On the supersymmetric non-abelian Born-Infeld action

E. A. Bergshoeff\textsuperscript{1}, M. de Roo\textsuperscript{1} and A. Sevrin\textsuperscript{2}

\textsuperscript{1} Institute for Theoretical Physics  
Nijenborgh 4, 9747 AG Groningen  
The Netherlands  
\textsuperscript{2} Theoretische Natuurkunde  
Vrije Universiteit Brussel  
Pleinlaan 2  
B-1050 Brussels, Belgium

Abstract: We review an iterative construction of the supersymmetric non-abelian Born-Infeld action. We obtain the action through second order in the fieldstrength. Kappa-invariance fixes the orderings which turn out to deviate from the symmetrized trace proposal.

1 Introduction

On a D\textsubscript{p}-brane there are eight propagating fermionic and eight propagating bosonic world-volume degrees of freedom. In the static gauge, the bosonic ones appear as a $U(1)$ vector field in $p+1$ dimensions and $9-p$ scalar fields. The former describe open strings attached to the brane while the latter describe the transversal positions of the brane. The effective action for slowly varying fields is known through all orders in $\alpha'$, it is the ten-dimensional $U(1)$ Born-Infeld theory dimensionally reduced to $p+1$ dimensions \cite{1}, \cite{2}. Its fully covariant supersymmetric generalization is known as well \cite{3}, \cite{4}, \cite{5}. It contains\textsuperscript{1} the embedding coordinates $X^\mu(\sigma)$ (of which, because of the worldvolume reparametrization invariance, only the transverse coordinates are physical degrees of freedom), a vector field $V_i(\sigma)$, and an $N=2$ spacetime fermionic field $\theta(\sigma)$. Because of the presence of a local fermionic symmetry, called $\kappa$-symmetry, half of the fermionic fields can be gauged away. This leaves, in the static gauge, the fieldcontent of a $U(1)$ supersymmetric Yang-Mills theory in $p+1$ dimensions which indeed describes $8+8$ degrees of freedom.

Once several, say $n$, D\textsubscript{p}-branes coincide, additional massless states appear enhancing the $U(1)^n$ gauge symmetry to a full $U(n)$ gauge symmetry \cite{6}. Both the fermions and the scalars transform now in the adjoint representation of $U(n)$. Already at the purely bosonic level there are difficulties in defining a non-abelian generalization of the Born-Infeld action. They arise from the fact that the notion of acceleration terms becomes

\textsuperscript{1}We denote by $\mu, \nu = 0,\ldots,9$ the spacetime indices and by $i, j = 0,\ldots,p$ the worldvolume coordinates $\sigma^i$. 
ambiguous as can be seen from

\[ D_i D_j F_{kl} = \frac{1}{2} \{ D_i, D_j \} F_{kl} - \frac{i}{2} [ F_{ij}, F_{kl} ]. \]  

(1)

Based on the results of a direct calculation of the action through order \( F^4 \) [7], [8], and assuming all terms proportional to anti-symmetrized products of fieldstrengths to be acceleration terms which are then ignored, a proposal was formulated for the non-abelian action [9]. The action assumes a form similar to the abelian case but, upon expanding it in powers of the fieldstrength, one first symmetrizes over all fieldstrengths and subsequently one performs the group theoretical trace. Other possibilities are discussed in [10]. However, by calculating the mass spectrum from the effective action in the presence of certain background fields and by comparing it to the spectrum predicted by string theory, it can be shown that the symmetrized trace proposal is flawed from order \( F^6 \) on [11], [12].

As a direct calculation of the effective action at higher orders in \( \alpha' \) seems out of reach, different approaches are called for. One possibility would be to use the mass spectrum as a guideline [13]. In the present paper we explore a suggestion in [12] and use \( \kappa \)-symmetry to fix the effective action. Full technical details can be found in [14].

2 The abelian case

In order to avoid additional complications coming from the presence of transversal coordinates, we focus throughout this paper on D9-branes. Lower dimensional branes can then be studied upon performing a suitable T-duality transformation [15]. The \( \kappa \)-symmetric lagrangian has the schematic form

\[ \mathcal{L} = - e^{-\phi} \sqrt{-\det (g + \mathcal{F})} + C e^{\mathcal{F}}, \]  

(2)

with \( \mathcal{F} = 2dV + B \). The first term is the Born-Infeld lagrangian \( \mathcal{L}_{BI} \) while the second is the Wess-Zumino term \( \mathcal{L}_{WZ} \). Both the NS-NS background fields \( g, \phi, B \) and all the R-R background fields \( C \) are superfields, i.e., they are functions of the superspace coordinates \( (X^\mu, \theta) \). The \( \kappa \)-symmetry acts on the fermions as

\[ \delta \bar{\theta}(\sigma) = \bar{\eta}(\sigma) \equiv \bar{\kappa}(\sigma) (1 + \Gamma), \]  

(3)

where \( \Gamma \) depends on worldvolume and background fields. The variation of the D-brane action is then

\[ \delta \mathcal{L} = - \bar{\eta} (1 - \Gamma) \mathcal{T}, \]  

(4)

where \( \mathcal{T} \) is some expression in terms of the worldvolume and background fields. Invariance is obtained provided \( \Gamma \) satisfies

\[ \Gamma^2 = 1. \]  

(5)

As can be shown by combining eqs. (3) and (5), and by using the fact that \( \text{tr} \Gamma = 0 \), we can use the \( \kappa \)-gauge invariance to eliminate half of the fermions. Working in a flat gravitational background and taking all other bulk backgroundfields to be zero, we can use the \( \kappa \)-symmetry to put

\[ \theta = \left( \begin{array}{c} \theta_1 \equiv \chi \\ \theta_2 = 0 \end{array} \right). \]  

(6)
Fixing the worldvolume reparametrizations by taking \( X^\mu = \delta^{i\mu} \sigma^i \), one finds that the complete Wess-Zumino term vanishes and the Born-Infeld lagrangian is given by [3]

\[
L_BI = - \sqrt{-\det (\eta_{\mu\nu} + F_{\mu\nu} + \bar{\chi} \Gamma_\mu \partial_\nu \chi + \frac{1}{4} \bar{\chi} \Gamma^a \partial_\nu \chi \bar{\chi} \Gamma_a \partial_\nu \chi)}.
\]  

(7)

This lagrangian has 16 linear and 16 nonlinear supersymmetries which are obtained from the original 32 supersymmetries of the N=2 superspace after implementing the fact that they get deformed with a field-dependent kappa-transformation upon fixing the kappa-gauge.

Taking a closer look at the origin of various terms in eq. (4), one finds that the \( \bar{\eta} \Gamma T \) term arises from the variation of the Wess-Zumino lagrangian \( L_{WZ} \), while the \( -\bar{\eta} T \) originates from the variation of the Born-Infeld lagrangian \( L_{BI} \). Expanding both \( \Gamma \) and \( T \) in powers of the fieldstrength \( F \),

\[
\Gamma = \sum_{i \geq 0} \Gamma_i, \quad T = \sum_{i \geq 0} T_i,
\]

(8)

we get

\[
\delta L_{WZ} = \bar{\eta} \Gamma_0 T_0 + \bar{\eta} \Gamma_0 T_1 + \bar{\eta} \Gamma_1 T_0 + \ldots,
\]

\[
\delta L_{BI} = -\bar{\eta} T_0 - \bar{\eta} T_1 + \ldots,
\]

(9)

(10)

where eq. (5) implies additional restrictions

\[
(\Gamma_0)^2 = 1, \quad \{\Gamma_0, \Gamma_1\} = 0, \ldots
\]

(11)

This structure suggests an iterative procedure for obtaining the D-brane action. Because of its topological nature, the form of the Wess-Zumino term is severely constrained. It is itself given as an expansion in powers of the fieldstrength \( F \). Varying the term independent of \( F \) in it gives the first term in eq. (9). Combining this with the first relation in eq. (11), both \( \Gamma_0 \) and \( T_0 \) get identified. Integrating \( T_0 \) then yields the lowest order in \( F \) of the Born-Infeld lagrangian \( L_{BI} \). Proceeding like this order by order in the fieldstrength \( F \), one fixes ambiguities in \( L_{WZ} \) and one constructs the Born-Infeld lagrangian \( L_{BI} \).

