Integrability and Scheme-Independence of Even Dimensional Quantum Geometry Effective Action

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Abstract

We investigate how the integrability conditions for conformal anomalies constrain the form of the effective action of even dimensional quantum geometry. We show that the effective action of 4DQG satisfying the integrability has a manifestly diffeomorphism invariant and regularization scheme independent form. We then generalize the arguments to 6 dimensions and propose a model of 6DQG. The expected form of 6DQG effective action is given.

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1 Introduction

Since quantum geometry (QG) is defined by the functional integrations over the metric fields, diffeomorphism invariance in QG is equivalently described as an invariance under any change of background-metric. This background-metric independence includes an invariance under a conformal change of the background-metric. Thus, in even dimensional QG well-defined on the background-metric [1]–[16], conformal anomalies [17]–[29] play an important role. Therefore, to preserve diffeomorphism invariance we must formulate even dimensional QG considering that conformal anomalies always exist [1]–[16].

Background-metric independence in 2 dimensions means that QG is described as a conformal field theory [2, 3]. The idea can generalize to arbitrary even dimensions [11, 13, 14, 15, 16]. But, as studied in recent works [13, 14], the generalization is not simple because the traceless mode becomes dynamical in higher dimensions so that higher dimensional QG is no longer described as a free theory. Furthermore, it has been understood that the integrability condition of conformal anomaly [21, 22] gives a strong constraint on even dimensional QG [7, 8, 14].

In this paper we further investigate how the integrability condition of conformal anomaly acts on even dimensional QG. We also settle the problem of regularization scheme dependence and show that the effective action has a manifestly diffeomorphism invariant and regularization scheme independent form.

This paper is organized as follows. In next section we present fundamental idea how to preserve diffeomorphism invariance in even dimensional QG and review how such an idea is realized in exactly solvable 2DQG [2, 3, 4]. In $D \geq 4$ dimensions, the integrability condition of the conformal anomaly not only restricts matter fields to be conformally invariant ones but also fixes many indefiniteness in gravity sector [14]. How the integrability condition acts on 4DQG is rediscussed in section 3. We then show that the effective action is written in a diffeomorphism invariant and scheme independent form. The generalization to 6 dimensions [15, 16] is studied in section 4. We show that the Duff’s scheme [19] is also useful to tame the trivial anomalies in 6 dimensions [23]–[27]. Based on the arguments of the integrability given in the study of 4DQG, we propose a model of 6DQG. Many indefinite coefficients accompany with the existence of many curvature invariants are fixed by the
integrability and the expected scheme-independent form of 6DQG effective action is given. Section 5 is devoted to conclusions and discussion.

Our curvature conventions are $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$ and $R^\lambda_{\mu\sigma\nu} = \partial_{\nu} \Gamma^\lambda_{\mu\sigma} - \cdots$.

2 Conditions of Diffeomorphism Invariance

In this section we briefly explain how to realize diffeomorphism invariance in even dimensional QG.

QG is defined by functional integration over the metric field as

$$Z = \int \frac{|g^{-1}dg|_g[dX]_g}{\text{vol}(\text{diff.})} \exp[-I(X,g)] , \quad (2.1)$$

where $I$ is an invariant action and $X$ is a matter field. In this paper we consider a conformal scalar without self-interactions, for example. The measure of the metric field is defined by the invariant norm

$$<dg, dg>_g = \int d^D x \sqrt{g} g^{\mu\nu} g^{\lambda\sigma} (dg_{\mu\lambda} dg_{\nu\sigma} + u g_{\mu\nu} g_{\lambda\sigma}) , \quad (2.2)$$

where $D = 2n$ and $u > -1/D$. This measure can orthogonally decompose into the conformal mode and the traceless mode as

$$<d\phi, d\phi>_g = \int d^D x \sqrt{\bar{g}} (d\phi)^2 , \quad (2.3)$$

$$<dh, dh>_g = \int d^D x \sqrt{\bar{g}} \text{tr}(e^{-h} de^h)^2 . \quad (2.4)$$

Here, the metric is decomposed as $g_{\mu\nu} = e^{2\phi} \bar{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu} = (\bar{g} e^h)_{\mu\nu}$, where $\text{tr}(h) = 0$ [6, 13, 14].

This definition is manifestly diffeomorphism invariant/background-metric independent. But, it is not well-defined because the measures of the metric fields defined by (2.3) and (2.4) have the metric dependence, $\sqrt{\bar{g}}$, in the measures itself so that we must integrate that dependence when we quantize the conformal mode, $\phi$.

Instead, we consider the measures defined on the background-metric as

$$<d\phi, d\phi>_{\bar{g}} = \int d^D x \sqrt{\bar{g}} (d\phi)^2 , \quad (2.5)$$

$$<dh, dh>_{\bar{g}} = \int d^D x \sqrt{\bar{g}} \text{tr}(e^{-h} de^h)^2 . \quad (2.6)$$
This replacement, however, violates diffeomorphism invariance. In fact, these norms conformally change under the general coordinate transformation, \( \delta g_{\mu\nu} = g_{\mu\lambda} \nabla_\nu \xi^\lambda + g_{\nu\lambda} \nabla_\mu \xi^\lambda \), which is decomposed as

\[
\delta \phi = \frac{1}{D} \tilde{\nabla}_\lambda \xi^\lambda + \xi^\lambda \partial_\lambda \phi ,
\]

\[
\delta \bar{g}_{\mu\nu} = \bar{g}_{\mu\lambda} \nabla_\nu \xi^\lambda + \bar{g}_{\nu\lambda} \nabla_\mu \xi^\lambda - \frac{2}{D} \bar{g}_{\mu\nu} \tilde{\nabla}_\lambda \xi^\lambda ,
\]

(2.7)

where \( \tilde{\nabla}_\lambda \xi^\lambda = \tilde{\nabla}_\lambda \xi^\lambda \) is used. Therefore, these measures produce conformal anomalies [20] under the general coordinate transformation.

As a lesson from 2DQG [3, 4, 5], in order to preserve diffeomorphism invariance, we must add an action, \( S \), as

\[
Z = \int \frac{[d\phi][e^{-h} d\tilde{e}^h][dX] \tilde{g}}{\text{vol}(\text{diff.})} \exp[-S(\phi, \tilde{g}) - I(X, g)] ,
\]

(2.8)

where the measures of the metric fields are now defined by (2.5) and (2.6).

Let us briefly see how background-metric independence constrains the theory (2.8). Background-metric independence for the traceless mode represents the condition that \( \tilde{g} \) and \( h \) always appear in the combination \( \bar{g} = \tilde{g} e^h \) in the theory (2.8) [13]. This condition guarantees, at most, that the effective action has an invariant form on the metric, \( \bar{g} \).

