for various values of the relative angles and distances among branes are discussed. The known configurations of parallel branes and brane-antibranes are obtained at extrema of the dynamical potential.

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1 Introduction

The $D$-branes at angles, viewed either as boundary states in the Fock space of the closed strings [1, 2, 3] or as solitons in the low energy limit of string theories [4, 5, 6, 7] can form interesting systems. Recently, they had been realized in super Yang-Mills theories [8, 9, 10, 11] and it was shown in [12, 13, 14, 15, 16] that the cancellation of the tadpole anomaly in type II theories compactified on $Z_N \times Z_M$ orbifolds requires the introduction of $D$-branes at angles in order to produce supersymmetric non-chiral field theories. On the other hand, there were found static solutions to Einstein’s equations corresponding to branes at angles that intersect on a three-brane in the context of the Randall-Sundrum model [17]. A deficit angle in the transverse space of branes was used in a tentative to motivate a critical cosmological constant [18]. Also, the branes at angles were employed in the modelling of black-holes [19]. Branes at angles on compact manifolds and in the Born-Infeld field theory were studied in [20, 21, 22, 23]. More recently, their connection with the noncommutative geometry has been investigated in [24, 25, 26].

The interacting potential between two $D_p$-branes depends on their relative angles [27, 28, 29] and it has relative and absolute extrema which describe a brane-antibrane system or a configuration of two parallel branes, respectively. Using several $D$-branes and $NS$-branes, some stable configurations of branes-antibranes can be obtained [30, 31] by compensating geometrically the interaction generated by the tachyonic fields [32]. However, it is interesting to see if there are any stable configurations of $D$-branes only. Intuitively, one would say that it is possible to find some values of the relative angles between branes for which the potential of the system reaches an extrema. The aim of this short note is to address the question of stability in the case of a system of four $D_p$-branes at angles. This system is more richer than a system of three branes since it includes the latter one and presents a configuration of two brane-antibrane pairs.

The outline of the paper is as follows. In Section 2 we review the basic features of a system formed by two branes at angles. In Section 3 we construct the dynamical potential of four branes and analyse their configurations. In Section 4 we present some configurations that describe brane-antibranes. In these two sections the effects due to the presence of the tachyons are ignored. The reason is that the tachyons should be described by an open string field theory and at present we do not know any such of theory that describes branes at angles (but see [39]). However, the tachyons play an important role in the dynamics and stability. Therefore, in Section 5 some general comments on the tachyons of the system are made. The last section is devoted to discussions.

2 Two branes at angles

Let us consider two $D_p$-branes in type II string theories that make a relative angle $\theta$ in the $(p, p + 1)$ plane and are separated by a vector $Z^\mu$. Consider an open string stretched between the branes with one end at $\sigma = 0$ on one brane and the other end at $\sigma = \pi$ on the other brane. The boundary condition are the usual ones on the directions outside this plane and are given by

$$X^{(p)} \sin \theta - X^{(p+1)} \cos \theta = 0$$
$$\partial_\sigma X^{(p)} \cos \theta + \partial_\sigma X^{(p+1)} \sin \theta = 0$$  (1)
in the plane, on the brane on which \( \sigma = \pi \). The fermionic boundary conditions follow from the requirement that the string be supersymmetric. In the RNS formalism in superconformal gauge they are given by the usual boundary conditions outside the \((p, p+1)\)-plane and by

\[
\bar{\epsilon} \rho^1 \rho^0 \psi^{(p)} = 0 \\
\bar{\epsilon} \rho^1 \rho^1 \psi^{(p+1)} = 0,
\]

for \( \sigma = 0 \) and

\[
\bar{\epsilon} \rho^1 \rho^0 (\cos \theta \psi^{(p)} + \sin \theta \psi^{(p+1)}) = 0 \\
\bar{\epsilon} \rho^1 \rho^1 (\cos \theta \psi^{(p)} - \sin \theta \psi^{(p+1)}) = 0,
\]

for \( \sigma = \pi \). Here, \( \rho^0 \) and \( \rho^1 \) are the complex Dirac matrices and \( \epsilon \) is an arbitrary two-dimensional Majorana spinor that parametrizes the supersymmetry transformation. The system can be quantized in the canonical formalism \([1, 2, 3]\) and the usual Ramond and Neveu-Schwarz sectors are obtained with the ranges of the indices of the Fourier modes in \( Z + \theta/\pi \) in \( R \) sector and \( Z + 1/2 + \theta/\pi \) in the \( NS \) sector.

The interaction between the branes can be calculated from the exchange of states from the closed string channel. The scattering amplitude of the massless states (\( \rightarrow 0 \) limit) is given in terms of the dual open string variables by the following relation \([1]\)

\[
A(\theta, Z) = V_p \int dt (8\pi^2 \alpha' t)^{-\frac{p}{2}} e^{-\frac{Z^2 t}{2\pi^2 \alpha'}} [8t^3 \tan(\frac{\theta}{2}) \sin^2(\frac{\theta}{2})]
\]

and it is computed as in the \( \theta = 0 \) case \([33]\).

The dynamical potential of the long range interactions has contributions from both \( R \) and \( NS \) sectors and its form can be read off Eq.(4). If we consider for simplicity that \( Z^\mu \) has just one component different from zero then the potential has the following form

\[
V(\theta, Z) = -V_p 4(4\pi \alpha')^{3-p} \pi^{\frac{p}{2}} \frac{p}{4} \Gamma(3 - \frac{p}{2}) Z^{p-6} \frac{(1 - \cos \theta)^2}{\sin \theta}.
\]

For \( \theta \in [0, \pi] \) the potential above has an absolute maximum at \( \pi \) where it blows up. This is interpreted as an unstable brane-antibrane configuration which eventually collapses to an brane of lower dimension. The reson for that is the presence of a tachyon that is not removed from the spectrum by the GSO projection \([35]\). Let us remark that one can extend the analysis for the case when \( \theta \in [0, 2\pi] \). The boundary conditions (1) remain the same if we replace the angle \( \theta \) by \( \theta + \pi \). Therefore, the solution of the bosonic equations of motion is the same and by supersymmetry we will obtain the same spinorial solutions. However, if we plot the potential for the full interval \([0, 2\pi] \), we see that \( \pi \) is in the same time an absolute minimum of the function (see Fig.(1)).

One possible interpretation is that for small angles above \( \pi \) the system will tend to assume the brane-antibrane configuration and at this point the system decays to a stable brane of lower dimension \([35]\). At \( 0 \) and \( 2\pi \) the systems has a zero potential. This corresponds to two parallel branes between which the \( NS \) and \( R \) contributions to potential cancell each other.

The system breaks all the supersymmetries of the background for arbitrary angle between branes \([34]\). In the non-supersymmetric configurations a tachyon appears for any value of the angle between branes at distances below some critical value.
3 Dynamical potential of four branes at angles

In this section we will discuss the dynamical potential of a system of four $Dp$-branes. This potential is obtained by integrating over the amplitude of exchange massless modes of closed strings.

Let us consider a system of four $Dp$-branes that make one relative angle between any of two of them in the $(p,p+1)$ plane. We denote the four branes be $p$, $\bar{p}$, $p'$ and $\bar{p}'$, respectively. In what follows we will consider only two brane processes in closed string tree level approximation. The sectors entering in this interaction and the general configuration of the system are parametrized by three angles and three relative distances which are chosen accordingly to the Table (3).

<table>
<thead>
<tr>
<th>Pair</th>
<th>Angles</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p - p'$</td>
<td>$\phi$</td>
<td>$Y$</td>
</tr>
<tr>
<td>$p - \bar{p}$</td>
<td>$\omega$</td>
<td>$L_1$</td>
</tr>
<tr>
<td>$p - p'$</td>
<td>$\phi + \chi$</td>
<td>$Y + L_2$</td>
</tr>
<tr>
<td>$p' - \bar{p}$</td>
<td>$\phi - \omega$</td>
<td>$Y - L_1$</td>
</tr>
<tr>
<td>$p' - \bar{p}'$</td>
<td>$\chi$</td>
<td>$L_2$</td>
</tr>
<tr>
<td>$\bar{p} - \bar{p}'$</td>
<td>$\phi + \chi - \omega$</td>
<td>$Y + L_2 - L_1$</td>
</tr>
</tbody>
</table>