In the next section, we will adopt this strategy in order to obtain the non-abelian Born-Infeld lagrangian.

3 The non-abelian case

When constructing the supersymmetric D9-brane action, one can aim for several goals. In order of increasing ambition they are given by:

1. The construction of the supersymmetric non-abelian Born-Infeld action with trivial IIB supergravity backgrounds in the static gauge. Besides the vector fields \( V_i(\sigma) \), there is an \( N = 1 \) spinor field \( \chi \) in the adjoint of \( U(n) \).

2. Repeat step 1, but now with a manifest worldvolume reparametrization invariance. In addition to the fields listed above, we have now the embedding coordinates \( X^\mu(\sigma) \) as well.

3. Repeat step 1, but making the action invariant under \( \kappa \)-symmetry. Instead of the \( N = 1 \) spinor \( \chi \) we get now an \( N = 2 \) spinor \( \theta \) transforming in the adjoint of \( U(n) \).

4. Combine the programmes listed under 2 and 3.
5. Repeat the previous programme in a background of non-trivial IIB supergravity bulkfields.

Some initial steps towards achieving point 1 were made in [16] in four dimensions. In step 2 one has to decide whether the worldvolume embedding coordinates are singlets under $U(n)$ or whether they transform, in analogy with transversal coordinates, in the adjoint representation. The analysis in [14] suggests that only the latter option is possibly consistent. However then another fundamental problem shows up. In order to be able to reach the static gauge, the structure of the worldvolume has to be adapted such as to obtain a sufficiently large reparametrization group. In order to proceed we opted for the programme in step 3 as steps 4 and 5 are presently out of reach.

We introduce fields $\theta^A(\sigma)$, which are an $(N = 2)$ doublet of Majorana-Weyl spinors for each $A$, satisfying $\Gamma_{11}\theta^A = \theta^A$. They transform as follows under supersymmetry ($\epsilon$), $\kappa$-symmetry ($\kappa$) and Yang-Mills transformations ($\Lambda^A$):

$$
\delta \tilde{\theta}^A(\sigma) = -\epsilon^A + \kappa^B(\sigma)(1\delta^{BA} + \Gamma^{BA}(\sigma)) + f_{AB}C\Lambda^B(\sigma)\tilde{\theta}^C(\sigma). 
$$

(12)

Here $\epsilon^A$ are constant, $\Gamma^{AB}$ depends on the worldvolume fields and it must satisfy

$$
\Gamma^{AB}\Gamma^{BC} = \delta^{AC}1. 
$$

(13)

From now on we use $\tilde{\eta}^A \equiv \kappa^B(\sigma)(1\delta^{BA} + \Gamma^{BA}(\sigma))$. Because $\epsilon^A$ is constant we find from the commutator of Yang-Mills and supersymmetry transformations that $f_{AB}C\epsilon^C = 0$. Therefore $\epsilon = \epsilon^AT_A$ must be proportional to the unit matrix, i.e., there is only one nonvanishing $\epsilon$ parameter. Only after $\kappa$-gauge fixing will all $\theta$’s transform under supersymmetry. The commutator of $\kappa$-symmetry and supersymmetry teaches us that $\Gamma^{AB}$ is a supersymmetry invariant.

We have now the tools at hand to start the programme outlined at the end of the previous section. We present the results and refer to [14] for details of the calculation. Through second order in $F$, the lagrangian is $L = L_{WZ} + L_{BI}$, with the Wess-Zumino term given by

$$
L_{WZ} = \epsilon^{i_1\cdots i_{10}}\left\{ \frac{1}{2\cdot 9!}\theta^A\sigma_1\gamma_{i_1\cdots i_9}D_{i_{10}}\theta^A 
- \frac{1}{4\cdot 7!}\bar{\theta}^A\mathcal{P}_{(1)}^{ABC}\gamma_{i_1\cdots i_5}D_{i_6}\theta^B F_{C}^{i_7i_{10}} 
+ \frac{1}{16\cdot 5!}\bar{\theta}^A(-\sigma_1S^{ABCD})\gamma_{i_1\cdots i_5}D_{i_6}\theta^B(F_C^D F^D)_{i_7\cdots i_{10}}\right\}, 
$$