Background-metric independence for the conformal mode requires that \( S \) should satisfy the Wess-Zumino condition [30] defined by

\[
S(\phi, \tilde{g}) = S(\omega, \bar{g}) + S(\phi - \omega, e^{2\omega} \bar{g}) .
\]

(2.9)

Such an action is given by integrating conformal anomaly within the interval \([0, \phi]\). So it satisfies the initial condition \( S(0, \bar{g}) = 0 \) and has a local form. In this paper we call this local action the Wess-Zumino action because condition (2.9) is essential in the arguments of diffeomorphism invariance. In 2 dimensions it is usually called the Liouville action [1]. On the other hand, the well-known non-local forms of integrated conformal anomaly are called Polyakov [1] and Riegert [7] action in 2 and 4 dimensions. Why we distinguish between the local and the non-local ones becomes clear soon below.

The Wess-Zumino condition fixes the form of \( S \), but some overall coefficients remain to be determined. These coefficients should be determined
from the requirement of diffeomorphism invariance in a self-consistent manner. The process to determine them is as follows.

Under the general coordinate transformation, $\delta I = 0$, while the Wess-Zumino action is not invariant and produces a conformal anomaly. This property comes from condition (2.9). Diffeomorphism invariance is now realized dynamically such that $\delta S$ cancels conformal anomalies calculated by loop effects of the combined theory, $I = S + I$. In other words, consider the regularized 1PI effective action, $\Gamma$, of the combined theory, $\mathcal{I}$, and require $\delta \Gamma = 0$ to determine $S$. This means that, although the tree action, $\mathcal{I}$, is not manifestly invariant, including loop effects, the effective action becomes an invariant form on the metric, $g$.

Here, it is worth making a comment on the difference of the Wess-Zumino action defined by (2.9) and non-local Polyakov/Riegert action. The former produces conformal anomalies under the general coordinate transformation, while the non-local Polyakov/Riegert action, which will appear in the effective action by loop effects, is generally defined by the condition that it produces conformal anomalies under a conformal change.

As an exercise, let us first discuss 2DQG coupled $N$ conformal scalars. The tree action in conformal gauge is given by [3, 4, 5]

$$I = \frac{b}{4\pi} \int d^2x \sqrt{\bar{g}} (\bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \bar{R} \phi) + I_{GF+FP} + I_M(X, \bar{g}), \quad (2.10)$$

where $I_M$ is invariant action of $N$ free scalars. The gauge-fixing term and the Faddeev-Popov (FP) ghost action are given by [31]

$$I_{GF+FP} = \frac{1}{4\pi} \int d^2x \sqrt{\bar{g}} (-iB_{\mu\nu}(\bar{g}^{\mu\nu} - \bar{g}^{\mu\nu}) + 2\bar{g}^{\mu\nu} b_{\mu\lambda} \bar{\nabla}_\nu c^\lambda), \quad (2.11)$$

where the reparametrization ghost $c^\mu$ is a contravariant vector. $B_{\mu\nu}$ and the anti-ghost $b_{\mu\nu}$ are covariant symmetric traceless tensors. The coefficient, $b$, is determined by diffeomorphism invariance uniquely.

Consider effective action of 2DQG, which has the following form:

$$\Gamma = \mathcal{I}(\phi, X, \bar{g}) + W(\bar{g}), \quad (2.12)$$

where $W$ is a loop effect, which depends only on $\bar{g}$ because the measure is now defined on $\bar{g}$. The condition of diffeomorphism invariance, $\delta \Gamma = 0$, is now given by

$$-\frac{b}{4\pi} \int d^2x \sqrt{\bar{g}} \omega \bar{R} + \delta_\omega W(\bar{g}) = 0, \quad (2.13)$$

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where \( \delta \omega \bar{g}_{\mu\nu} = 2 \omega \bar{g}_{\mu\nu} \) and \( \omega = -\frac{1}{2} \nabla_\lambda \xi^\lambda \). Here, \( \delta W = \delta \omega W \) because \( W \) does not depend on the conformal mode, \( \phi \). The second term of l.h.s is nothing but conformal anomaly of the theory, \( \mathcal{I} \).

From one-loop calculations using the tree action, \( \mathcal{I} \), we obtain the well-known non-local Polyakov action [1]

\[
W(\bar{g}) = \frac{N - 25}{96\pi} \int d^2 x \sqrt{\bar{g}} R^{1/2} R ,
\tag{2.14}
\]

where \( N \) comes from scalar matter fields and \(-26\) from the ghosts. The change of the coefficient from \( N - 26 \) to \( N - 25 \) is due to the contribution from the conformal mode.

As mentioned above, diffeomorphism invariance determines the coefficient, \( b \), uniquely as [3]

\[
b = \frac{25 - N}{6} .
\tag{2.15}
\]

Using the relation

\[
-\frac{1}{24\pi} \int d^2 x \sqrt{\bar{g}} (\bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \bar{R} \phi) + \frac{1}{96\pi} \int d^2 x \sqrt{\bar{g}} R^{1/2} R
\]

\[
= \frac{1}{96\pi} \int d^2 x \sqrt{\bar{g}} R^{1/2} R ,
\tag{2.16}
\]

the effective action can be re-expressed in a manifestly invariant form:

\[
\Gamma = \frac{N - 25}{96\pi} \int d^2 x \sqrt{\bar{g}} R^{1/2} R + I_M(X, g) .
\tag{2.17}
\]

Here we use the fact that matter action is conformally invariant such that \( I_M(X, \bar{g}) = I_M(X, g) \).

### 3 4D Quantum Geometry

Recently, we showed that there is a model of diffeomorphism invariant 4DQG [13, 14]. This model has many advantages in physics. It is renormalizable and asymptotically free. It will solve the cosmological constant problem dynamically without fine-tuning [9, 10]. It naturally describes our 4
dimensional universe at the long distance or at the large $N$. On the other hand, the unitarity problem is unsolved. In this paper we do not discuss the unitarity problem, which is expected to be solved dynamically [32, 33, 35, 14].

3.1 Tree action

The tree action of 4DQG [13] is given by a proper combination of the Wess-Zumino action [7, 8] and invariant action required by the integrability conditions discussed in [14] and also in the following subsection 3.3 as

$$\mathcal{I} = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \left\{ \frac{1}{t^2} \bar{F} + a \bar{F} \phi + 2b \phi \bar{\Delta}_4 \phi + b \left( \bar{G} - \frac{2}{3} \bar{\Box} \bar{R} \right) \phi \\
+ \frac{1}{36} (2a + 2b + 3c) \bar{R}^2 + \mathcal{L}_{GF+FP} \right\} + I_{LE}(X, g). \quad (3.1)$$

where $\mathcal{L}_{GF+FP}$ is the gauge-fixing term and the FP ghost Lagrangian defined below. $I_{LE}$ represents lower-derivative actions which include actions of conformally invariant matter fields, the Einstein-Hilbert action and the cosmological constant term. The lower-derivative gravitational actions are treated in the perturbation of the massive constants [10, 12, 14].

