Canonical Theory of 2+1 Gravity

M.Kenmoku 1, T.Matsuyama 2, R.Sato 3 and S.Uchida 4

1 Department of Physics, Nara Women’s University, Nara 630-8506, Japan
2 Department of Physics, Nara University of Education, Takabatake-cho, Nara 630-8528, Japan
3 4 Graduate School of Human Culture, Nara Women’s University, Nara 630-8506, Japan

Recently 2+1 dimensional gravity theory, especially AdS$_3$ has been studied extensively [1, 2]. It was shown to be equivalent to the 2+1 Chern-Simon theory [3] and has been investigated to understand the black hole thermodynamics, i.e. Hawking temperature [4] and others. The purpose of this report is to investigate the canonical formalism of the original 2+1 Einstein gravity theory instead of the Chern-Simon theory. For the spherically symmetric space-time, local conserved quantities(local mass and angular momentum) are introduced and using them canonical quantum theory is defined. Constraints are imposed on state vectors and solved analytically. The strategy to obtain the solution is followed by our previous work [5].

1 kenmoku@phys.nara-wu.ac.jp
2 matsuyat@nara-edu.ac.jp
3 reika@phys.nara-wu.ac.jp
4 satoko@phys.nara-wu.ac.jp
1 Canonical formalism

We start to consider the Einstein-Hilbert action with cosmological constant $\lambda$ in 2+1 dimensional space-time,

$$I = \frac{1}{16\pi G_2} \int d^3x \sqrt{-g} g(R - 2\lambda). \quad (1)$$

The gravitational constant in 2+1 dimension is set to $G_2 = 1/4$ in the following.

The metrics in polar coordinate are expressed in ADM decomposition [6] as

$$ds^2 = -N^2 dt^2 + \Lambda^2 (dr + N^r dt)^2 + R^2 (d\phi + N^\phi dt)^2$$

$$+ 2C (dr + N^r dt)(d\phi + N^\phi dt), \quad (2)$$

where all metrics are assumed to be function of time $t$ and radial coordinate $r$.

In the following, dot and dash denotes the derivative with respect to $t$ and $r$.

The action in canonical formalism is in the form

$$I = \int dt dr \left[ P_\Lambda \dot{\Lambda} + P_R \dot{R} + P_C \dot{C} - (NH + N^r H_r + N^\phi H_\phi) \right]$$

$$- \int dt dr \left( [(\Lambda P_\Lambda + CP_C)N^r]' + [\frac{C}{\Lambda} P_\Lambda + R^2 P_C] N^\phi ]' \right), \quad (3)$$

where canonical momenta are

$$P_\Lambda = \frac{\partial L}{\partial \dot{\Lambda}} = \frac{2\Lambda R(N^r R' - \dot{R})}{N\sqrt{h}}, \quad (4)$$

$$P_R = \frac{\partial L}{\partial \dot{R}} = \frac{2R (CN^\phi' + \Lambda((\Lambda N^r)' - \dot{\Lambda}))}{N\sqrt{h}}, \quad (5)$$

$$P_C = \frac{\partial L}{\partial \dot{C}} = -\frac{(N^r C)' + R^2 N^\phi' - \dot{C}}{N\sqrt{h}}, \quad (6)$$

and the Hamiltonian and the momentum constraints are defined as

$$H = -\frac{\sqrt{h}}{2} \left( \frac{P_\Lambda P_R}{\Lambda R} - P_C^2 \right) - 2\left( -\frac{R^2 + RR''}{\sqrt{h}} + \frac{RR'h'}{2h\sqrt{h}} \right) - 2\lambda \sqrt{h}, \quad (7)$$

$$H_r = P_R R' - CP_C' - \Lambda P_\Lambda', \quad (8)$$

$$H_\phi = -\left( \frac{C}{\Lambda} P_\Lambda + R^2 P_C \right)' . \quad (9)$$
It is essential to introduce the local conservation quantities, the angular momentum $J$ and the mass function $M$ as follows.

\[
J := - \int dr H \phi = \frac{C}{\Lambda} P_\Lambda + R^2 P_C, \quad (10)
\]

\[
M := - \int dr \left( \frac{R R'}{\sqrt{\hbar}} H + \frac{P_\Lambda}{\Lambda} H_r + P_C H \phi \right) = \frac{1}{2} \left( P_\Lambda^2 + \frac{2 C P_\Lambda P_C}{\Lambda} + R^2 P_C^2 - \frac{(R R')^2}{h} - \lambda R^2 \right), \quad (11)
\]

We make transformation from old variables $\Lambda, R$ and $C$ into new variables

\[
\begin{pmatrix}
\Lambda \\
R \\
C
\end{pmatrix} \longrightarrow \begin{pmatrix}
\bar{\Lambda} \\
\bar{R} \\
\bar{C}
\end{pmatrix} = \begin{pmatrix}
\sqrt{\Lambda^2 - C^2 R^{-2}} \\
R \\
C R^{-2}
\end{pmatrix}. \quad (12)
\]

The corresponding momenta are transformed as

\[
\begin{pmatrix}
P_\Lambda \\
P_R \\
P_C
\end{pmatrix} \longrightarrow \begin{pmatrix}
P_{\bar{\Lambda}} \\
P_{\bar{R}} \\
P_{\bar{C}}
\end{pmatrix} = \begin{pmatrix}
\bar{\Lambda}^{-1} P_{\Lambda} \\
C^2 \Lambda^{-1} R^{-3} P_\Lambda + P_R + 2 C R^{-1} P_C \\
C \Lambda^{-1} P_\Lambda + R^2 P_C
\end{pmatrix}. \quad (13)
\]

2 Quantum solutions

Next we proceed the quantum theory in the Schrödinger picture and the quantized operators are denoted by the notation hat. Our strategy is to solve the eigenvalue equation for $\hat{J}$, $\hat{M}$ and the constraint equation for $H_r$ step by step instead of solving the constraint equations $\hat{H} \Psi = 0$, $\hat{H}_r \Psi = 0$ and $\hat{H}_\phi \Psi = 0$.

Step 1: Angular momentum eigen equation

The eigenvalue equation of the local angular momentam (Eq. (10))

\[
\hat{J} \Psi = \hat{P}_C \Psi = j \Psi \quad (14)
\]

is solved with the eigenvalue $j$ and the eigen function is obtained in the form

\[
\Psi = e^{i j \Phi} u(\bar{\Lambda}, \bar{R}), \quad (15)
\]
with
\[ \Phi = \int dr \bar{C}(r) . \]  

(16)

Step 2: Momentum constraint equation

The radial momentum constraint equation
\[ \hat{H}_r \Psi = (\bar{R}' \hat{P}_{\bar{R}} - \bar{\Lambda}(\hat{P}_{\bar{\Lambda}}')) \epsilon^{ij} \Phi u(\bar{\Lambda}, \bar{R}) = 0 , \]

(17)

restricts the functional form of the wave function as
\[ \Psi = \epsilon^{ij} \Phi u(Z) , \]

(18)

where we introduce variable \( Z \)
\[ Z = \int dr \bar{\Lambda} f(\bar{R}, \chi) = \int dr \int \bar{\Lambda}(r) \bar{\Lambda} f(\bar{R}, \chi) , \]

(19)

with
\[ \chi := R^2 \bar{\Lambda}^{-2} . \]

(20)

The arbitrary function \( f \) and \( \bar{f} \) are related each other:
\[ f(\bar{R}, \chi) = - \int \chi d\chi \frac{\bar{f}(\bar{R}, \chi)}{2\chi} . \]

(21)

Step 3: Mass eigen equation

The local mass operator \( \hat{M} \) is defined as
\[ \hat{M} - m = \frac{1}{2} A \hat{P}_{\bar{\Lambda}} A^{-1} \hat{P}_{\bar{\Lambda}} + \frac{1}{2} (-\chi + \hat{F}(\bar{R})) , \]

(22)

where
\[ \hat{F}(\bar{R}) = 1 - 2m - \lambda \bar{R}^2 + \frac{1}{4} j^2 \bar{R}^{-2} , \]

(23)
and

\[ A = A_Z(Z) \bar{A}(\bar{R}, \chi) , \]  

which is called ordering factor. We take \( \bar{A} \) as

\[ \bar{A} = \frac{\delta Z}{\delta \Lambda} = \bar{f} = \sqrt{\chi - F_j(\bar{R})} , \]  

where

\[ F_j(\bar{R}) := \hat{F} \mid_{j=\bar{J}}(\bar{R}) . \]  

Then using the mass operator for each eigenvalue of angular momentum \( j \)

\[ \hat{M}_j := \hat{M} \mid_{j=\bar{J}} , \]  

the mass eigen equation

\[ \hat{M}_j u_{j,m}(Z) = m u_{j,m}(Z) , \]  

can reduce to the equation with respect to \( Z \)

\[ \frac{d^2 u_{j,m}(Z)}{dZ^2} - A_Z^{-1} \frac{\delta A_Z}{\delta Z} \frac{d u_{j,m}(Z)}{dZ} + u_{j,m}(Z) = 0 . \]  

If we choose the remaining ordering factor as \( A_Z = Z^{2\nu-1} \), the above equation becomes the Bessel equation

\[ \frac{d^2 u_{j,m}(Z)}{dZ^2} - \frac{2\nu - 1}{Z} \frac{d u_{j,m}(Z)}{dZ} + u_{j,m}(Z) = 0 , \]  

and the solution is

\[ u_{j,m}^{(\nu)}(Z) = Z^\nu [b_1 H^{(1)}_\nu(Z) + b_2 H^{(2)}_\nu(Z)] , \]  

where \( H_\nu(Z) \) is the Hankel function.
3 Summary

In conclusion, the general form of quantum wave function is obtained

$$\Psi(Z) = \sum_{j,m} c_{j,m} e^{ij\Phi} u^{(\nu)}_{j,m}(Z), \quad (32)$$

where $c_{j,m}$ are the expansion coefficients, $u^{(\nu)}_{j,m}(Z)$ is expressed by Eq. (31) and $Z$ is expressed using Eqs. (19) and (25) as

$$Z = \int dr \int \Lambda^{(\nu)} d\sqrt{\chi - F_j(\bar{R})}$$
$$= \int dr \left( \Lambda \sqrt{\chi - F_j(\bar{R})} - \bar{R}' \ln \left| \sqrt{\chi + \sqrt{\chi - F_j(\bar{R})}} \right| \right), \quad (33)$$

where $\chi$ and $F_j$ are given in Eqs. (20) and (26). It is worthwhile to note that the analytic solution in Eq.(32) is shown to satisfy the original constraint equations as well as the Wheeler-DeWitt equation. Therefore we have successfully obtained the analytic solution for the Wheeler-DeWitt equation. The interpretation for the wave function will be appeared in separate paper.

References