The strings stretching between any two branes have boundary conditions of the type (1) in the $(p,p+1)$ plane. The interactions between branes are superpositions of two brane interactions. Therefore, the total potential is just a sum of the potentials from all sectors above

$$V(\theta_i, L_i) = \sum_i V_i(\theta_i, L_i), \quad (6)$$

where $\theta_i \in \{\phi, \omega, \phi + \chi, \phi - \omega, \chi, \phi + \chi - \omega\}$ and $L_i \in \{Y, L_1, Y + L_2, Y - L_1, L_2, Y + L_2 - L_1\}$. Explicitly, the dynamical potential for long range interactions has the following form

$$V \sim \frac{(1 - \cos \phi)^2}{\sin \phi} Y^{p-6} + \frac{(1 - \cos \omega)^2}{\sin \omega} L_1^{p-6} + \frac{(1 - \cos \chi)^2}{\sin \chi} L_2^{p-6}$$
More general solutions of (8) will depend on certain values of the \( \phi \). In general, the solutions of the system (8) will depend on the parameters \( Y, L_1, L_2 \). However, as it is easy to see, there are some solutions independent of all distances. These describe configurations with arbitrary combinations of the following values of angles

\[
\phi = 0, 2\pi , \quad \omega = 0, 2\pi , \quad \chi = 0, 2\pi .
\]

(9)

The result is known and it says that parallel branes form a stable system. In these configurations the value of the potential is zero.

### 3.1 Configurations in which the potential has an extrema

In general, the solutions of the system (8) will depend on the parameters \( Y, L_1, L_2 \). However, as it is easy to see, there are some solutions independent of all distances. These describe configurations with arbitrary combinations of the following values of angles

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### 3.2 \( \phi = 0 \)

More general solutions of (8) will depend on certain values of the \( Y, L_1 \) and \( L_2 \). Due to the fact that the system above is degenerate, we investigate some particular cases.

Let us assume that \( \phi = 0 \) which implies that \( F(\phi) \) is also zero. A nontrivial solution, i.e. a solution in which not all of the remaining \( G \)'s are zero, can exist only if the following equation is satisfied

\[
(Y + L_2)^{p-6}[L_1^{p-6} - (Y - L_1)^{p-6}](Y + L_2 - L_1)^{p-6} + \\
(Y - L_1)^{p-6}[L_2^{p-6} + (Y + L_2)^{p-6}](Y + L_2 - L_1)^{p-6} + \\
(Y + L_2 - L_1)^{p-6}[L_1^{p-6} - (Y - L_1)^{p-6}][L_2^{p-6} + (Y + L_2)^{p-6}] = 0.
\]

(10)

Once (10) solved, one can find, in principle, the values of \( F \)'s for which the potential has an extrema, and from them one can deduce the angles of the configuration, if any.

An obvious solution of the equation (10) is given by \( Y = L_1 - L_2 \) (we consider only positively definite distances.) A general solution is a root of the equation of degree \( 3(p-6) \) in one distance, say \( Y \), in function of the other two. In the limit when \( L_1 \to 0, L_2 \to 0 \), the extrema can be obtained only if \( Y \to 0 \), that is if the system collapses. In the case of six branes the system is stable only if the angles \( \chi \) and \( \omega \) are simultaneously zero or \( 2\pi \).
3.3 \( \phi = 0, \chi = \omega \)

To obtain nontrivial solutions in this case one has to set all the distances to zero. The other possibility is to have the following relationship between \( F \)'s satisfied

\[
F(\chi) = F(\omega) \frac{L_1}{L_2}^{p-6},
\]

which holds only for \( p \neq 6 \). For \( p = 6 \) the solution is given by (9). In addition, the distances \( L_1 \) and \( L_2 \) should satisfy the following relation

\[
(2L_1 + L_2)^{p-6} = -L_2^{p-6},
\]

while \( Y = 2L_1 \). For four, five and eight branes the configurations are easily read off these conditions.