The invariants, $F$ and $G$, are defined by

$$F = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2, \quad (3.2)$$

$$G = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2. \quad (3.3)$$

In 4 dimensions they are the square of the Weyl tensor and the Euler density, respectively. $\Delta_4$ is the conformally covariant 4th order operator [7]

$$\Delta_4 = \Box^2 + 2 R^{\mu\nu} \nabla_{\mu} \nabla_{\nu} - \frac{2}{3} R \Box + \frac{1}{3} (\nabla^{\mu} R) \nabla_{\mu}, \quad (3.4)$$

which satisfies $\Delta_4 = e^{-4\phi} \bar{\Delta}_4$ locally for a scalar.

Above, we introduce the dimensionless coupling, $t$, only for the traceless mode as $\bar{g}_{\mu\nu} = (\bar{g} e^{\phi})_{\mu\nu}$ and consider the perturbation of $t$. The kinetic term of the conformal mode comes from the Wess-Zumino action. Since the

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2Contrary to 2DQG in which the classical limit is given by $N \to -\infty$, the positive large $N$ gives the correct classical limit in 4DQG [10].
invariant $R^2$ terms cancel out in our model, the self-interactions of $\phi$ appear only in the lower-derivative actions in the exponential form, which treated exactly order by order of $t$ [14].

The gauge-fixing term and the FP ghost action are given by [33, 34].

$$\mathcal{L}_{GF+FP} = 2iB^\mu N_\mu \chi^\nu - \zeta B^\mu N_\mu B^\nu - 2i\tilde{c}^\mu N_\mu \tilde{\nabla}^\lambda \delta_B h^\nu_\lambda ,$$

(3.5)

where $\chi^\nu = \tilde{\nabla}^\lambda h^\nu_\lambda$ and $N_\mu$ is a symmetric 2nd order operator. The BRST transformations are given by

$$\delta_B h^\mu_\nu = i\left\{\tilde{\nabla}^\mu c_\nu + \tilde{\nabla}_\nu c^\mu - \frac{1}{2} \delta^\mu_\nu \tilde{\nabla}^\lambda c_\lambda + t c^\lambda \tilde{\nabla}_\lambda h^\mu_\nu + \frac{t}{2} h^\mu_\lambda \left(\tilde{\nabla}_\nu c^\lambda - \tilde{\nabla}^\lambda c_\nu\right) + \frac{t}{2} h^\lambda_\nu \left(\tilde{\nabla}^\mu c_\lambda - \tilde{\nabla}_\lambda c^\mu\right) + \cdots \right\} ,$$

$$\delta_B \phi = it c^\lambda \partial_\lambda \phi + i \frac{t}{4} \tilde{\nabla}^\lambda c_\lambda ,$$

$$\delta_B \tilde{c}^\mu = B^\mu , \quad \delta_B B^\mu = 0 ,$$

$$\delta_B c^\mu = it c^\lambda \tilde{\nabla}_\lambda c^\mu .$$

The first two are obtained by replacing $\xi^\mu/t$ in the general coordinate transformation (2.7) with the contravariant vector ghost field, $ic^\mu$. The kinetic term of the ghost action then becomes $t$-independent. This BRST transformation is nilpotent. Using this transformation, the gauge-fixing term and the FP ghost action can be written as $\mathcal{L}_{GF+FP} = 2i\delta_B \{\partial^\mu N_\mu (\chi^\nu + \frac{1}{2} \zeta B^\nu)\}$ [36].

The important property of this tree action is that it transforms under the general coordinate transformation (2.7) as

$$\delta I = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \omega \left\{-a \left(F + \frac{2}{3} \Box R\right) - bG - c \Box R\right\} ,$$

(3.7)

where

$$\omega = -\frac{1}{4} \tilde{\nabla}_\lambda \xi^\lambda .$$

(3.8)

In the case of the BRST transformation, $\xi^\mu$ is replaced with $itc^\mu$. \(^3\)

3Even in 2DQG, although we can set $\delta_B I = 0$ if we take the flat background-metric and integrate the $B_{\mu\nu}$ field out, the nilpotency of the BRST charge at the quantum level after all requires condition (2.15). Thus, the BRST invariance in even dimensional QG is realized dynamically.
The $\Box \bar{R}$ terms in (3.7) will depend on regularization scheme. We here use the Duff’s scheme [19] of dimensional regularization characterized by the equations

$$\delta \phi \int d^D x \sqrt{g} F = (D - 4) \int d^D x \sqrt{g} \phi \left( F + \frac{2}{3} \Box R \right), \quad (3.9)$$

$$\delta \phi \int d^D x \sqrt{g} G = (D - 4) \int d^D x \sqrt{g} \phi G. \quad (3.10)$$

When we define the tree action, $I$, it is taken into account that the Duff’s scheme will be used for computing loop effects of the effective action later. As shown below, the scheme-dependent terms cancel out and we obtain a scheme-independent effective action.

### 3.2 Effective action

As investigated in [14], the regularized effective action of the theory, $I$, has the following form:

$$\Gamma = I(X, \phi, \bar{g}) + V_{NS}(\phi, \bar{g}) + W_F(\bar{g}, \mu) + W_G(\bar{g}) + W_{\Box R}(\bar{g}). \quad (3.11)$$

Here, the first term of r.h.s. is the tree action. $V_{NS}$ and $W_F, G, \Box R$ come from loop diagrams. The former represents corrections to the Wess-Zumino action, and the latter three represent corrections to the traceless mode $h^\mu_\nu$.

Let us first consider corrections to the traceless mode. Here, $W_F$ is the part which associates to the conformally invariant counterterm to $\bar{F}$, and it can be determined by computing two-point diagrams of the traceless mode. In the Duff’s scheme, it has the following scale-dependent form:

$$W_F(\bar{g}, \mu) = \frac{f}{(4\pi)^2} \int d^4 x \sqrt{\bar{g}} \left\{ -\frac{1}{4} C_{\mu\nu\lambda\sigma} \log \left( \frac{\Delta^C_4}{\mu^4} \right) \bar{C}^{\mu\nu\lambda\sigma} - \frac{1}{18} \bar{R}^2 \right\}. \quad (3.12)$$

Here, the appearance of the $\bar{R}^2$ term is due to using the Duff’s scheme. $C$ is the Weyl tensor and $\Delta^C_4 = \Box^2 + \cdots$ is an appropriate conformally covariant operator for the Weyl tensor. The explicit form of $\Delta^C_4$ is unknown, but it is known that there is $W_F$ which satisfies the equation [18, 23, 25]

$$\delta W_F(\bar{g}, \mu) = \delta_{\omega} W_F(\bar{g}, \mu) = \frac{f}{(4\pi)^2} \int d^4 x \sqrt{\bar{g}} \omega \left( \bar{F} + \frac{2}{3} \Box \bar{R} \right), \quad (3.13)$$

\[ \text{Some errors in the form of the effective action in section 3.3 of ref. [14] are corrected in this section.} \]
where $\delta \omega \bar{g}_{\mu\nu} = 2 \omega \bar{g}_{\mu\nu}$ with (3.8). Thus, $W_F$ produces the type B anomaly in the classification of [23].

$W_G$ is the part which associates to the conformally invariant counterterm to $\bar{G}$. It is what is called non-local Riegert action which produces the type A anomaly, or the Euler density in the classification of [23] and has the following form:

$$W_G(\bar{g}) = \frac{e}{(4\pi)^2} \int d^4x \sqrt{\bar{g}} \left\{ \frac{1}{8} \bar{G} \frac{1}{\Delta_4} \bar{G} - \frac{1}{18} \bar{R}^2 \right\}.$$  \hspace{1cm} (3.14)

where

$$\mathcal{G} = G - \frac{2}{3} \Box R.$$  \hspace{1cm} (3.15)

As announced above, $W_G$ produces the type A anomaly as

$$\delta W_G(\bar{g}) = \delta \omega W_G(\bar{g}) = \frac{e}{(4\pi)^2} \int d^4x \sqrt{\bar{g}} \omega \bar{G}.$$  \hspace{1cm} (3.16)

The $\bar{R}^2$ term is needed to realize equation (3.16). The presence of this term leads to the vanishing of $o(h^2)$ corrections to $W_G$ in the flat background. This is consistent with the direct loop calculations of two-point diagrams of $h$. Hence, $W_G$ is related to more than $o(h^3)$ vertex corrections.

The coefficients $f$ and $e$ are scheme-independent. They are expanded by the renormalized coupling $t_r$ as

$$f = f_0 + f_1 t_r^2 + \cdots, \quad e = e_0 + e_1 t_r^2 + \cdots.$$  \hspace{1cm} (3.17)

Here, $f_0$ and $e_0$ have already been computed by one-loop diagrams as

$$f_0 = -\frac{N}{120} - \frac{199}{30} + \frac{1}{15}, \quad e_0 = \frac{N}{360} + \frac{87}{20} - \frac{7}{90},$$  \hspace{1cm} (3.18)

where the first contributions of each coefficient come from $N$ conformal scalar fields [19]. The second and the last ones come from the traceless mode [33] and the conformal mode [11], respectively. The coefficients $f_1$ and $e_1$ are given by functions of $a$ and $b$, to which not only two-loop diagrams but also one-loop, but order $t_r^2$ diagrams contribute [14].

The beta function for the coupling $t_r$ is given by $\beta = f \frac{t_r^2}{2}$. Since $f_0$ is negative, 4DQG is asymptotically free. Here, note that, although background-metric independence includes an invariance under any conformal change of the background-metric, usual $\beta$-function is not needed to vanish. This nature is owing to that there is a conformal anomaly, or the Wess-Zumino action.
The last one in (3.11) is a scheme-dependent part defined by

\[
W_{\square R}(\bar{g}) = -\frac{u}{12(4\pi)^2} \int d^4x \sqrt{\bar{g}} \bar{R}^2 .
\]  

(3.19)

It is unknown whether this term is really necessary or not. Anyway, the coefficient, \(u\), is at most order \(t^2\) such that \(u = u_1t^2 + \cdots\).

As computed in [14], the correction, \(V_{NS}\), is scale-independent and merely changes the coefficients \(a\) and \(b\) in the tree action into \(\tilde{a} = a(1 + v_a)\) and \(\tilde{b} = b(1 + v_b)\), where \(v_a\) and \(v_b\) are order \(t^2\) at the one-loop level. The meanings of this fact will be explained in the following subsection.

Now, the conditions of diffeomorphism invariance are given by the following equations [14]:

\[
\tilde{a} = f , \quad \tilde{b} = e , \quad c = u .
\]  

(3.20)

Since \(f_1\) and \(e_1\) are functions of \(a\) and \(b\), while \(f_0\) and \(e_0\) are the constants independent of \(a\) and \(b\), we can solve these equations in the perturbation of \(t_r\). Note that one-loop coefficients of \(v_a\) and \(v_b\) are related to the order \(t^2\) coefficients, \(f_1\) and \(e_1\), of \(W_F\) and \(W_G\). This is reasonable because the Wess-Zumino action originally comes from the measure so that it is essentially a quantum effect. Thus, one-loop contributions given by quantizing the Wess-Zumino action are related to two-loop contributions.

Substituting the solutions of (3.20), the \(\bar{R}^2\) terms cancel out and we obtain the scheme-independent and manifestly invariant effective action \(^5\)

\[
\Gamma = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \left\{ -\frac{f}{4} C_{\mu\nu\lambda\sigma} \log \left( \frac{\Delta^4}{\mu^4} \right) C^{\mu\nu\lambda\sigma} + \frac{e}{8} G \frac{1}{\Delta^4} G \right\} + I_{LE}(X,g) .
\]  

(3.21)

Here, the Weyl action \(F\) is absorbed in the scale, \(\mu\).

### 3.3 Two-loop integrability

Here, we summarize the conditions of diffeomorphism invariance discussed in ref. [14].

The condition to be able to make a theory diffeomorphism invariant is that, in the effective action, there is no action which produce a term that

\(^5\)It is not exclude that the invariant \(R^2\) term appears in the effective action. There is a possibility that such a term appears in \(V_{NS}\) at order \(t^4\).
does not appear in the variation of the tree action, $\delta I$ (3.7). Namely, diffeomorphism invariance states that the following action is not allowed:

$$W_{R^2}(\bar{g}, \mu) = \frac{r}{(4\pi)^2} \int d^4x \sqrt{\bar{g}} \bar{R} \log \left( \frac{\bar{\Delta}_4}{\mu^4} \right) \bar{R},$$

(3.22)

because this action produces $\bar{R}^2$ under the general coordinate transformation. And also, a scale-dependent action including the conformal mode, $\phi$, for example,

$$V_S(\phi, \bar{g}, \mu) = \frac{s}{(4\pi)^2} \int d^4x \sqrt{\bar{g}} \phi \bar{\Delta}_4 \log \left( \frac{\bar{\Delta}_4}{\mu^4} \right) \phi$$

(3.23)

is not allowed because this action can not be absorbed in the Wess-Zumino action by changing the coefficients $a$ and $b$ and produces the term not being in $\delta I$ under the general coordinate transformation.

In general, parts of the effective actions other than the type B anomaly producing one, $W_F$, must be independent of the scale, $\mu$, such as $V_{NS}$, $W_G$ and $W_{\square R}$. The vanishing of $r$ and $s$, at least to order $t^2$, was shown in the previous paper [14]. It is as follows.

Expand $r$ and $s$ as $r = r_0 + r_1 t^2 + \cdots$ and $s = s_0 + s_1 t^2 + \cdots$. The vanishing of $r_0$ is guaranteed in our model because at this order only conformally invariant vertices contribute to the one-loop diagrams. This is the consequence that the invariant $R^2$ terms cancel out such that self-interactions of the conformal mode, $\phi$, do not appear in the tree action $I$ except in the lower-derivative terms such as the cosmological constant in the exponential form.

The vanishing of $s_0$ is proved directly by showing the finiteness of the self-energy diagram of $\phi$ [14]. Here, the fact that there is no interactions of $R^2$ is essentially used. Note that we can not explain this result by using conformal invariance. It can be explained only by diffeomorphism invariance/background-metric independence.

The background-metric independence for the conformal mode implies that $W_{R^2}$ and $V_S$ is related each other so that $s = 0$ means $r = 0$. Now, we introduce the coupling, $t$, only for the traceless mode so that $s_0$ is related to $r_1$. Thus, $r_1 = 0$ is indirectly shown.

More direct check of $r_1 = 0$ is as follows. Since there is no self-interactions of $\phi$, two-loop diagrams that contribute to $f_1$, $g_1$ and $r_1$ are derived from the conformally invariant vertices of $2b\phi \bar{\Delta}_4 \phi$ and $\frac{1}{t^2} \bar{F}$ so that the contributions
of two-loop diagrams to \( r_1 \) vanish. But, there are contributions from one-loop, but order \( t^2 \) diagrams, which include the vertices of \( a\bar{F}\phi \), \( b(\bar{G} - \frac{2}{3} \square \bar{R})\phi \) and \( \frac{1}{32}(2a + 2b + 3c)\bar{R}^2 \). Here, these vertices except the first one are non-conformally invariant so that we must pay attention to such one-loop contributions.

As shown in [13, 5], the variation of the one-loop contributions to the effective action of our model is given by

\[
\delta_{\omega}W^{(1)}(\bar{g}) = -2Tr(\omega e^{-\epsilon K}),
\]

(3.24)

where \( \epsilon \) is a cutoff. The matrix operator, \( K \), is defined by the kinetic term \( \frac{1}{2}\Phi^t\mathcal{K}\Phi \) on arbitrary background-metric, \( \hat{g} \), where \( \Phi = (\phi, h_{\mu\nu}, X) \). The \( t \)-independent diagonal parts gives the coefficients, \( f_0 \) and \( e_0 \). The off-diagonal parts as well as \( t \)-dependent diagonal parts gives contributions of order \( t^2 \). Note that, unlike for matter fields, we do not use the condition of conformal invariance for gravitational fields to derive this expression. We merely use the facts that \( K \) is a 4th order operator and there is no self-interactions of the conformal mode. If there is the invariant \( R^2 \) term, we can not describe \( \delta_{\omega}W^{(1)} \) in such a simple form because we do not introduce the coupling, \( t \), for the conformal mode, \( \phi \). This is a general property of \( 2n \)-th order operators in \( 2n \) dimensions and \( \delta_{\omega}W^{(1)} \) is shown to be integrable [13, 14]. Thus, our model satisfies \( r_1 = 0 \).

In 4 dimensions the integrability gives strong constraints on QG. It seems that there is no other 4DQG that overcomes the interability conditions than our model. So 4DQG may be fixed uniquely according to conformal matter contents.

4 6D Quantum Geometry

In this section we see that the arguments of the integrability in 4DQG can generalize to the 6 dimensional case. Since there are many curvature invariants in 6 dimensions, many indefinite coefficients will appear to define 6D action. However, we see below that many of them will be fixed by the integrability.
4.1 Duff’s scheme in 6 dimensions

Recently, 6 dimensional conformal anomalies have been studied in detail [22]–[29]. In this subsection we summarize their results and then show that we can apply the Duff’s scheme to 6 dimensional case also.

In 6 dimensions there are 17 independent curvature invariants. We here use the following bases [22, 27]:

\[ K_1 = R^3 \, , \quad K_2 = R R_{\mu\nu} R^{\mu\nu} \, , \quad K_3 = R R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} \, , \]
\[ K_4 = R_{\mu}^{\nu} R_{\lambda}^{\nu} R_{\lambda}^{\mu} \, , \quad K_5 = R_{\mu\nu} R_{\lambda\sigma} R_{\mu\nu\lambda\sigma} \, , \quad K_6 = R_{\mu\nu} R_{\alpha\beta\gamma} R^{\mu\alpha\beta\gamma} \, , \]
\[ K_7 = R_{\mu\nu} R_{\sigma}^{\lambda} R_{\lambda\sigma}^{\mu\nu} \, , \quad K_8 = R_{\mu\nu\alpha}\beta R_{\alpha\lambda\sigma}^{\mu\nu} R_{\lambda\sigma}^{\alpha\beta} \, , \quad K_9 = R R \, , \]
\[ K_{10} = R_{\mu\nu} \nabla R^{\mu\nu} \, , \quad K_{11} = R_{\mu\nu\lambda\sigma} \nabla R_{\mu\nu\lambda\sigma} \, , \quad K_{12} = R_{\mu\nu} \nabla_{\mu} \nabla_{\nu} R \, , \]
\[ K_{13} = (\nabla_{\lambda} R_{\mu\nu}) \nabla_{\lambda} R^{\mu\nu} \, , \quad K_{14} = (\nabla_{\lambda} R_{\mu\nu}) \nabla_{\mu} R^{\nu\lambda} \, , \]
\[ K_{15} = (\nabla_{\lambda} R_{\alpha\beta\gamma\delta}) \nabla_{\lambda} R^{\alpha\beta\gamma\delta} \, , \quad K_{16} = R^2 \, , \quad K_{17} = \Box R \, . \] (4.1)

The results for conformal anomalies are summarized as follows. There are 10 independent integrable curvature invariants [27]. They give a basis for the conformal anomalies in 6 dimensions. In the classification of ref. [23], the type A anomaly is unique and given by the Euler density

\[ G_6 = -K_1 + 12K_2 - 3K_3 - 16K_4 + 24K_5 + 24K_6 - 4K_7 - 8K_8 \, . \] (4.2)

Here, we normalize it as

\[ G_6 = -\frac{1}{8} \epsilon_{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3} \epsilon_{\lambda_1 \sigma_1 \lambda_2 \sigma_2 \lambda_3 \sigma_3} R_{\lambda_1 \sigma_1}^{\mu_1 \nu_1} R_{\lambda_2 \sigma_2}^{\mu_2 \nu_2} R_{\lambda_3 \sigma_3}^{\mu_3 \nu_3} \, . \] (4.3)

