Lagrangian Noether symmetries as canonical transformations

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Abstract

We prove that, given a time-independent Lagrangian defined in the first tangent bundle of configuration space, every infinitesimal Noether symmetry that is defined in the \( n \)-tangent bundle and is not vanishing on-shell, can be written as a canonical symmetry in an enlarged phase space, up to constraints that vanish on-shell. The proof is performed by the implementation of a change of variables from the the \( n \)-tangent bundle of the Lagrangian theory to an extension of the Hamiltonian formalism which is particularly suited for the case when the Lagrangian is singular. This result proves the assertion that any Noether symmetry can be canonically realized in an enlarged phase space. Then we work out the regular case as a particular application of this ideas and rederive the Noether identities in this framework. Finally we present an example to illustrate our results.

1 Introduction.

1.1 Noether transformations

In gauge theories, the conserved quantities associated with infinitesimal Noether transformations exhibit some features that are absent in a regular (non-gauge) theory. In
we have studied exhaustively the characterization of such conserved quantities in some specific contexts. We always consider as our starting point a time-independent first order Lagrangian $L(q, \dot{q})$ defined in configuration-velocity space $TQ$, that is, the tangent bundle of some configuration manifold $Q$. Gauge theories rely on Lagrangians whose Hessian matrix with respect to the velocities ($q$ stand for local coordinates in $Q$)

$$W_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j},$$

is not invertible (singular Lagrangians); this fact implies the existence of constraints in phase space. The canonical treatment of these theories was first solved by Dirac [2].

For the $n$-th tangent bundle of $Q$ (we leave $n$ undetermined but finite in order to keep locality), the most general infinitesimal Noether transformations that can be constructed for either a singular or regular Lagrangian $L(q, \dot{q})$, is a set of functions of the form

$$\delta^L q^i(q, \dot{q}, \ddot{q}, \ldots; t),$$

where $q, \dot{q}, \ddot{q}, \ldots$ are the local coordinates in the $n$-th tangent bundle. These functions are such that they induce a transformation $\delta L$ satisfying

$$\delta L = \frac{dL}{dt} F,$$

for some infinitesimal function $F(q, \dot{q}, \ddot{q}, \ldots; t)$. Property (3) completely characterizes the Noether symmetries. The total time derivative operator in the $n$ tangent bundle is defined as

$$\frac{dL}{dt} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} + \ldots .$$

Equation (3) can be equivalently written as

$$[L]_i \delta^L q^i + \frac{dL}{dt} G^L = 0,$$

where $[L]_i$ stands for the Euler-Lagrange equations

$$[L]_i := \alpha_i - W_{is} \ddot{q}^s,$$

with

$$\alpha_i := - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^s} \ddot{q}^s + \frac{\partial L}{\partial q^i}.$$ 

The conserved quantity is then $G^L = (\partial L/\partial \dot{q}^i) \ddot{q}^i - F$. In a gauge theory the infinitesimal Noether transformations (2) may contain arbitrary infinitesimal functions; these are the Noether gauge transformations. In such case the associated conserved quantity is zero on-shell, that is, it is a combination of -first class- constraints.

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1 In most of the paper we will ignore the coordinate labels for $q$, that is, we will often write $q$ instead of $q^i$. 

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1.2 Noether transformations in the enlarged formalism.

The enlarged formalism is a Lagrangian treatment of the Hamilton variational principle which is particularly convenient for singular Lagrangians. This formalism takes as a starting point the canonical Lagrangian \( L_c \). Once a canonical Hamiltonian \( H_c \) and a complete set of independent primary constraints \( \phi_\mu \) are determined out of the original Lagrangian \( L \), \( L_c \) is defined as follows,

\[
L_c(q, p, \lambda; \dot{q}, \dot{p}, \dot{\lambda}) := p\dot{q} - H_c(q, p) - \lambda^\mu \phi_\mu(q, p).
\]  

(6)

The new configuration space for \( L_c \) is the old phase space enlarged with the Lagrange multipliers \( \lambda^\mu \) as new independent variables. The dynamics given by \( L_c \) is nothing but the constrained Dirac’s Hamiltonian dynamics for a system with canonical Hamiltonian \( H_c \) and a number of primary constraints \( \phi_\mu \).

We will consider Noether transformations for \( L_c \) that are canonically generated for what regards the variables \( q \)'s and \( p \)'s, but that can depend also on the Lagrange multipliers \( \lambda^\mu \) and their time derivatives at any finite order. In [1] we have established the condition for a function \( G^c(q, p, \lambda, \dot{\lambda}, \ddot{\lambda}, \ldots; t) \) to be a Noether generator for \( L_c \) according to the rules

\[
\delta_c q = \{q, G^c\} \quad \delta_c p = \{p, G^c\}.
\]  

(7)

The condition is

\[
\frac{D G^c}{D t} + \{G^c, H_D\} = pc,
\]  

(8)

where \( pc \) stands for an arbitrary linear combination of the primary constraints \( \phi_\mu \); \( H_D \) is the Dirac Hamiltonian defined by

\[
H_D = H_c + \lambda^\mu \phi_\mu,
\]

and we have also introduced the notation

\[
\frac{D}{D t} := \frac{\partial}{\partial t} + \dot{\lambda}^\mu \frac{\partial}{\partial \lambda} + \ddot{\lambda}^\mu \frac{\partial}{\partial \dot{\lambda}} + \ldots.
\]  

(9)

The right hand side of (8) contains the prescription to define \( \delta_c \lambda \). Indeed, if “\( pc \)” in the right hand side of (8) is \( C^\mu \phi_\mu \), then \( \delta_c \lambda^\mu = C^\mu \). We reproduce in the appendix the proof of the condition (8) as given in [1].

In [3] an algorithm was introduced to obtain gauge generators for theories with only first class constraints. One can easily check that our expression (8) condenses all the contents of this algorithm. The generality of (8) lies in the fact that it extends the formalism of [3] to a general gauge theory -with first and second class constraints- and that it covers the case of rigid Noether transformations as well.

One might wonder whether these Noether transformations for the enlarged Lagrangian \( L_c \) are still Noether transformations for the Lagrangian \( L \). The answer is in the positive. It is proved in [1] that we can obtain a transformation \( \delta^L q \) of the type (2) and satisfying (3) out of our \( \delta_c q \). Indeed, to go back to the original Lagrangian \( L \) and to establish its corresponding Noether transformation associated with \( \delta_c q \) of (7),
we must produce a pullback from phase space to configuration-velocity space in order to substitute the momenta $p$ by their Lagrangian definition $\dot{\hat{p}} := \partial L / \partial \dot{q}$, but we must also substitute the variables $\lambda, \dot{\lambda}, \ddot{\lambda}, \ldots$ by their Lagrangian counterparts. This is easily done by noticing that the Euler-Lagrange equations of (6) for $p$ and $\lambda$ can be used to isolate these variables in terms of $q$ and $\dot{q}$. In fact, from the equations

$$[L_c]_p = \dot{q} - \frac{\partial H_c}{\partial \dot{p}} - \lambda^\mu \frac{\partial \phi_\mu}{\partial \dot{p}} = 0,$$

and

$$[L_c]_{\lambda^\mu} = -\phi_\mu = 0,$$

we get $p = \dot{\hat{p}}(q, \dot{q})$ and a Lagrangian determination for $\lambda^\mu$ that defines a new set of functions $v^\mu$ such that $\lambda^\mu = v^\mu(q, \dot{q})$. These functions $v^\mu(q, \dot{q})$, with some properties, were already introduced in [4]. We arrive at the definition for $\delta^L q$

$$\delta^L q(q, \dot{q}, \ddot{q}; \ldots; t) := \delta_c q(q, \dot{\hat{p}}, v^\mu, \dot{v}^\mu, \ddot{v}^\mu, \ldots; t),$$

(10)

where $\dot{v}^\mu = dL v^\mu / dt$, etc.. It is proved in [1] that this $\delta^L q$ defined in (10) is indeed a Noether transformation for $L$ having

$$G^L_c(q, \dot{q}, \ddot{q}, \ldots; t) := G^c(q, \dot{\hat{p}}, v^\mu, \dot{v}^\mu, \ddot{v}^\mu, \ldots; t),$$

as its associated conserved quantity.

The scope of the present paper is to show that this type of transformations (7), generated by functions $G^c$ satisfying (8), is general enough to encompass the most general Lagrangian Noether transformation of the type (2). We will prove that the transformations (10) are, up to the addition of constraints –that vanish on shell–, the most general transformations satisfying (3), that is, the most general Noether transformations. The proof will be based on a change of variables that is particularly suited to connect the Hamiltonian and the Lagrangian formalism in the singular case. These new variables are introduced in the next section, whereas in section 3 we prove the main result of this paper. A simple application to the regular case is given in section 4. As a bonus, in section 5, we rederive in this framework the Noether identities for singular Lagrangians and in section 6 we present an example to illustrate our result. Finally, we reproduce in the appendix the proof of some results quoted in the introduction.

2 From the old to the new variables

As a warm up, let us observe that, in a regular case, that is, when (1) is invertible, we can use the equations of motion, that are expressible in normal form as $\ddot{q} = W^{-1} \alpha$, to eliminate the higher order dependences in (2); this elimination results in a new transformation $\delta^L_0 q$ in $TQ \times R$, given by

$$\delta^L_0 q(q, \dot{q}; \ldots; t) \equiv \delta^L q(q, \dot{q}, \ddot{q}; \ldots; t)|_{[L]=0},$$

where in $[L] = 0$ we include its time derivatives at any order that is needed.

Now the question: is $\delta^L_0 q$ a Noether transformation for $L$? The answer is yes and the proof will come out as a particular case of the general results we will obtain below.
2.1 The new variables.

In a regular case, the coordinates in $TQ$ may be traded with those in $T^*Q$ by means of the Legendre map, which is invertible. Instead, in a singular case (the gauge case), the Lagrangian definition of the momenta, $\hat{p} := \partial L/\partial \dot{q}$ does not allow for the determination of $\dot{q}$ in terms of phase space coordinates. In fact, the definition of the momenta determines a certain number of independent constraints $\phi_\mu(q, p)$, $\mu = 1, ..., m$, in phase space, that is, functions such that their pullbacks $\phi_\mu(q, \hat{p})$ vanish identically. These functions locally define the primary constraint surface $M_0 \in T^*Q$. Due to the presence of these constraints, in order to trade the coordinates in $TQ$ with those in this surface, we need to enlarge the phase space with $m$ new coordinates in $R^m$ (in fact, since all our results are local, all what we need is a open subset of $R^m$). For purposes that are made clear later on, we shall call $\lambda^\mu$ these new coordinates. The change of coordinates will be given by:

\[
TQ \leftrightarrow M_0 \times R^m
\]  
(11)

\[
q^i, \dot{q}^i \leftrightarrow q^i, p_i, \lambda^\mu, \quad \text{with} \quad \phi_\mu(q, p) = 0,
\]  
(12)

with the following specific definition of one set of coordinates in terms of the other. For $TQ \rightarrow M_0 \times R^m$ we have

\[
p_i = \hat{p}_i(q, \dot{q}), \quad \lambda^\mu = v^\mu(q, \dot{q}),
\]  

with $v^\mu$ as defined in subsection 1.2; and for $TQ \leftarrow M_0 \times R^m$, we have

\[
\dot{q}^i = \frac{\partial H_c}{\partial p_i} + \lambda^\mu \frac{\partial \phi_\mu}{\partial p_i} = \frac{\partial H_D}{\partial p_i} \quad \text{with} \quad \phi_\mu(q, p) = 0.
\]  
(13)

The complete change of variables includes higher order time derivatives corresponding to higher order tangent structures. Thus, at the next level we have

\[
\begin{array}{c}
\text{old variables} \\
q^i, \dot{q}^i, \ddot{q}^i
\end{array} \leftrightarrow \begin{array}{c}
\text{new variables} \\
q^i, p_i, \lambda^\mu, \dot{p}_i, \dot{\lambda}^\mu
\end{array}
\]  
(14)

with $\phi_\mu(q, p) = 0$ and $\dot{\phi}_\mu(q, p, \lambda, \dot{p}) = 0$ for the new variables in the right hand side. Observe that $\dot{q}$ does not appear in the right hand side because it is substituted by use of (13). The new restrictions $\dot{\phi}_\mu$ are nothing but the time derivatives of $\phi_\mu$, that can be written, using (13), as

\[
\dot{\phi}_\mu(q, p, \lambda, \dot{p}) = \frac{\partial \phi_\mu}{\partial q} \frac{\partial H_D}{\partial p} + \dot{p} \frac{\partial \phi_\mu}{\partial p}.
\]

It is of great advantage to use, instead of $\dot{p}$, a new variable $l$ defined as

\[
l := \dot{p} + \frac{\partial H_D}{\partial q}.
\]  
(16)

When we undo the change of variables, we discover that $l = -[L] = -(\alpha - W\ddot{q})$, that is, the Euler-Lagrange derivative of $L$. This means that setting $l = 0$ will correspond
to satisfying the Euler-Lagrange equations. Summing up, we have now the change of variables

\begin{equation}
\text{old variables} \quad \longleftrightarrow \quad \text{new variables} \tag{17}
\end{equation}

\begin{align*}
q^i, \dot{q}^i, \ddot{q}^i &\quad \longleftrightarrow \quad q^i, p_i, \lambda^\mu, l_i, \dot{\lambda}^\mu, \tag{18}
\end{align*}

with \( \phi^{(0)}_\mu(q, p) := \phi_\mu(q, p) = 0 \) and \( \phi^{(1)}_\mu(q, p, \lambda, l) := \dot{\phi}_\mu(q, p, \lambda, \dot{p}) = 0, \tag{19} \)

with the new relations

\begin{align*}
\ddot{q}^i &= \left\{ \frac{\partial H_D}{\partial p_i}, H_D \right\} + l \frac{\partial H_D}{\partial p} + \dot{\lambda} \frac{\partial H_D}{\partial \dot{p} \partial \lambda},
\end{align*}

on one direction (←), and

\begin{align*}
l &= -[L] = \dot{p} - \frac{\partial L}{\partial q}, \quad \dot{\lambda} = q \frac{\partial v}{\partial q} + \dot{q} \frac{\partial v}{\partial \dot{q}},
\end{align*}

on the other (→). Restrictions \( \phi^{(1)}_\mu \) can be written as

\begin{align*}
\phi^{(1)}_\mu &= \left\{ \phi_\mu, H_D \right\} + l \frac{\partial \phi_\mu}{\partial p}, \tag{20}
\end{align*}

and they restrict the number of independent \( l \) variables in the same way as \( \phi^{(0)}_\mu := \phi_\mu \) restricts the number of independent momenta.

The next order is easily developed:

\begin{equation}
\text{old variables} \quad \longleftrightarrow \quad \text{new variables} \tag{21}
\end{equation}

\begin{align*}
q^i, \dot{q}^i, \ddot{q}^i, \dddot{q}^i &\quad \longleftrightarrow \quad q^i, p_i, \lambda^\mu, l_i, \dot{\lambda}^\mu, \dot{l}_i, \ddot{\lambda}^\mu, \ddot{l}_i, \tag{22}
\end{align*}

with \( \phi^{(0)}_\mu(q, p) = 0 \), \( \phi^{(1)}_\mu(q, p, \lambda, l) = 0 \), and \( \phi^{(2)}_\mu(q, p, \lambda, l, \dot{\lambda}, \dot{l}) = 0, \tag{23} \)

with \( \phi^{(2)}_\mu \) being the time derivative of \( \phi^{(1)}_\mu \). The change of coordinates at any order is defined along the same lines. General formulas for \( \phi^{(n)} \) will be given below.

The total time derivative operator, written as (4) in terms of the old variables, takes with the new variables the form:

\begin{align*}
\frac{d}{dt} &= \mathcal{D} + l \frac{\partial \phi_\mu}{\partial p} + \dot{l} \frac{\partial \phi_\mu}{\partial l} + \ddot{l} \frac{\partial \phi_\mu}{\partial \dot{l}} + \ldots, \tag{24}
\end{align*}

where \( \mathcal{D} \) is defined as

\begin{align*}
\mathcal{D} := \frac{D}{Dt} + \left\{ -, H_D \right\} \tag{25}
\end{align*}

with \( D / Dt \) already defined in (9). But we must still bear in mind that the variables \( p, l, \dot{l}, \) etc., are restricted by \( \phi^{(0)}_\mu = 0, \phi^{(1)}_\mu = 0, \phi^{(2)}_\mu = 0, \) etc. We call these functions “restrictions” instead of constraints because all them vanish identically when they are expressed in terms of the old variables.

Notice the advantage of using the new variables in a non-regular theory: whereas with the old variables it is difficult to set the Euler-Lagrange equations to zero –i.e., to go on shell– because these equations can not be written in normal form, it is trivial to do so when working with the new set variables –if one takes into account the restrictions appropriately.
2.2 Restrictions for the new variables, general formulas.

As it will be seen later, we do not need for our purposes the complete expressions for the restrictions but only their expansion in terms the variables \(l, \dot{l}, \ddot{l}, \ldots\), up to quadratic terms. The definition

\[
\phi^{(n+1)}_{\mu} := \frac{d}{dt} (\phi^{(n)})
\]

allows for a recursion formula that can be written as follows

\[
\phi^{(n+1)}_{\mu} = D^{n+1} \phi_{\mu} + \sum_{k=0}^{n} l^{(k)}(n,k) \phi_{\mu} + \text{(quadratic terms in } l, \dot{l}, \ddot{l}, \ldots\text{)},
\]

(26)

where \(D^{m}\) is the \(m\) times composition of the operator (25), \(\alpha(n,k)\) is a composition of operators

\[
\alpha(n,k) := \sum_{m=0}^{n-k} \left( \begin{array}{c} m + k \\ m \end{array} \right) D^{m} \circ \frac{\partial}{\partial p} \circ D^{n-k-m},
\]

that satisfies, for \(k > 0\), the relation

\[
\alpha(n,k) + \alpha(n,k - 1) = \alpha(n + 1,k),
\]

and \(l^{(k)}\) stands for the \(k\)-derivative of \(l\), with \(l^{(0)} = l\).

As we have said before, all these restrictions \(\phi^{(n)}_{\mu}\) become identically zero when they are expressed in terms of the old variables. However, if we set \(l, \dot{l}, \ddot{l}, \ldots\) to zero within \(\phi^{(n)}_{\mu}\), we get constraints,

\[
D^{n} \phi_{\mu} \approx 0,
\]

(27)

that are not identically zero in terms of the old variables —except for \(\phi_{\mu}\) itself. These constraints will play a relevant role in our developments. We use Dirac’s weak equality, \(\approx\), for them because when viewed in terms of the old variables \(q, \dot{q}, \ddot{q}, \ldots\), they are just combinations of the Euler-Lagrange equations and its derivatives. Thus, for \(n = 0\) we have \(\phi_{\mu}(q, \dot{p}) = 0\) identically; for \(n = 1\), using (20),

\[
(D^{1} \phi_{\mu})(q, \dot{p}, v_{\mu}) = [L]_{i} \frac{\partial \phi_{\mu}}{\partial p_{i}} = \alpha_{i} \frac{\partial \phi_{\mu}}{\partial p_{i}},
\]

that are the primary Lagrangian constraints (Notice [4] that \(\partial \phi_{\mu}/\partial p_{i}, \mu = 1 \cdots m\) form a basis for the null vectors of the Hessian matrix (1)), and so on.

2.3 Relation with Dirac’s constraints.

Constraints (27) are not in the form of Dirac constraints, as obtained in Dirac’s stabilization algorithm for constrained system [2], [4], [5], but they have the same content. Dirac’s algorithm is more refined than what we need here, and is able to reformulate the whole set of constraints (27) as a) some standard Dirac constraints of the type \(\psi(q, p)\), and b) the determination of some of the Lagrange multipliers \(\lambda\) as functions in phase space. Dirac’s clever trick relies in the classification of constraints as first and second class. Let us see how it works for our “secondary” constraints \(D \phi_{\mu}\).
If we distinguish, among the primary constraints $\phi_\mu$, those that are first class, $\phi_{\mu_0}$, and those second class, $\phi_{\mu'}_0$, then

$$D\phi_{\mu_0} = \{\phi_{\mu_0}, H_c\} + \lambda^\nu \{\phi_{\mu_0}, \phi_\nu\},$$

and since the first class condition makes the second piece, $\{\phi_{\mu_0}, \phi_\nu\}$, to vanish in the surface of the primary constraints, we are left with the secondary Dirac constraints

$$\phi^1_{\mu_0} := \{\phi_{\mu_0}, H_c\}.$$

On the other hand, the requirement

$$D\phi_{\mu'}_0 = \{\phi_{\mu'}_0, H_c\} + \lambda^\nu \{\phi_{\mu'}_0, \phi_\nu\} = 0,$$

allows for the canonical determination of $\lambda^\nu_0$ as a function in phase space, $\lambda^\nu_0 = \lambda^\nu_0(c(q,p))$, because the matrix $\{\phi_{\mu'}_0, \phi_\nu\}$ is regular as implied by the second class condition. In the standard Dirac’s method, this determination of some of the variables $\lambda$ is then introduced into the dynamics and the operator $D$ is modified to

$$D' := \frac{D'}{Dt} + \{-, H'_D\}$$

where $H'_D = H'_c + \lambda^\nu_0 \phi_{\mu_0}$ and $H'_c = H_c + \lambda^\nu_0 \phi_{\mu'}_0$. Now $\frac{D'}{Dt}$ is the adaptation of equation (9) to the Lagrange multipliers that are left undetermined, that is, $\lambda^\nu_0$.

Once the dynamics has been adapted to the partial knowledge of the Lagrange multipliers, we only need to care about the new constraints $\phi^1_{\mu_0}$ and require $D'\phi^1_{\mu_0} = 0$ as a new set of constraints. Then the whole mechanism starts again. This is Dirac method.

For what the constraints (in the enlarged formalism) of the form $\lambda^\nu_0 - \lambda^\nu_0(c(q,p)) = 0$ are concerned, the application of the new evolution operator $D'$ leads to new constraints involving the time derivative of $\lambda^\nu_0$, that is,

$$\dot{\lambda}^\nu_0 - \{\lambda^\nu_0(c(q,p)), H'_D\} = 0,$$

and so on. This information is irrelevant from the point of view of the Dirac method because the variables $\lambda^\nu_0$ have been substituted by their canonical determinations $\lambda^\nu_0$ and thus have disappeared from the formalism.

### 3 Reformulation of the Noether condition with the new variables.

Consider a theory defined by a Lagrangian $L$ and with a Noether transformation $\delta^L q$ of the type (2) for which there exists a function $G^L(q, \dot{q}, \ddot{q}, ...; t)$ satisfying (5). Now we express $\delta^L q$ and $G^L$ in terms of the new variables. We end up with (for definiteness we fix the highest order arguments, the superscript $E$ stands for reference to the enlarged formalism)

$$\delta^E q(q, p, \lambda, \dot{\lambda}, \ddot{\lambda}, ..., \lambda^{(I+1)}, \lambda^{(I+2)}), \quad G^E(q, p, \lambda, \dot{\lambda}, \ddot{\lambda}, ..., \lambda^{(I)}, \lambda^{(I+1)}), \quad (28)$$
such that

\[
\delta^L q(q, \dot{q}, \ddot{q}, \ldots; t) = \delta^E q(q, \dot{p}, v^\mu, -[L], \frac{dL_v}{dt}, \ldots; t),
\]

(29)

\[
G^L(q, \dot{q}, \ddot{q}, \ldots; t) = G^E(q, \dot{p}, v^\mu, -[L], \frac{dL_v}{dt}, \ldots; t).
\]

(30)

Notice that \( \delta^E q \) and \( G^E \) are not uniquely defined, for we can add to them any linear combination of the restrictions \( \phi^{(n)}_\mu \) with no effect in \( \delta^L q \) and \( G^L \). These arbitrariness will be fixed below at their due moment. Expression (5) now becomes, in terms of the new variables,

\[
-l\delta^E q + \frac{dG^E}{dt} = restr_{pc},
\]

where \( restr_{pc} \) stands for an equality under the restrictions \( \phi^{(n)}_\mu = 0, n = 0, 1, \ldots, f + 2 \). According to the theory of Lagrange multipliers, this means that there exist functions \( a^n_\mu \) such that

\[
-l\delta^E q + \frac{dG^E}{dt} = \sum_{n=0}^{f+2} a^n_\mu \phi^{(n)}_\mu,
\]

(31)

with all the variables now taken as independent.

Notice that, for a given \( n > 0 \),

\[
a^n_\mu \phi^{(n)}_\mu = a^n_\mu \frac{d}{dt} \phi^{(n-1)}_\mu = \frac{d}{dt} \left( a^n_\mu \phi^{(n-1)}_\mu \right) - \left( \frac{d}{dt} a^n_\mu \right) \phi^{(n-1)}_\mu.
\]

(32)

Therefore, if we define a new set of functions,

\[
b^m_\mu = \sum_{n=m+1}^{f+2} (-1)^{n-m-1} a^n_\mu \phi^{(n-1)}_\mu,
\]

for \( m = 0, \ldots, f + 1 \) and with \( a^n_\mu \) being the \( k \)-th time derivative of \( a^n_\mu \), then (31) can be rewritten as

\[
-l\delta^E q + \frac{d}{dt} \left( G^E + \sum_{m=0}^{f+1} b^m_\mu \phi^{(m)}_\mu \right) = restr_{pc}.
\]

Here \( pc \) stands, as usual, for a linear combination of the primary constraints \( \phi_\mu \) that define the surface \( M_0 \in T^*Q \). Now we take advantage of the arbitrariness of the choice of \( G^E \) under the addition of pieces linear in the restrictions \( \phi^{(n)}_\mu \) to absorb the piece \( \sum_{m=0}^{f+1} b^m_\mu \phi^{(m)}_\mu \) into \( G^E \). We can conclude that a function \( G^E \) exists, satisfying (30), such that, together with a given function \( \delta^E q \) satisfying (29), the following relation holds:

\[
-l\delta^E q + \frac{d}{dt} G^E = restr_{pc}.
\]

(33)

Notice that (33) reduces to (8) when we enforce \( l = 0 \) into it. Let us expand (33) up to quadratic terms in \( l \) and its time derivatives. Defining the expansions for \( \delta^E q \) and \( G^E \),
as
\[ \delta^E q = \delta_0 q + \text{(linear terms in } l, \dot{l}, ...) \],
\[ G^E = G^c + \sum_{k=0}^f l^{(k)} G_k + \text{(quadratic terms in } l, \dot{l}, ...) \],
\[ (34) \]

(now \( \delta_0 q, G^c \), and \( G_k \) only depend on \( q, p, \lambda, \dot{\lambda}, \ldots \), that is, they do not depend on the variables \( l^{(k)} \)) we get, for (33),
\[ -l \delta_0 q + \frac{d}{dt} \left( G^c + \sum_{k=0}^f l^{(k)} G_k \right) + \text{(quadratic terms in } l, \dot{l}, ...) = pc. \]
\[ (36) \]

Using (24), (36) gives
\[ -l \delta_0 q + \frac{d}{dt} \left( G^c + \sum_{k=0}^f l^{(k)} G_k \right) + \text{(quadratic terms in } l, \dot{l}, ...) = pc, \]
\[ (37) \]

with the new definitions \( G_{f+1} := 0 \) and \( G_{-1} := 0 \). Now, isolating in (37) the pieces with no dependence on \( l \), we get
\[ DG^c = pc, \]
which is nothing but the condition (8) for a function \( G^c(q, p, \lambda, \dot{\lambda}, \ldots) \) to be a canonical generator of a Noether transformation within the enlarged formalism. The expansion for \( l^{(k)} \) in (37) gives, for \( k > 0 \),
\[ DG_k + G_{k-1} = c^\mu_k \phi_\mu, \]
for some functions \( c^\mu_k \). Notice then that this result, together with the identity
\[ G_0 = \sum_{k=0}^f (-1)^{(k)} D^k (DG_{k+1} + G_k), \]
allows to write \( G_0 \) as
\[ G_0 = \sum_{k=0}^f (-1)^{(k)} D^k (c^\mu_k \phi_\mu), \]
that is, \( G_0 \) is a combination of the constraints \( D^k \phi_\mu \) introduced in (27). Finally, the expansion for \( l \) in (37) gives
\[ -\delta_0 q + \frac{\partial G^c}{\partial p} + DG_0 = c^\mu_0 \phi_\mu, \]
for some functions \( c^\mu_0 \). But since \( G_0 \) is a combination of constraints, so it is \( DG_0 \), and therefore we arrive at
\[ \delta_0 q = \frac{\partial G^c}{\partial p} - \sum_{n=0}^{f+1} d^\mu_n D^n \phi_\mu, \]
\[ (38) \]
for some functions $d_n^\mu$. Now, as we did before with $G^E$, we can use the arbitrariness that equation (29) allows for the choice of $\delta^E q$ and add to it a linear combination of the restrictions, namely $d_n^\mu(n)$. The new $\delta_0 q$ deduced from the expansion (34) for the new $\delta^E q$ will absorb the piece $d_n^\mu(n)|_{(k)=0} = d_n^\mu D^\mu \phi_n$ in (38) and therefore it will satisfy,

$$\delta_0 q = \frac{\partial G^c}{\partial p} = \{q, G^c\}. \quad \text{(39)}$$

Equation (39) is the main result of our paper. To summarize, we have considered, as a starting point, a Noether transformation $\delta L q$ satisfying (3) of the most general type (2), and we have obtained the objects $\delta_0 q$ and $G^c$ in the enlarged formalism such that they satisfy the Noether condition (8) and such that $\delta_0 q$ is canonically generated by $G^c$, according to (39). The results of [1] guarantee that $\delta_0 q|_{\lambda^\mu = v^\mu(q,\dot{q})}^2$ is a Noether transformation for the Lagrangian $L$.

Taking into account equations (29) and (30), and the expansions (34) and (35), we observe that this new Noether transformation $\delta_0 q|_{p=\hat{p}, \lambda^\mu = v^\mu(q,\dot{q})}$ differs from the original one, $\delta^L q$, at most by a combination of the equations of motion, $[L]$, and its time derivatives. They coincide on-shell. The same is true for the relation between $G^L$ and $G^c|_{p=\hat{p}, \lambda^\mu = v^\mu(q,\dot{q})}$.

Our conclusion can be stated as follows:

**Theorem 1** Given a first order Lagrangian $L(q, \dot{q})$, and a generalized Noether transformation for it,

$$\delta^L q(q, \dot{q}, \ddot{q}, \ldots; t),$$

with its associated conserved quantity

$$G^L(q, \dot{q}, \ddot{q}, \ldots; t),$$

there always exists a Noether transformation,

$$\delta_0 q(q, p, \lambda, \dot{\lambda}, \ldots; t),$$

and a conserved quantity,

$$G^c(q, p, \lambda, \dot{\lambda}, \ldots; t),$$

in the enlarged formalism (with Lagrangian $L_c$) such that

$$\delta_0 q = \frac{\partial G^c}{\partial p},$$

and that

$$\delta^L_0 q := \delta_0 q|_{p=\hat{p}, \lambda^\mu = v^\mu(q,\dot{q})},$$

and

$$G^L_0 := G^c|_{p=\hat{p}, \lambda^\mu = v^\mu(q,\dot{q})},$$

$^2$Substitution of $\lambda, \dot{\lambda}, \ldots$ is also understood in this kind of expressions.
are a Noether transformation and its associated conserved quantity for the Lagrangian \( L \), and that they differ from \( \delta^L q \) and \( G^c \), respectively, by terms that are, at most, a combination of the equations of motion and its time derivatives – that is, they coincide on shell.

This theorem can be summarized as follows:

Any Noether symmetry \( \delta^L q(q, \dot{q}, \ddot{q}, ...; t) \) can be expressed, up to terms that vanish on-shell, as a canonical transformation in a phase space enlarged with the variables \( \lambda, \dot{\lambda}, \ddot{\lambda}, ... \), where \( \lambda \) are the Lagrange multipliers associated with the primary constraints.

4 Application to a regular theory.

Now we can give an answer to the question raised at the beginning of section 2. In the case of a regular theory, that is, when the Hessian (1) is regular, there are no constraints and hence the variables \( \lambda \) in the enlarged formalism do not appear. This means that \( \delta_0 q \) will be only \( \delta_0 q(q, p; t) \) and \( G^c \) will be \( G^c(q, p; t) \). Since the Euler-Lagrange equations \( [L] = 0 \) allow for isolating

\[
\ddot{q}^i = (W^{(-1)})^{ij} \alpha_j,
\]

what we find is that

\[
\delta^L q|_{\ddot{q}=(W^{(-1)})_\alpha} = \delta_0 q|_{p=\dot{p}},
\]

where it is understood that the higher time derivatives in \( \delta^L q \) are also substituted by lower ones by the repeated use of (40) and its time derivatives.

It is also true that, in this case,

\[
G^L|_{\ddot{q}=(W^{(-1)})_\alpha} = G^c|_{p=\dot{p}}.
\]

We can conclude that in a regular theory defined by a first order Lagrangian, the most general Noether transformation coincides on-shell with a Noether transformation that is canonically generated in phase space.

5 Revisiting the Noether identities.

According to the results of the previous sections, the most general continuous Noether transformations for a given Lagrangian is associated, up to constraints of the formalism, to a generator \( G(q, p, \lambda, \dot{\lambda}, \ddot{\lambda}, ...; t) \) \( (G^c \text{ in section 3}) \) that satisfies (8),

\[
DG = pc.
\]

Consider now the particular case of a gauge transformation, this means that \( G \) can be expanded as

\[
G = \sum_{i=0}^{M} \epsilon^{(i)} G_i,
\]
with $\epsilon^{(0)}$ an arbitrary function and

$$\epsilon^{(i+1)} = \frac{d}{dt}(\epsilon^{(i)}).$$

Plugging this expansion into (8) we get

$$G_M = pc,$$

$$G_{i-1} + DG_i = pc, \quad i = 1, \ldots, M$$

$$DG_0 = pc.$$  \hfill (43)

If we define $G_{M+1} = G_{-1} = 0$, this expression can be summarized as $G_{i-1} + DG_i = pc$ for $i = 0, \ldots, M + 1$. Then, taking into account (24), we can define the quantities $K_i$ as

$$K_i := G_{i-1} + \frac{d}{dt}(G_i) - l \frac{\partial G_i}{\partial p} = pc = e^\mu \phi_\mu,$$  \hfill (44)

for some coefficients $e^\mu$. The quantities $K_i$ are therefore primary constraints. Using (44), we can construct the following relation

$$\sum_{i=0}^{M+1} (-1)^i \left(\frac{d}{dt}\right)^i (K_i) = \sum_{i=0}^{M+1} (-1)^i \left(\frac{d}{dt}\right)^i (G_{i-1} + \frac{d}{dt}(G_i))$$

$$- \sum_{i=0}^{M} (-1)^i \left(\frac{d}{dt}\right)^i (l \frac{\partial G_i}{\partial p}).$$  \hfill (45)

Since the first piece in the right hand side of (45), vanishes identically, use of (44) yields

$$\sum_{i=0}^{M} (-1)^i \left(\frac{d}{dt}\right)^i (l \frac{\partial G_i}{\partial p}) = e^{\mu}_n \phi^{(n)}_\mu,$$

for some coefficients $e^{\mu}_n$. Now we can take advantage of the fact that in terms of the old variables, the restrictions $\phi^{(n)}_\mu$ vanish identically and the variables $l$ become the Euler-Lagrangian equations $-[L]$, to get

$$\sum_{i=0}^{M} (-1)^i \left(\frac{d}{dt}\right)^i ([L]f_i) = 0,$$  \hfill (46)

identically, where

$$f_i := \left. \frac{\partial G_i}{\partial p} \right|_{p=p, \lambda^\nu = \nu^\nu(q, \dot{q})}.$$

Equation (46) is the Noether identity corresponding to the gauge transformation

$$\delta q = \frac{\partial G}{\partial p} = \sum_{i=0}^{M} \epsilon^{(i)} f_i.$$
6 Example

Here we will consider as an example an equivalent formulation of the conformal particle, first introduced in [6] and developed in [7]. This system has attracted attention recently [8] by gauging the $Sp(2,\mathbb{R})$ invariance of a zero Hamiltonian system whose canonical Lagrangian gives an equivalent way to define the conformal particle. It is then possible to write down an action that have interesting properties like rigid $SO(2,D)$ symmetry and a rich structure in the reduced phase space. This properties has been used recently in connection with string theory [9], AdS space time and in connection to the so called M-theory [10]. As another application the system can be used for isometric embedding of BPS branes in flat spaces with two times [11]. Here we will consider the equivalent system defined by the Lagrangian

$$L = \frac{\dot{X}^2}{2e} - \frac{1}{2}X^2,$$  \hspace{1cm} (47)

in the configuration space defined by the variables $(e, X^M), M = 0, 1, 2, ... D$ in a space-time with “two times” with metric $\eta = \text{diag}(-1, -1, 1, ..., 1)$. These “two times” are necessary to solve the constraints without ghosts. A generalized Lagrangian symmetry for this system is

$$\delta^L X = -2 \left( \frac{\dot{X}}{e} \right) \epsilon - \left( \frac{\dot{X}}{e} \right) \dot{\epsilon} + \frac{\ddot{X}}{e} \epsilon, \quad (48)$$

and

$$\delta^L e = \ddot{\epsilon} + 4\dot{\epsilon} + 2\epsilon \dot{\epsilon},$$

whose associated Lagrangian generator is

$$G^L = -\epsilon \left( \frac{1}{e} \ddot{X} + X - \frac{1}{e^2} \dot{e} \dot{X} \right)^2 + (\ddot{\epsilon} + 2\epsilon \dot{\epsilon}) \frac{\dot{X}^2}{2e^2} + \frac{\dot{\epsilon}}{e} X \cdot \dot{X} + \epsilon X^2,$$  \hspace{1cm} (49)

where $\epsilon$ is an arbitrary parameter and the dot is a short notation for $\frac{d}{dt}$ (see eq. (4)). This generator, $G^L$, and its associated symmetries $\delta^L e, \delta^L X$ satisfy (5). Notice that (48) and (49) depends on higher order derivatives of the variables $X^M$ with respect to time. Nevertheless, we will show that it is possible to associate this generalized Lagrangian symmetry with a canonical symmetry in the enlarged phase space (containing the Lagrange multipliers as variables). In turn, this canonical symmetry generates the correct $Sp(2,\mathbb{R})$ symmetry in the sector $(X^M, P^M)$ of the initial conditions in the enlarged phase space.

To construct the associated canonical formalism for the Lagrangian (47) we start from the definition of the canonical momenta

$$P_e = 0, \quad P = \frac{\dot{X}}{e},$$

so, $P_e = 0$ is a primary constraint. The Dirac Hamiltonian is

$$H_D = \frac{1}{2} e P^2 + \frac{1}{2} X^2 + \lambda P_e,$$
where $\lambda$ is the Lagrange multiplier associated with the primary constraint $P_e = 0$ whose successive stabilization produces the Dirac first class constraints $P^2 = 0$, $X \cdot P = 0$ and $X^2 = 0$ that close under $Sp(2,R)$ Lie algebra. To associate with the given Lagrangian symmetry a symmetry in the enlarged space we will use the change of variables introduced in section 2. The dictionary is given by

$$
\dot{e} = \lambda, \quad \dot{X} = eP,
$$

$$
\ddot{e} = \dot{\lambda}, \quad \ddot{X} = \lambda P + e(-X + l_X),
$$

$$
\dddot{e} = \ddot{\lambda}, \quad \dddot{X} = \lambda P + 2\lambda(-X + l_X) + e(-eP + \dot{l}_X),
$$

where

$$
l_X = \dot{P} + \frac{\partial H_D}{\partial X},
$$

(see eq. (16)). Using this change of variables we construct the objects (28), that satisfy (29) and (30) and, consequently, (31). With the appropriate addition of restrictions we can prepare $\delta^E q$ and $G^E$ to satisfy (33). The result is

$$
G_E = G^c - \epsilon l_X^2,
$$

with

$$
G^c = (\dot{\epsilon} + 4\epsilon \dot{\epsilon} - 2\epsilon \lambda)P_e + \frac{1}{2}(\dot{\epsilon} + 2\epsilon \dot{\epsilon})P^2 + \dot{\epsilon} X \cdot P + \epsilon X^2,
$$

and

$$
\delta^E X = (\dot{\epsilon} + 2\epsilon \dot{\epsilon})P + \dot{\epsilon} X - 2\dot{\epsilon} l_X - \epsilon l_X,
$$

$$
\delta^E \epsilon = \ddot{\epsilon} + 4\epsilon \dot{\epsilon} + 2\lambda \epsilon.
$$

Now we can apply our theorem as stated in section 3 by noticing that the sector that does not depends on $l_X$ in (50) generates a canonical symmetry that will reduce to the original symmetry in the $n$-th tangent space up to constraints. Indeed,

$$
\delta_0 X = \{X, G^c\} = (\dot{\epsilon} + 2\epsilon \dot{\epsilon})P + \dot{\epsilon} X,
$$

$$
\delta_0 \epsilon = \{\epsilon, G^c\} = \ddot{\epsilon} + 4\epsilon \dot{\epsilon} + 2\lambda \epsilon,
$$

is a canonical symmetry for the system under consideration. This can be checked explicitly by noticing that $G^c$ solves (8) in the phase space. When this symmetry is displayed in the initial conditions surface, the arbitrary function $\epsilon$ and its derivatives $\dot{\epsilon}$, $\ddot{\epsilon}$, $\dddot{\epsilon}$ become all of them arbitrary parameters. The gauge symmetry contains then three independent generators that form an $Sp(2,R)$ algebra in the $(X, P)$ sector of phase space that corresponds to the algebra of the first class constraints $P^2, P \cdot X$ and $X^2$, as expected.

For completeness we list the restrictions associated with this problem and the Noether identity that follows from them. The first four restrictions are

$$
\phi^{(0)} = P_e,
$$

$$
\phi^{(1)} = \frac{1}{2}P^2 + l_e = 0,
$$

$$
\phi^{(2)} = X \cdot P - P \cdot l_X + \dot{l}_e,
$$

$$
\phi^{(3)} = eP^2 - X^2 + 2X \cdot l_X - l_X^2 - P \cdot \dot{l}_X + \dot{l}_e.
$$
Notice that the Dirac constraints can be obtained from these restrictions by enforcing $l_c$ and $l_X$ to zero. The Noether identity is

$$2\dot{e}[L]_e + 4e[\dot{L}]_e + 3X \cdot [\dot{L}]_X - 3[\dot{L}]_X \cdot [L]_X - \frac{\ddot{X}}{e} \cdot [\ddot{L}]_X + [\ddot{L}]_e = 0.$$  

This identity can be obtained as follows. First we get the stabilization of the last restriction under the evolution operator (24), and then we substitute the terms that contain Dirac constraints by use of the previous restrictions and taking into account that $l_X = -[L]_X$, $l_c = -[L]_e$ and $\lambda = \dot{e}$. