We discuss QCD with two light flavors at large baryon chemical potential $\mu$. Color superconductivity leads to partial breaking of the color SU(3) group. We show that the infrared physics is governed by the gluodynamics of the remaining SU(2) group with an exponentially soft confinement scale $\Lambda_{\text{QCD}} \sim \Delta \exp(-a\mu/(g\Delta))$, where $\Delta \ll \mu$ is the superconducting gap, $g$ is the strong coupling, and $a = 2\sqrt{2}\pi/11$. We estimate that at moderate baryon densities $\Lambda_{\text{QCD}}$ is $\mathcal{O}(10\text{ MeV})$ or smaller. The confinement radius increases exponentially with density, leading to “asymptotic deconfinement.” The velocity of the SU(2) gluons is small due to the large dielectric constant of the medium.

Soon after the discovery of asymptotic freedom in QCD [1] a hypothesis has been put forward that at high baryon densities quarks, which are normally confined in hadrons by strong forces, are liberated, i.e., nuclear matter transforms into deconfined quark matter [2]. In recent years, our knowledge of dense quark matter has considerably expanded. We now understand that in reality dense matter shows more intricate features than in the original picture of [2]. In particular, quark matter at high densities exhibits the phenomenon of color superconductivity [3,4], which determines the symmetry of the ground state and the infrared dynamics.

The number of relevant light quark flavors $N_f$ turns out to play a crucial role. In most applications, e.g. at densities characteristic of neutron stars, $N_f = 2$ (up and down quarks). The following picture emerges in perturbation theory, as well as in instanton-inspired models. The condensation of color antitriplet up-down diquarks breaks the color SU(3) down to an SU(2) subgroup. As a result, five of the original eight gluons acquire “masses” by the Meissner effect [4,5], similar to the Higgs mechanism. The remaining three gluons are massless in perturbation theory. Due to Cooper pairing, the spectrum of quark degrees of freedom carrying nontrivial SU(2) color charge has a gap $\Delta$. At small coupling (large densities), this gap is exponentially smaller than $\mu$. In order to understand the physics below the energy scale $\Delta$ we must examine the pure gluodynamics in the remaining unbroken SU(2) sector. As we shall see, the process of high-density “deconfinement” is quite nontrivial in this case.

Below the scale $\Delta$, we expect that the heavy degrees of freedom decouple and the remaining fields can be described by a local effective Lagrangian. At first sight, it might seem that this effective Lagrangian is simply $L = -\frac{i}{4} F_{\mu\nu}^a$, i.e., the SU(2) Yang-Mills Lagrangian with the coupling matching the running coupling in the original theory at the scale $\Delta$. However, a closer look shows that the situation is somewhat more complicated, and in fact more interesting.

First, the absence of color degrees of freedom below the scale $\Delta$ implies that the medium is transparent with respect to the gluons. In particular, there is no Debye screening and Meissner effect for the SU(2) gluons. Mathematically, the polarization tensor $\Pi^{\mu\nu}(q)$ vanishes at $q = 0$. This fact can be checked by a direct calculation of $\Pi$ at small $q$, as was done in Ref. [5]. The absence of Debye screening means that a static color charge inserted into the medium cannot be completely screened as it is in hot plasmas. This is easy to understand since all quarks carrying SU(2) color are bound into Cooper pairs, which are SU(2) singlets. Analogously, the absence of the Meissner effect is also a consequence of the neutrality of the Cooper pairs: superconducting currents, which are coherent motions of the condensate, cannot screen the magnetic field, since the condensate is SU(2) neutral.

However, although a static SU(2) charge cannot be completely Debye screened by SU(2) neutral Cooper pairs, it can still be partially screened if the medium is polarizable, i.e., if it has a dielectric constant $\epsilon$ different from unity. If $\epsilon > 1$, the Coulomb potential between two static color charges is $g^2/(er)$, i.e., the gauge coupling is effectively reduced by a factor of $\epsilon^{1/2}$. As explained in more detail below, this is exactly the situation occurring in the color-superconducting phase. Analogously the medium can, in principle, have a magnetic permeability $\lambda \neq 1$. (We denote the permeability by $\lambda$ instead of the more standard $\mu$, since the latter symbol is already used for the chemical potential.) The dynamics of gluons is thus modified by the dielectric constant and the magnetic permeability of the medium. Hence, one needs to develop a theory of “gluodynamics of continuous media”, which, as far as we know, has never been encountered before. This theory, in contrast to its U(1) counterpart (the electrodynamics of continuous media), is an interacting theory.
Fortunately, even without explicit calculation we can already write down the effective Lagrangian of SU(2) gluons from rather general arguments. It should satisfy the requirements of locality (since the quarks that have been integrated out have gaps) and gauge invariance. It does not need to be Lorentz invariant, since this invariance is already violated by the presence of the high-density medium, but it should be rotationally invariant and conserve parity. Thus the effective action must have the following form

$$S_{\text{eff}} = \int d^4x \left( \frac{\kappa}{2} E^a \cdot E^a - \frac{1}{2\lambda} B^a \cdot B^a \right),$$  

where $E_i^a \equiv F_0^a$ and $B_i^a \equiv \frac{1}{2} \epsilon_{ijk} F_j^a k^b$. Higher-order corrections (in powers of fields and derivatives) to (1) are irrelevant for the infrared physics and have been neglected. The constants $\epsilon$ and $\lambda$ in Eq. (1) have the meaning of the dielectric constant and the magnetic permeability in the regime where the gluon fields are linear. In particular, $v = 1/\sqrt{\lambda}$ is the speed of gluons in this regime.

To find $\epsilon$ and $\lambda$ one has to calculate (1) by integrating out the quark degrees of freedom in the QCD Lagrangian. This amounts to computing the one-loop polarization operator $\Pi(q)$ and the gluon vertices $\Gamma_s(q_1, q_2)$, $\Gamma_s(q_1, q_2, q_3)$, etc. This procedure is the same as the one giving rise to the hard thermal loop (HTL) and hard dense loop (HDL) effective actions [6]. The situation here is simpler than in the HTL and HDL cases: in the regime where all gluon momenta $q$ are much smaller than $\Delta$, the functions $\Pi$ and $\Gamma_s$ can be expanded in powers of $q$, yielding a local effective Lagrangian. (In contrast, the HTL and HDL actions are non-local, since the fermions do not have gaps.) The gauge invariance of the effective Lagrangian greatly simplifies our task: in order to know $\epsilon$ and $\lambda$ one needs to compute only the polarization tensor $\Pi$ of the SU(2) gluons. The leading contribution at large density comes from the superconducting quark loop. The calculation of the polarization tensor was done in Ref. [5]; from Eq. (99a) of that paper one can derive the following expression for $\Pi^{ab}_{\Delta}(q_0, \mathbf{q})$, $a, b = 1, 2, 3$:

$$\Pi^{ab}_{\Delta}(q_0, \mathbf{q}) = -\delta_{ab} g^2 \mu^2 \frac{\Delta}{q^2} \int_0^\infty dz \int_0^{q/2\Delta} dy \left( 1 - \frac{z^2 - y^2 + 1}{u_+ u_-} \right)$$

$$\times \left( 1 + \frac{1}{u_+ + u_- + q_0/\Delta} + \frac{1}{u_+ + u_- - q_0/\Delta} - \frac{2}{u_+ + u_-} \right),$$  

where $u_\pm = \sqrt{(z \pm y)^2 + 1}$, and we assume that $q_0$ and $q \equiv |\mathbf{q}|$ are much smaller than $\mu$ so that the dominant contribution comes only from particles near the Fermi surface. For $|q_0| > 2\Delta$, one should replace $q_0$ by $q_0 + i\epsilon$. Expanding Eq. (2) to quadratic order in $q_0$ and $q$ around $q_0 = q = 0$ one finds

$$\Pi^{ab}_{\Delta}(q_0, \mathbf{q}) = -\kappa q^2 \delta_{ab} = \frac{g^2 \mu^2}{18\pi^2 \Delta^2},$$  

where

$$\kappa = \frac{g^2 \mu^2}{18\pi^2 \Delta^2}. (4)$$

The appearance of factor $\mu^2$ in Eq. (4) is due to the fact that the loop integral is dominated by the momentum region near the Fermi surface, whose area is proportional to $\mu^2$. Similarly one obtains from Eq. (99b) of [5]

$$\Pi^{ab}_0(q_0, \mathbf{q}) = -\kappa q^0 q^i \delta_{ab}. (5)$$

The computation of $\Pi^{ij}_{\Delta}(q_0, \mathbf{q})$ is facilitated by writing

$$\Pi(q_0, \mathbf{q}) \equiv [\Pi(q_0, \mathbf{q}) - \Pi(0, \mathbf{q})] + [\Pi(0, \mathbf{q}) - \Pi_{\text{HDL}}(0, \mathbf{q})] + \Pi_{\text{HDL}}(0, \mathbf{q}), (6)$$

where $\Pi_{\text{HDL}}$ is the standard HDL gluon self-energy, which vanishes for $q_0 = 0$. The term $\Pi(0, \mathbf{q}) - \Pi_{\text{HDL}}(0, \mathbf{q})$ can be shown to be of order $O(\Delta^2$ and thus negligible compared to the first term $\Pi(q_0, \mathbf{q}) - \Pi(0, \mathbf{q})$. The reason for the manipulation (6) is to get rid of the antiparticle contributions in this term. With Eq. (99c) of [5], the result for $\Pi^{ij}_{\Delta}$ can be written analogously to Eq. (2) as

$$\Pi^{ij}_{\Delta}(q_0, \mathbf{q}) = -\delta_{ij} \frac{g^2 \mu^2 \Delta}{q^2} \int_0^\infty dz \int_0^{q/2\Delta} dy \frac{dy}{2\pi} \frac{dz}{2\pi} k^i k^j$$

$$\times \left( 1 - \frac{z^2 - y^2 - 1}{u_+ u_-} \right)$$

$$\times \left( 1 + \frac{1}{u_+ + u_- + q_0/\Delta} + \frac{1}{u_+ + u_- - q_0/\Delta} - \frac{2}{u_+ + u_-} \right),$$  

where $k = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$, and $\cos \theta = 2y\Delta/q$. Expanding to quadratic order in $q_0$ and $q$ one finds

$$\Pi^{ij}_{\Delta}(q_0, \mathbf{q}) = -\kappa q_0 q^i \delta^{ij} \delta_{ab}. (8)$$

Note that: (i) the polarization tensor satisfies current conservation, $q_\mu \Pi^{\mu\nu}_{ab} = 0$; (ii) at this order, there is no spatial transverse contribution $\sim q^2 \delta^{ij} - q^i q^j$ to $\Pi^{ij}$, although such a term is not forbidden by the symmetries.

After the quark loop has been integrated out, the quadratic term in the effective Lagrangian becomes $A^\mu(-q)(D_{ab}^{-1}(q) + \Pi^{ab}_{\Delta}(q))A^\nu(q)$, where $D_0$ is the bare gluon propagator. Comparing with the quadratic terms in Eq. (1), we obtain

$$\epsilon = 1 + \kappa = 1 + \frac{g^2 \mu^2}{18\pi^2 \Delta^2}, (9)$$

$$\lambda = 1. (10)$$

Equation (10) is due to the absence of the spatial transverse term in $\Pi^\Delta$. At high densities, the gap $\Delta$ is exponentially suppressed compared to the chemical potential $\mu$ [7],

$$\Delta = \mu_0 g^{-5c}/g, \quad c = \frac{3\pi^2}{\sqrt{2}},$$  

where

$$\mu_0 = \frac{g^2 \mu^2}{18\pi^2 \Delta^2}. (11)$$
where $g$ is the gauge coupling at the scale $\mu$, and $b$ is some numerical constant. According to Eq. (4), $\kappa \gg 1$ and we can write

$$\epsilon \approx \frac{g^2 \mu^2}{18\pi^2 \Delta^2} \gg 1,$$

(12)

which means that the dielectric constant of the medium is very large. Hence, the Coulomb potential between SU(2) color charges is greatly reduced. This can be interpreted as a consequence of the fact that the Cooper pairs have large size (of order $1/\Delta$) and so are easy to polarize. The magnetic permeability, in contrast, remains close to 1 due to the absence of mechanisms that would strongly screen the magnetic field.