There are three type B anomalies. They are locally conformally invariant in 6 dimensions:

\[ F_1 = \frac{19}{800} K_1 - \frac{57}{160} K_2 + \frac{3}{40} K_3 + \frac{7}{16} K_4 - \frac{9}{8} K_5 - \frac{3}{4} K_6 + K_8 \, , \] (4.4)
\[ F_2 = \frac{9}{200} K_1 - \frac{27}{40} K_2 + \frac{3}{10} K_3 + \frac{5}{4} K_4 - \frac{3}{2} K_5 - 3K_6 + K_7 \, , \] (4.5)
\[ F_3 = -\frac{11}{50} K_1 + \frac{27}{10} K_2 - \frac{6}{5} K_3 - K_4 + 6K_5 + 2K_7 - 8K_8 \]
\[ + \frac{3}{5} K_9 - 6K_{10} + 6K_{11} + 3K_{13} - 6K_{14} + 3K_{15} \, . \] (4.6)
Here, $F_1$ and $F_2$ correspond to two independent combinations of the Weyl tensors, $C_{\alpha\mu\nu}C^\mu^\lambda\nu^\alpha_\lambda^\beta$ and $C_{\alpha\beta}C^\mu_\lambda\sigmaC^\alpha_\lambda\sigma$, respectively. $F_3$ gives the kinetic term of the traceless mode and expressed, up to the total derivative term, as $C_{\mu\alpha\beta}(\Box\delta_\mu^\nu+4R_\mu^\nu-\frac{6}{5}R\delta_\mu^\nu)C^{\nu\alpha\beta\gamma}$.

The other 6 combinations are given by

\begin{align*}
M_5 &= 6K_6 - 3K_7 + 12K_8 + K_{10} - 7K_{11} - 11K_{13} + 12K_{14} - 4K_{15} , \\
M_6 &= -\frac{1}{5}K_9 + K_{10} + \frac{2}{5}K_{12} + K_{13} , \\
M_7 &= K_4 + K_5 - \frac{3}{20}K_9 + \frac{4}{5}K_{12} + K_{14} , \\
M_8 &= -\frac{1}{5}K_9 + K_{11} + \frac{2}{5}K_{12} + K_{15} , \\
M_9 &= K_{16} , \\
M_{10} &= K_{17} .
\end{align*}

These are classified in the trivial conformal anomalies.

In order to treat the trivial anomalies, $M_{5,\ldots,10}$, unambiguously, we use dimensional regularization. Consider the conformal variations of the functions, $G_6$ and $F_{1,2,3}$, defined by the combinations listed above. In $D$ dimensions we obtain the following equations:

\begin{equation}
\delta_{\phi} \int d^D x \sqrt{g}G_6 = (D - 6) \int d^D x \sqrt{g}\phi G_6 \tag{4.13}
\end{equation}

and

\begin{equation}
\delta_{\phi} \int d^D x \sqrt{g}F_i = (D - 6) \int d^D x \sqrt{g}\phi \left( F_i + \sum_{n=5}^{10} z_{i,n}M_n \right) \quad (i = 1, 2, 3) , \tag{4.14}
\end{equation}

where

\begin{align*}
[z_{1,1}, z_{1,6}, z_{1,7}, z_{1,8}, z_{1,9}, z_{1,10}] &= \left[ \frac{1}{16}, \frac{71}{80}, \frac{15}{16}, \frac{13}{40}, \frac{159}{3200}, 0 \right] \tag{4.15} \\
[z_{2,5}, z_{2,6}, z_{2,7}, z_{2,8}, z_{2,9}, z_{2,10}] &= \left[ -\frac{1}{4}, -\frac{1}{20}, -\frac{3}{4}, -\frac{7}{10}, -\frac{51}{800}, 0 \right] \tag{4.16} \\
[z_{3,5}, z_{3,6}, z_{3,7}, z_{3,8}, z_{3,9}, z_{3,10}] &= \left[ 1, \frac{1}{5}, 3, \frac{14}{5}, \frac{39}{200}, \frac{3}{5} \right] . \tag{4.17}
\end{align*}

Here, note that the r.h.s. of equation (4.14) is expanded by $F_i$ itself and the trivial conformal anomalies. This equation suggests that the Duff’s scheme works well in 6 dimensions also.
4.2 Tree action

Let us first look for the conformally covariant 6th order operator in 6 dimensions [15]. It is expanded by 21 independent operators apart from the $\Box^3$ term as

$$\Delta_6 = \Box^3 + v_1 R^\mu\nu \nabla_\mu \nabla_\nu \Box + v_2 R \Box^2 + v_3 (\nabla^\lambda R^\mu\nu) \nabla_\lambda \nabla_\mu \nabla_\nu +$$

$$v_4 (\nabla^\lambda R) \nabla_\lambda \Box + v_5 (\nabla^\mu \nabla^\nu R) \nabla_\mu \nabla_\nu + v_6 (\Box R^\mu\nu) \nabla_\mu \nabla_\nu +$$

$$v_7 (\Box R) \Box + v_8 R^\mu_{\alpha\beta\gamma} \nabla_\mu \nabla_\nu + v_9 R^\mu\nu\lambda\sigma R^\nu\mu\lambda\sigma \Box$$

$$+ v_{10} R^\mu\nu R^\mu\nu R_{\alpha\beta} \nabla_\mu \nabla_\nu + v_{11} R^{\mu\nu\lambda} R^\nu R^\nu_{\lambda} R_{\mu\nu} \nabla_\mu \nabla_\nu + v_{12} R^{\mu\nu} R_{\mu\nu} \Box$$

$$+ v_{13} R R^\mu\nu \nabla_\mu \nabla_\nu + v_{14} R^{\mu\nu} \Box + v_{15} (\nabla^\lambda \Box R) \nabla_\lambda + v_{16} R_{\alpha\beta\gamma\mu}(\nabla^\mu R^\alpha\beta\gamma\nu) \nabla_\nu +$$

$$v_{17} R^{\mu\nu\lambda\sigma}(\nabla_\mu R_{\nu\lambda}) \nabla_\sigma + v_{18} R_{\mu\nu}(\nabla^\mu R^{\nu\lambda}) \nabla_\lambda + v_{19} R_{\mu\nu}(\nabla^\lambda R^{\nu\mu}) \nabla_\lambda +$$

$$v_{20} R^{\mu\nu}(\nabla_\mu R) \nabla_\nu + v_{21} R(\nabla^\lambda R) \nabla_\lambda$$

(4.18)