(14)

and the Born-Infeld lagrangian by,

$$
L_{BI} = -\left\{ 1 + \frac{1}{2}\bar{\theta}^A \gamma^j D_j \theta^A - \frac{1}{2}\bar{\theta}^A \sigma_1 \mathcal{P}_{(1)}^{ABC} \gamma_{[i} D_{j]} \theta^B F^{ij}C 
+ \frac{1}{4}F^{ij}A F_{ij} - \frac{1}{2}\bar{\theta}^A S_{ABCD}\gamma_{[i}(D_{j]}\theta^B\{F_{ik}^C F^j_{k}^D + \frac{1}{2}\eta_{ij}F_{kl}^C F^{kl}D]\} 
+ \frac{1}{4}\bar{\theta}^A A^{ABCD}\gamma_{ijk}\{D^k\theta^B F^{il}C F^j_{l}^D - D_l\theta^B F^{ij}C F^{kl}D\}\right\}, 
$$

(15)

where,

$$
\mathcal{P}_{(1)}^{ABC} = (i\sigma_2) d^{ABC},
$$

(16)

$^2$The $U(n)$ generators $T_A$, $A \in \{1, \cdots, n^2\}$, are Hermitian $n \times n$ matrices normalized as $tr T_A T_B = \delta_{AB}$. The product of two $U(n)$ generators is given by $T_A T_B = (d_{ABC} + i f_{ABC}) T_C$, where $d$ and $f$ are symmetric and antisymmetric in $AB$, respectively.
\[ S_{ABCD} = P_{(1)}^{AE(C} P_{(1)}^{BD)E} = -d^{AE(C} d^{BD)E}, \]
\[ A_{ABCD} = P_{(1)}^{AE(C} P_{(1)}^{BD)E} = -d^{AE[C} d^{BD]E}. \]

The global supersymmetry and local \( \kappa \)-transformations are given by,
\[ \delta \bar{\theta}^A = -\bar{\epsilon}^A + \bar{\eta}^A, \]
\[ \delta V_i^A = \frac{1}{2} (\bar{\epsilon}^B + \bar{\eta}^B) \sigma_1 P_{BCA} \gamma_i \delta^C + \frac{1}{2} (\bar{\epsilon}^B + \bar{\eta}^B) S_{BCDA} \gamma_k \delta^C F^{klD} + \frac{1}{4} (\bar{\epsilon}^B + \bar{\eta}^B) A_{BCDA} \gamma_{ikl} \delta^C F^{klD}, \]
where as mentioned before, \( \epsilon^A \) satisfies \( f_{ABC} \epsilon^C = 0 \). Finally, \( \Gamma^{AB} \) which was introduced in eq. (12) is given by,
\[ \Gamma^{AB} = \Gamma^{(0)} \left\{ \sigma_1 \delta \bar{\theta}^A + P_{(1)}^{ABC} \frac{1}{2} \gamma^{kl} F_{kl}^C - \sigma_1 S_{ABCD} (\frac{1}{8} \gamma_{ijkl} F_{ijkl}^D - \frac{1}{4} f_{kl}^C F^{klD}) - \sigma_1 A_{ABCD} \frac{1}{2} \gamma_{ij} F_{ijkl}^C F_{i}^D \right\}. \]

We proceed with fixing the \( \kappa \)-symmetry. Writing out the \( N = 2 \) doublets explicitly,
\[ \Gamma = \begin{pmatrix} 0 & \gamma \\ \bar{\gamma} & 0 \end{pmatrix}, \]
we find that eq. (13) implies \( \gamma \bar{\gamma} = \bar{\gamma} \gamma = 1 \). Here \( \gamma, \bar{\gamma} \) are \( 32 \times 32 \) matrices, with in addition indices \( AB \), where \( A, B \) run from 1 to \( n^2 \). Separating the fermions into \( N = 1 \) fermions we get for eq. (19),
\[ \delta \bar{\theta}_1^A = -\bar{\epsilon}_1^A + \bar{\eta}_1^A, \quad \delta \bar{\theta}_2^A = -\bar{\epsilon}_2^A + \bar{\eta}_2^A, \]
where using the relation between \( \eta \) and \( \kappa \) we get,
\[ \bar{\eta} = (\bar{\eta}_1 \quad \bar{\eta}_2) = (\bar{\kappa}_1 + \bar{\kappa}_2 \bar{\gamma} \quad \bar{\kappa}_2 + \bar{\kappa}_1 \gamma). \]
Using the \( \kappa \)-symmetry, we can put \( \bar{\theta}_2 = 0 \). As a consequence \( \kappa_2 \) is fixed, \( \bar{\kappa}_2 = \bar{\epsilon}_2 - \bar{\kappa}_1 \gamma \).

Combining this with eqs. (23) and (21), we obtain the supersymmetry transformations of the fermions,
\[ \delta \bar{\chi}^A = -\bar{\epsilon}_1^A - \bar{\epsilon}_2^A + \bar{\epsilon}_2^B \left\{ d^{BAC} \frac{1}{2} \gamma^{kl} F_{kl}^C + S_{BACD} (\frac{1}{8} \gamma_{ijkl} F_{ijkl}^D - \frac{1}{4} f_{kl}^C F^{klD}) \right\}, \]
where we called \( \chi^A \equiv \theta_1^A \). Implementing the gauge choice in eq. (20), we get the transformation rules for the gauge fields as well,
\[ \delta V_i^A = \frac{1}{2} (\bar{\epsilon}_1^B - \bar{\epsilon}_2^B) d^{BCA} \gamma_{i} \chi^C - \frac{1}{2} \bar{\epsilon}^B d^{BED} d^{ECA} \gamma_{kl} \chi^C F^{klD} + \frac{1}{4} (\bar{\epsilon}_1^B - \bar{\epsilon}_2^B) S_{BCDA} \gamma_{ikl} \chi^C F^{klD}. \]

After gauge fixing, the Wess-Zumino term vanishes since it was off-diagonal in the fermions \( \theta_1 \) and \( \theta_2 \). The Born-Infeld term is given by
\[ \mathcal{L}_{BI} = -\left\{ 1 + \frac{1}{2} \bar{\chi}^A \gamma^i D_i \chi^A + \frac{1}{2} d_{ABC} \bar{\chi}^A \gamma_i D_{ij} \chi^B F^{ijC} + \frac{1}{4} f_{ij}^A F_{ij}^A \right\} + \frac{1}{2} d^{AE} d^{BDE} \bar{\chi}^A \gamma^i (D_{ij}) \chi^B \left\{ F^{i}^j C F_{ikl}^D + \frac{1}{2} \eta_{ij} F_{kl}^C F^{klD} \right\} - \frac{1}{4} d^{AE[C} d^{BD]E} \bar{\chi}^A \gamma_{ijk} \left\{ D^k \chi^B F^{i}^l C F_{i}^j D - D_i \chi^B F^{ijC} F^{klD} \right\}. \]
It is clear that the terms of the form $\bar{\chi} \partial \chi F^2$ are not symmetric traces of $U(n)$ generators. The symmetric trace is given by,

$$\text{tr} T_{(A T_B T_C T_D)} = \frac{1}{3} (d_{ABE}d_{CDE} + d_{CAE}d_{BDE} + d_{BCE}d_{ADE}),$$  

while the second line in (27) contains only two of the three contributions needed for the symmetric trace, the last line contains explicit anti-symmetrizations and can be rewritten in terms of structure constants,

$$d_{AEC}d_{BDE} - d_{AED}d_{BCE} = f_{ABE}f_{CDE}.$$  

4 Conclusions

In this paper we have obtained the non-abelian generalization of the Born-Infeld action up to terms quartic in the Yang-Mills field strength, and including all fermion bilinear terms up to terms cubic in the field strength. The terms of the form $\bar{\chi} \partial \chi F^2$ deviate from the symmetric trace conjecture. The precise structure of the non-abelian Born-Infeld action remains an enigma. One clue is provided by the fact that in the abelian case $\Gamma$ factorizes into a part that is polynomial in $F$, and the inverse of the Born-Infeld action, which expands to an infinite series in $F$. While such a factorization will be more complicated in the non-abelian case [14], we need to pursue this programme to higher order in the fieldstrength [17] in order to see some pattern appearing. In addition, having the supersymmetric Born-Infeld at higher order, would allow us to study non-abelian BPS states.

The simplest of non-abelian BPS configurations arises as follows [18]. Taking two $D_p$-branes in the $(2, 4, \cdots, 2p)$ directions, we keep one of them fixed and rotate the other one subsequently over an angle $\phi_1$ in the $(23)$ plane, over an angle $\phi_2$ in the $(45)$ plane, ..., over an angle $\phi_p$ in the $(2p 2p+1)$ plane. The following table summarizes for various values of $p$ the BPS conditions on the angles (which are different from zero) and the number of remaining supercharges.

<table>
<thead>
<tr>
<th>$p$</th>
<th>BPS condition</th>
<th>susy’s</th>
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<tbody>
<tr>
<td>2</td>
<td>$\phi_1 = \phi_2$</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>$\phi_1 = \phi_2 + \phi_3$</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>$\phi_1 = \phi_2 + \phi_3 + \phi_4$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$\phi_1 = \phi_2, \phi_3 = \phi_4$</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>$\phi_1 = \phi_2 = \phi_3 = \phi_4$</td>
<td>6</td>
</tr>
</tbody>
</table>

T-dualizing in the $3, 5, \cdots, 2p + 1$ directions yields two coinciding D2$p$ branes with magnetic fields, $F_{2i 2i+1}, i \in \{1, \cdots, p\}$, turned on. In the simplest case we have $F_{2i 2i+1} \equiv f_i \sigma_3$, with $f_i$ constant. The relation between magnetic fields and angles is $\tan(\phi_i/2) = 2\pi \alpha’ f_i$. Translating the BPS conditions on the angles in conditions on the fieldstrengths, we get,

<table>
<thead>
<tr>
<th>$p$</th>
<th>BPS condition</th>
<th>fieldstrengths</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\phi_1 = \phi_2$</td>
<td>$f_1 = f_2$</td>
</tr>
<tr>
<td>3</td>
<td>$\phi_1 = \phi_2 + \phi_3$</td>
<td>$f_1 = f_2 + \frac{f_3}{f_1} f_1 f_2 f_3$</td>
</tr>
<tr>
<td>4</td>
<td>$\phi_1 = \phi_2 + \phi_3 + \phi_4$</td>
<td>$f_1 = f_2 + f_3 + f_4 + (2\pi \alpha’)^2 (f_1 f_2 f_3 f_4 + f_1 f_2 f_4 - f_2 f_3 f_4)$</td>
</tr>
<tr>
<td></td>
<td>$\phi_1 = \phi_2, \phi_3 = \phi_4$</td>
<td>$f_1 = f_2, f_3 = f_4$</td>
</tr>
<tr>
<td></td>
<td>$\phi_1 = \phi_2 = \phi_3 = \phi_4$</td>
<td>$f_1 = f_2 = f_3 = f_4$</td>
</tr>
</tbody>
</table>
In several cases we get $\alpha^{'2m}$ corrections. Therefore we will have to go at least to order $F^3$ in the supersymmetry transformation rules in order to be able to compare our results to these predictions. In particular we will then also be able to analyze non-diagonal BPS configurations. As shown in [19], the knowledge of $\Gamma$ is sufficient to elegantly perform this analysis.

Another intriguing point is the apparent incompatibility between $\kappa$-symmetry and worldvolume reparametrisation invariance. The analysis in [14] suggests that also the worldvolume embedding coordinates transform in the adjoint of $U(n)$. Needless to say a better understanding of this would have profound implications in the understanding of D-brane geometry and might facilitate the coupling to curved backgrounds.

Finally, it would be interesting to investigate whether the superembedding techniques developed in [20] or the analysis of [22] can be applied to the problem at hand.

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