3.4 \( \omega = 0 \)

The system (8) is degenerate in this case, too. However, the corresponding relation

\[
L_2^{p-6}([Y + L2]^{p-6} + (Y + L_2 - L_1)^{p-6}] - L_2^{p-6}(Y + L_2 - L_1)^{p-6}[Y^{p-6} + (Y - L_1)^{p-6}] = 0
\]

admits solution for \( p = 6 \) if \( Y - L_1 \neq 0 \) and \( Y + L_2 - L_1 \neq 0 \), in contrast with the previous case (10). For any of the limits \( L_1 \rightarrow 0 \) or \( L_2 \rightarrow 0 \), the relation cancels for any values of the remaining two parameters.

3.5 \( \chi = 0 \)

The relation that resolve the degeneracy of the system is given by

\[
L_1^{p-6}(Y + L_2)^{p-6}[(Y - L_1)^{p-6} + (Y + L_2 - L_1)^{p-6}] - L_1^{p-6}(Y + L_2 - L_1)^{p-6}[Y^{p-6} + (Y + L_2)^{p-6}] = 0.
\]

In this case, there are solutions for \( p = 6 \) for any value of all parameters. In the limit where \( L_1 \rightarrow 0 \), the potential can have an extrema for any \( Y \) and \( L_1 \). At \( L_2 \rightarrow 0 \) we have solutions for arbitrary \( Y \) and \( L_1 \). If we set now the angles \( \phi = \omega \) we see that the potential can have an extrema only if the system collapses.

We may ask what happens when the potential is varied with respect to the parameters \( Y \), \( L_1 \) and \( L_2 \), respectively. Since the derivative of the potential with respect to the spatial variable is the definition of the force, we may rephrase this question by writing the condition of stability of the system in terms of forces acting on branes. In this situation, the system would be stable if each of the brane is in equilibrium, that is if all forces acting of each brane cancel each other.

This condition can be casted into the following form for \( p \neq 6 \)

\[
\begin{align*}
G(\omega) L_1^{p-5} + G(\phi) Y^{p-5} + G(\phi + \chi)(Y + L_2)^{p-5} = 0 \\
- G(\omega) L_1^{p-5} + G(\phi - \omega)(Y - L_1)^{p-5} + G(\phi + \chi - \omega)(Y + L_2 - L_1)^{p-5} = 0 \\
- G(\phi) Y^{p-5} - G(\phi - \omega)(Y - L_1)^{p-5} + G(\chi) L_2^{p-5} = 0 \\
- G(\phi + \chi)(Y + L_2)^{p-5} - G(\phi + \chi - \omega)(Y + L_2 - L_1)^{p-5} - G(\chi) L_2^{p-5} = 0,
\end{align*}
\]

5
where $G(\theta_i)$ is the term from the potential that depends on $\theta_i$. The system above is degenerate, therefore the best one can do is to express three of the terms containing distances in terms of the other three, containing the angles. Again, particular solutions can be obtained by setting some parameters to constants. For example, for the angle $\phi = 0$ and the other angles undetermined we obtain a relation between the spatial parameter

$$\frac{(Y + L_2)^{p-5}}{L_1^{p-5}} = \frac{L_2^{p-5}}{(Y - L_1)^{p-5}} = \frac{L_2^{p-5} + (Y + L_2)^{p-5}}{L_1^{p-5} + (Y - L_1)^{p-5}} = 0. \quad (16)$$

It is easy to see that for $L_2^2 + L_1^2 + 6L_1L_2 > 0$ there are acceptable solution of (16) which give the expression of $Y$ in terms of $L_1$ and $L_2$.

In a similar way we can discuss the configurations at fixed angles $\omega = 0$ and $\chi = 0$. The corresponding relations are given by

$$\frac{(Y + L_2)^{p-5}}{Y^{p-5}} = \frac{(Y + L_2 - L_1)^{p-5}}{(Y - L_1)^{p-5}} = -\frac{(Y + L_2 - L_1)^{p-5} + (Y + L_2)^{p-5}}{Y^{p-5} + (Y - L_1)^{p-5}} \quad (17)$$

and

$$\frac{(Y - L_1)^{p-5}}{Y^{p-5}} = \frac{(Y + L_2 - L_1)^{p-5}}{(Y + L_2)^{p-5}} = \frac{(Y + L_2 - L_1)^{p-5} + (Y - L_1)^{p-5}}{Y^{p-5} + (Y + L_2)^{p-5}}, \quad (18)$$

respectively. We consider in all relations above that the denominators do not vanish. The (16), (17) and (18) represent necessary conditions for the stability of the system. However, they should not be compatible to each other since they were established for different values of angles.

### 4 Brane-antibrane pairs

For each relative angle that equals $\pi$ there is an antibrane in the system. The non-equivalent configurations of branes-antibranes are the ones containing one or two antibranes. For three antibranes the system is equivalent with a system with one antibrane.

One antibrane can be obtained by setting $\omega = 0$. If $\phi$ and $\chi$ are left arbitrary, the system will contain one brane-antibrane pair and two branes at arbitrary angle. We assume that these angles do not equal $\pi$. The potential $V(\phi, \omega = \pi, \chi)$ for any finite distances between branes is as in Fig.(1). However, for certain behaviour of the parameters $\omega$ and $L_1$ the potential is finite as $\omega \to \pi$ and $L_1 \to 0$. To see this, we assume that $L_1$ and $\omega$ vary towards 0 and $\pi$, respectively, with the same parameter $t$ which we pick up to be between $[0, 1]$. The dependence of the two parameters on $t$ should be $L_1 = tL_0$ and $\omega = (1-t)\pi$ where $L_0$ is a constant. Then for $p = 7, 8, 9$ the potential has a finite value at $t = 0$ as shown in Fig.(2),(3) and (4). The typical behaviour of $p \leq 6$ branes is illustrated in the Fig. (5). In plotting the potentials above, the distance $L_0$ was chosen positive and greater than one. A special case is obtained when $\phi = \chi = 0$. The configuration described in this case is that of three parallel branes and one antibrane among them. Thus there are three brane-antibrane pairs. The potential blows up in the neighbourhood of $\pi$ for any values of the parameters $Y$, $L_1$ and $L_2$ least the following relation is satisfied

$$(Y + L_2 - L_1)^{p-6} + (Y - L_1)^{p-6} - L_1^{p-6} = 0 \quad (19)$$
Figure 2: The behaviour of the dynamical potential of $D7$-branes with one brane-antibrane pair in the interval $t \in [0, 1]$ where $\omega = (1 - t)\pi$ and $L_1 = tL_0$.

Figure 3: The behaviour of the dynamical potential of $D8$-branes with one brane-antibrane pair in the interval $t \in [0, 1]$ where $\omega = (1 - t)\pi$ and $L_1 = tL_0$.

Figure 4: The behaviour of the dynamical potential of $D9$-branes with one brane-antibrane pair in the interval $t \in [0, 1]$ where $\omega = (1 - t)\pi$ and $L_1 = tL_0$. 
Figure 5: The typical behaviour of the dynamical potential of $Dp$-branes, $p \leq 6$, with one brane-antibrane pair in the interval $t \in [0, 1]$ where $\omega = (1 - t)\pi$ and $L_1 = tL_0$.

for which its value is undetermined. We notice that when the brane-antibrane are one a top of the other, the potential actually continues to be infinite. If the two branes are on the top of the other, there are two real solutions for $Y$ in terms of $L_1$: $Y_{1,2} = (2 \pm \sqrt{3})L_1$.

A configuration of two brane-antibrane pairs is obtained when two relative angles are set to $\pi$. If we choose $\omega = \chi = \pi$ and leave $\phi$ arbitrary, we see that, as in the previous case, the potential will diverge in this configuration due to terms of the form $(1 + 1)^2/0$. The only possibility of making these terms finite is when their coefficients go to zero as the angles go to $\pi$. This implies that the following equation should be satisfied by the relative distances between branes

$$L_1^{p-6} + L_2^{p-6} + (Y + L_2)^{p-6} + (Y - L_1)^{p-6} = 0. \quad (20)$$

Again, branes of different dimensionality will allow different solutions for Eq.(20). For $p = 6$ there is no possibility of getting a finite potential, while for $p = 7$ one should set $Y = -L_2$ for any value of $L_1$.

5 Effects of tachyons

In the previous sections we have analysed the configurations of four $Dp$-branes that make relative arbitrary angles and are spaced by relative arbitrary distances and we discussed some general configurations given by the extrema of the dynamical potential. However, it is known that for arbitrary values of angles and distances, a pair of $Dp$-branes is in a non-supersymmetric configuration [34]. In this situation, since the $NS$ ground state of the system depends on the relative distance and angle between the branes its mass is given by [34]

$$\alpha' m^2 = \frac{Z^2}{4\pi^2\alpha'} + \frac{\theta}{2\pi}. \quad (21)$$

Thus, for any given angle $\theta \in [0, 2\pi]$ there is a critical distance under which the ground state is a tachyon. The tachyon potential will affect the stability of the system. As was conjectured
in [35] the system will evolve until the potential will reach to a minimum. For a brane anti-brane system, one on the top of the other, the minimum will be reached in a configuration that represents a $Dp$-brane of lower dimensionality.

In the case of four $Dp$-branes at angles, the same reasoning applies to all pairs and six tachyons may appear if the relative distances among branes are smaller than the critical values. For $0 \leq \theta_i \leq \pi$ these are given by the following relations:

$$L_{ic}^2 = (2\pi\alpha')_1 \theta_i. \quad (22)$$

The equations above impose lower bound limits for the validity of the results obtained in the previous sections. If any of $L_i^2 < L_{ic}^2$ then a tachyon appears in the corresponding sector and the dynamics will be determined by the tachyon potential. However, the tachyon is an off-shell degree of freedom of string, and therefore its dynamics cannot be described by a first quantized string theory. If some of the relative angles $\theta_a = \pi$ and if the interaction among the tachyons is neglected, the tachyon potential is given by [36]

$$V(T_a) = \sum_a e^{-\frac{1}{4} T_a^2} = 1 - \sum_a \frac{1}{4} T_a^2 + \cdots, \quad (23)$$

and the tachyons condensate at $T_a \to \infty$ leaving behind lower dimensional branes in that sectors. However, the value of the tachyonic potential is not known for general angles. We hope to be able to say more on this topic in a future work [39].

6 Discussions

From the configurations analysed above, we can see that the extremum of the interacting potential between four branes at angles can be expressed as a condition between relative angles and distances among branes. For fixed angles, the condition reduced to some relations between distances, while for fixed distances, the extremum of the potential implies some relations between functions of cosine of angles. In general the angular factors $F(\theta_i)$ and $G(\theta_i)$ cannot be determined uniquely from the degenerate systems in which they enter, but two of them can be determined as functions of the third one. This leads to some algebraic equations which ranks depend on the dimensionality of the brane. Somewhat special are the cases when the angular part blows up and the spatial part goes to zero. If one assumes some relation between the way in which the two of them vary towards these limits, this implies for $p = 7, 8, 9$ some configurations in which the potential can have an extremum, while for lower branes the potential continues to have an infinite value. Nevertheless, when talking about stable configuration, we see that the situation is different. For example if one consider a system that contains one antibrane and three parallel branes, the antibrane might be in equilibrium due to the fact that the forces on it from various branes may compensate each other, but its force on any brane cannot be compensated. Consequently, there is a collapse due to the presence of three tachyonic fields in the system which in general interact among each other. If the tachyons are considered as non-interacting fields, their potential is given by Eq.(23). However, in general case, its form is unknown. Due to the fact that the tachyons are off-shell degrees of freedom, this potential should be obtained from an open string field theory [36, 37, 38].
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