From the requirement that $\delta_\phi (\sqrt{g} \Delta_6 Y) = 0$ is satisfied locally for a scalar $Y$, the coefficients are determined as follows:

$$v_1 = 4 , \quad v_2 = -1 , \quad v_3 = 4 , \quad v_4 = 0 , \quad v_5 = 0 , \quad v_6 = 4 ,$$

$$v_7 = -\frac{3}{5} , \quad v_8 = \zeta_1 , \quad v_9 = \zeta_2 , \quad v_{10} = \zeta_1 , \quad v_{11} = 6 - \frac{3}{4} \zeta_1 ,$$

$$v_{12} = -1 + \frac{1}{8} \zeta_1 - \zeta_2 , \quad v_{13} = -2 + \frac{1}{4} \zeta_1 , \quad v_{14} = \frac{9}{25} - \frac{1}{10} \zeta_1 + \frac{1}{10} \zeta_2 ,$$

$$v_{15} = \frac{2}{5} , \quad v_{16} = \zeta_1 + 4 \zeta_2 , \quad v_{17} = -\zeta_1 , \quad v_{18} = 6 + \frac{1}{4} \zeta_1 ,$$

$$v_{19} = -2 - \frac{3}{4} \zeta_1 - 2 \zeta_2 , \quad v_{20} = 1 - \frac{1}{8} \zeta_1 , \quad v_{21} = -\frac{7}{25} + \frac{3}{40} \zeta_1 + \frac{1}{5} \zeta_2 .$$

(4.19)

In 6 dimensions, $\Delta_6$ is not unique and two constants, $\zeta_1$ and $\zeta_2$, are not determined from the conformal property alone. The terms with these arbitrary constants are collected, using the Weyl tensor, in the forms, $\zeta_1 \nabla^\mu (C_{\mu\alpha\beta\gamma} C^{\alpha\beta\gamma\nu} \nabla_\nu)$ and $\zeta_2 \nabla^\lambda (C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} \nabla_\lambda)$, respectively [15].

Next, we look for the combination of $G_6$ and $M_n$ that satisfies the following conformal property locally:

$$\delta_\phi \left\{ \sqrt{g} \left( G_6 - \sum_{n=5}^{10} w_n M_n \right) \right\} = 6 \sqrt{g} \Delta_6 \phi .$$

(4.20)
This equation determines the coefficients, \(w_n\), uniquely for each \(\Delta_6\) with \(\zeta_1\) and \(\zeta_2\) as

\[
\begin{align*}
    w_5 &= 1 + \frac{1}{4} \zeta_1, \quad w_6 = 11 + \frac{1}{2} \zeta_1 - 3 \zeta_2, \quad w_7 = -6 - \frac{3}{4} \zeta_1, \\
    w_8 &= 1 + \zeta_1 + 3 \zeta_2, \quad w_9 = -\frac{21}{100} + \frac{9}{160} \zeta_1 + \frac{3}{20} \zeta_2, \quad w_{10} = \frac{3}{5} \\
\end{align*}
\]

(4.21)

Using equation (4.20), the Wess-Zumino action defined by integrating the conformal anomalies within the interval \([0, \phi]\) is expressed in the form:

\[
S(\phi, \bar{g}) = \frac{1}{(4\pi)^3} \int d^6 x \int_0^\phi d\phi \sqrt{g} \left\{ \sum_{i=1}^3 a_i \left( F_i + \sum_{n=5}^{10} z_{i,n} M_n \right) + b G_6 + \sum_{n=5}^{10} c_n M_n \right\} \\
= \frac{1}{(4\pi)^3} \int d^6 x \sqrt{g} \left\{ \sum_{i=1}^3 a_i \bar{F}_i \phi + 3b \phi \bar{\Delta}_6 \phi + b \left( G_6 - \sum_{n=5}^{10} w_n \bar{M}_n \right) \phi \right\} \\
+ \sum_{n=5}^{10} \frac{\sum_{i=1}^3 a_i z_{i,n} + bw_n + c_n}{(4\pi)^3} \int d^6 x (\sqrt{\bar{g}} L_n - \sqrt{g} \bar{L}_n).
\]

Here, \(L_n\)'s are local functions given by integrating \(M_n\)'s as

\[
\delta \phi \int d^6 x \sqrt{g} L_n = \int d^6 x \sqrt{g} \delta \phi M_n
\]

such that

\[
\begin{align*}
    L_5 &= \frac{1}{30} K_1 - \frac{1}{4} K_2 + K_6, \quad L_6 = \frac{1}{100} K_1 - \frac{1}{20} K_2, \\
    L_7 &= \frac{37}{6000} K_1 - \frac{7}{150} K_2 + \frac{1}{75} K_3 - \frac{1}{10} K_5 - \frac{1}{15} K_6, \\
    L_8 &= \frac{1}{150} K_1 - \frac{1}{20} K_3, \quad L_9 = -\frac{1}{30} K_1, \quad L_{10} = \frac{1}{300} K_1 - \frac{1}{20} K_9. \\
\end{align*}
\]

(4.23)

As discussed in 4DQG, the integrability suggests that 6th order parts of the invariant action \(I\) should be chosen such that the invariant \(L_n\) terms cancel out in the sum \(\mathcal{I} = S + I\). Hence, we obtain 6DQG tree action analogous to 4DQG as

\[
\mathcal{I} = \frac{1}{(4\pi)^3} \int d^6 x \sqrt{g} \left\{ -\frac{1}{t^2} \left( \bar{F}_3 + \alpha_1 \bar{F}_1 + \alpha_2 \bar{F}_2 \right) + \sum_{i=1}^3 a_i \bar{F}_i \phi \right\}
\]

(4.24)
\[ +3b\phi \bar{\Delta}_6 \phi + b \left( \bar{G}_6 - \sum_{n=5}^{10} w_n \bar{M}_n \right) \phi \]  
\[ - \sum_{n=5}^{10} \left( \sum_{i=1}^{3} a_i z_{i,n} + bw_n + c_n \right) \bar{L}_n \right\} + I_{LE}(X, g) . \]

Here, we introduce the dimensionless coupling, \( t \), as in 4DQG. In 6 dimension, extra dimensionless constants, \( \alpha_1 \) and \( \alpha_2 \), in addition to \( \zeta_1 \) and \( \zeta_2 \) in \( \Delta_6 \) and \( w_n \), appear, which do not fixed by the arguments of the integrability. The constants, \( t \), \( \alpha_1 \) and \( \alpha_2 \) will receive renormalization, but \( \zeta_1 \) and \( \zeta_2 \) may not do.