Now that the effective Lagrangian (1) has been obtained, one can use it to investigate the infrared dynamics of the gluons. First, one notices that (1) possesses a modified Lorentz symmetry in which the speed of light $c = 1$ is replaced by

$$v = \frac{1}{\sqrt{\epsilon}}.$$

(13)

One can make this symmetry manifest by rescaling the time coordinate, the gauge field and the coupling in Eq. (1),

$$x^0 \to x^0' = \frac{x^0}{\sqrt{\epsilon}},$$

$$A_0^a \to A_0^{a'} = \epsilon^{3/4} A_0^a,$$

$$A_1^a \to A_1^{a'} = \epsilon^{1/4} A_1^a,$$

$$g \to g' = \frac{g}{\epsilon^{1/4}}.$$  

(14)

After the rescaling (14), the action (1) obtains the familiar Lorentz-invariant form in the new coordinates,

$$S = -\frac{1}{4} \int d^4x' F_{\mu\nu}^a F^{a\mu\nu},$$

(15)

where

$$F_{\mu\nu}^a = \partial_{\mu}' A_{\nu}'^a - \partial_{\nu}' A_{\mu}'^a + g' f^{abc} A_{\mu}'^b A_{\nu}'^c.$$  

(16)

The coupling in the action (15) is not $g$ but $g'$ which is smaller by a factor of $\epsilon^{1/4}$. This means that the small parameter that controls the perturbative expansion in the theory (1) is not $\alpha_s = g^2/(4\pi)$ but rather

$$\alpha'_s = \frac{g^2}{4\pi \sqrt{\epsilon}},$$

(17)

which is much smaller than $\alpha_s$, since $\epsilon$ is large.

Another way to derive Eq. (17) is by restoring the factors of $\hbar$ and $c$ in the expression for the strong coupling constant $\alpha_s$ which is $g^2/(4\pi\hbar c)$ in the vacuum. In our dielectric medium, the Coulomb potential between two static charges separated by $r$ is $g^2/(cr)$. Consequently, we have to replace $g^2$ by $g'^2 = g^2/\epsilon$. The velocity of light $c$ also needs to be replaced by the velocity of gluons $v$. This gives

$$\frac{g'^2}{4\pi \hbar v} = \frac{g^2}{4\pi \sqrt{\epsilon}},$$

(18)

since $\hbar = 1$ in our unit system. Eq. (18) coincides with $\alpha'_s$ in Eq. (17), as one expects.

Using Eq. (12), one can express the coupling $\alpha'_s$ in terms of the gap $\Delta$,

$$\alpha'_s = \frac{3}{2\sqrt{2}} \frac{g\Delta}{\mu}.$$  

(19)

Equations (17) and (19) define the coupling in our effective theory at the matching scale with the original microscopic theory, i.e., at the scale $\Delta$. The coupling increases logarithmically as one moves to lower energies, since the SU(2) pure Yang-Mills theory is asymptotically free. This coupling becomes large at the confinement scale $\Lambda_{QCD}'$, which is the mass scale of SU(2) glueballs. The spectrum of these glueballs is known from lattice studies of SU(2) Yang-Mills theory [8], except that the role of the speed of light is now played by $v$, Eq. (13). The scale $\Lambda_{QCD}'$ is necessarily very small, since the coupling $\alpha'_s$ at the $\Delta$ scale is tiny, Eq. (19). Using the one-loop beta function, one can estimate

$$\Lambda_{QCD}' \sim \Delta \exp \left( -\frac{2\pi}{\beta_0 \alpha'_s} \right) \sim \Delta \exp \left( -\frac{2\sqrt{2} \pi}{11} \frac{\mu}{g\Delta} \right),$$

(20)

where $\beta_0$ is the first coefficient in the beta function and is equal to 22/3 in SU(2) gluodynamics.

We can draw a few immediate conclusions from Eq. (20). First, $\Lambda_{QCD}'$ depends very sensitively on the gap $\Delta$, in particular on the numerical value of the constant $b$ in Eq. (11). Unfortunately, the latter is not exactly known. The uncertainty in the value of gap $\Delta$ translates into a huge variation of $\Lambda_{QCD}'$. For example, if one uses the value

$$b = 512\pi^4,$$

(21)

which is obtained by solving the one-loop gap equation where the exchanged gluon propagator is replaced by the HDL expression [9], then with $\Lambda_{QCD} = 200$ MeV we find $\Lambda_{QCD}' \sim 10$ MeV at $\mu = 600$ MeV. However, if we use

$$b = 512\pi^4 \exp \left( -\frac{\pi^2 + 4}{8} \right),$$

(22)

which is obtained if one assumes the BCS ratio between the critical temperature $T_c$ and the gap $\Delta$, and computes $T_c$ taking into account the fermion wave-function renormalization [10], then, at the same chemical potential, $\Lambda_{QCD}'$ is reduced to a mere 0.3 keV! Regrettfully, neither Eq. (21) nor (22) seems to be entirely correct,
since there are physical effects that they do not take into account (e.g., the Meissner effect). Clearly, any attempt to give even the roughest numerical estimate for $\Lambda_{\text{QCD}}'$ requires an accurate determination of the gap $\Delta$. It has been argued that to compute $\Delta$ one needs a better understanding of the issue of gauge invariance [11]. Regardless of all these uncertainties, the exponential dependence of $\Lambda_{\text{QCD}}'$ on $\mu/\Delta$ makes it safe to predict that, even at moderate values of $\mu$, the confinement scale $\Lambda_{\text{QCD}}$ is very small, much smaller than $\Lambda_{\text{QCD}}$.

Second, as the density is increased, $\Lambda_{\text{QCD}}'$ decreases exponentially fast due to the factor $\mu/\Delta$ in the exponent, which vanishes as $\mu \to \infty$. We arrive at the following picture of how deconfinement occurs at large densities. At any given value of the chemical potential, the theory is, strictly speaking, confined. However, the radius of confinement grows exponentially as one increases the density. Therefore, if one looks at the physics at some large, but fixed, distance scale, then there is a crossover chemical potential where effectively the color degrees of freedom become deconfined at that scale. We call this mechanism “asymptotic deconfinement.”

We have not discussed other low energy excitations in high density 2-flavor QCD. The unpaired quark of the third color gives rise to a fermion (isospin doublet) mode carrying baryon charge. This mode is neutral from the point of view of the SU(2) gluons and does not affect the picture we described. There is also a light pseudoscalar isoscalar mode, similar to the $\eta$ meson, which can mix with the pseudoscalar glueball. This mode is also neutral and it acquires a small mass due to the anomalous breaking of the global $U(1)_A$ symmetry. This mass is suppressed by a power of $\mu$, while, as we have seen, the glueball masses are much (i.e., exponentially) smaller at large $\mu$.

So far, we have considered QCD with two light quarks at finite baryon densities. At sufficiently large $\mu$, the number of relevant light quarks becomes $N_f = 3$, and the ground state is the color-flavor-locking (CFL) state where the SU(3) color symmetry is broken completely [12]. Correspondingly, the Debye and Meissner masses do not vanish in the limit of constant fields $q \to 0$ [13]. There is no pure Yang-Mills gauge sector, and hence no glueballs, in the infrared. Instead, the only light modes are the Goldstone bosons which arise from the spontaneous breaking of global symmetries [12]. The dynamics of these modes are described by a non-linear sigma model [14], whose parameters can be determined by a perturbative computation [15]. In the real world the formulas derived in this paper are valid for $\mu$ below the “unlocking” phase transition [16].

Another regime of QCD with asymptotic deconfinement is that of large isospin density [17]. In this regime the gauge group SU(3) remains unbroken by the condensate, which is color neutral. The physics below the gap scale $\Delta$ is described by pure SU(3) gluodynamics of the type (1), and asymptotic deconfinement occurs at large isospin chemical potential.

The authors are indebted to R.D. Pisarski and A.R. Zhitnitsky for stimulating discussions. We thank RIKEN, Brookhaven National Laboratory, and U.S. Department of Energy [DE-AC02-98CH10886] for providing the facilities essential for the completion of this work. The work of DTS is supported, in part, by a DOE OJI Award.