Under the general coordinate transformation, this action changes as

\[ \delta I = \frac{1}{(4\pi)^3} \int d^6x \sqrt{\bar{g}} \omega \left\{ - \sum_{i=1}^{3} a_i \left( \bar{F}_i + \sum_{n=5}^{10} z_{i,n} \bar{M}_n \right) - b \bar{G}_6 - \sum_{n=5}^{10} c_n \bar{M}_n \right\} , \]  
\( (4.25) \)

where

\[ \omega = -\frac{1}{6} \hat{\nabla}_\lambda \xi^\lambda . \]  
\( (4.26) \)

### 4.3 Effective action

It is expected that the effective action of this model has the following form:

\[ \Gamma = \bar{I}(X, \phi, \bar{g}) + W_G(\bar{g}) + \sum_{i=1}^{3} W_{F_i}(\bar{g}, \mu) + \sum_{n=5}^{10} W_{M_n}(\bar{g}) , \]  
\( (4.27) \)

where the tilde on \( I \) denotes the inclusions of the finite corrections to the Wess-Zumino action described by \( V_{NS} \) in 4 dimensional model. Here, \( W_{G_6} \) is the generalization of the Polyakov-Riegert non-local and scale-independent action [24, 25]. We find its complete form as

\[ W_{G_6}(\bar{g}) = \frac{e}{(4\pi)^3} \int d^6x \sqrt{\bar{g}} \left\{ \frac{1}{12} \bar{G}_6 \frac{1}{\Delta_6} \bar{G}_6 + \sum_{n=5}^{10} w_n \bar{L}_n \right\} , \]  
\( (4.28) \)

where

\[ \mathcal{G}_6 = \mathcal{G}_6 - \sum_{n=5}^{10} w_n M_n . \]  
\( (4.29) \)

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This produces the type A anomaly,
\[ \delta W_{G_6}(\bar{g}) = \delta_\omega W_{G_6}(\bar{g}) = \frac{e}{(4\pi)^3} \int d^6x \sqrt{\bar{g}} \, \omega \bar{G}_6, \]  
(4.30)
where \( \delta_\omega \bar{g}_{\mu\nu} = 2\omega \bar{g}_{\mu\nu} \) with (4.26). This equation is realized for arbitrary values of \( \zeta_1 \) and \( \zeta_2 \). These constants as well as \( e \) will be determined according to matter contents.

The action, \( W_{F_i} \), which produces the type B anomaly in the Duff’s scheme, is defined by
\[ W_{F_i}(\bar{g}, \mu) = f_i \left( W'_{F_i}(\bar{g}, \mu) + \sum_{n=5}^{10} \frac{z_{i,n}}{(4\pi)^3} \int d^6x \sqrt{\bar{g}} \bar{L}_n \right). \]  
(4.31)
The \( \bar{L}_n \) terms will appear in the Duff’s scheme. \( W'_{F_i} \) is the scale-dependent part defined through the equation
\[ \delta_\omega W'_{F_i}(\bar{g}, \mu) = \frac{1}{(4\pi)^3} \int d^6x \sqrt{\bar{g}} \, \omega \bar{F}_i. \]  
(4.32)
It is known that the coefficients, \( e \) and \( f_i \), are regularization scheme-independent.

The remaining one, \( W_{M_n} \), is a scheme-dependent part defined by
\[ W_{M_n}(\bar{g}) = \frac{u_n}{(4\pi)^3} \int d^6x \sqrt{\bar{g}} \bar{L}_n. \]  
(4.33)
This action produces the trivial anomaly, \( \bar{M}_n \). As in 4DQG, it is unknown whether this action is really necessary or not. Since the vertices of tree action at zeroth order of \( t \) is conformally invariant, the coefficients \( u_n \) will be at most order \( t^2 \).

The conditions of diffeomorphism invariance are now given by
\[ \tilde{a}_i = f_i, \quad \tilde{b} = e, \quad c_n = u_n, \]  
(4.34)
where the tildes on \( a_i \) and \( b \) represent the inclusions of corrections to the Wess-Zumino action. As in 4DQG, the scheme-dependent terms, \( \bar{L}_n \), cancel out and the final expression will be an invariant and scheme-independent form
\[ \Gamma = \frac{e}{(4\pi)^3} \int d^6x \sqrt{\bar{g}} \frac{1}{12} \bar{G}_6 \frac{1}{\Delta_6} \bar{G}_6 + \sum_{i=1}^{3} f_i W'_{F_i}(g, \mu) + I_{LE}(X, g). \]  
(4.35)
The matter contributions to the coefficients, \( e \) and \( f_i \), have computed in refs. [26, 28, 29].
5 Conclusions and Discussion

In this paper we studied how the integrability conditions for conformal anomalies constrain the form of the effective action of even dimensional QG. We showed that the effective action of 4DQG satisfying such integrability conditions has a manifestly diffeomorphism invariant and scheme-independent form. We then generalized the arguments to 6 dimensions and propose a model of 6DQG. The expected scheme-independent form of the effective action was presented.

Now, the role of conformal anomalies in even dimensional QG is naturally understood in terms of background-metric independence/diffeomorphism invariance. In $D = 2n$ ($\geq 4$) dimensions, unlike 2DQG, there is no critical matter contents where the Wess-Zumino action vanishes. Thus, we can not avoid that $2n$-dimensional QG is to be $2n$-th order by diffeomorphism invariance.

Background-metric independence does not require the vanishing of usual beta functions in $D \geq 4$ dimensions though it includes the invariance under any conformal change of the background-metric. This nature is owing to that there are conformal anomalies, or the Wess-Zumino action in even dimensions. We think that conformal invariance in physics should be re-interpreted in terms of diffeomorphism invariance. In this case the problem of regularization scheme dependence will disappear.

In odd dimensions there is no conformal anomaly so that background-metric independence seems to require the theory to be finite. In 3 dimensions the Einstein-Hilbert+cosmological constant action is written in the Chern-Simons action and its quantum theory is expected to be topological [37]. But, for $D \geq 5$, it is unknown whether odd dimensional QG exists or not. Since, in odd dimensions, we can not introduce a dimensionless coupling constant, it seems to be necessary to make a theory to be super-renormalizable.

There is another approach to QG based on the dynamical triangulation in 2 [38, 39, 40] and 4 dimensions [41, 42, 43, 40]. It is expected that our model is given by a continuum limit of such a simplicial QG. In this paper we do not discuss quantum corrections of the lower-derivative gravitational actions. The anomalous dimensions of the gravitational constant and the cosmological constant are needed to compare two methods [10, 14]. A project of detailed comparison in 4DQG between them has started [43].

Finally, we give comments on dimensional regularization. Dimensional
regularization violates conformal invariance in general so that it is not suitable regularization to the theory in which conformal invariance plays an important role. Nevertheless, dimensional regularization is still much useful, because the violation is quite small and it is expected to give correct results to higher loops enough [44].

On the other hand, there is an assertion that, as far as using dimensional regularization, we can regularize QG defined by (2.1) in a manifestly diffeomorphism invariant way if we take a great care on conformal mode dependence [6]. At present, the relation between this approach and ours is unknown. Detailed analyses of this matter are important to prove renormalizability to all orders.

Anyway, the beautiful relations among integrable curvature invariants in D dimensions seems to suggest a validity of dimensional regularization. As for our model, at least up to order $t^2$, it will give correct results because of the finiteness of the self-energy diagrams of $\phi$ which implies that our model is rather insensitive to the conformal mode dependence. Whether the derived effective action at higher order is acceptable or not will be decided by the condition that it has a scheme-independent form and does not have the terms which violate diffeomorphism invariance such as (3.22) and (3.23).

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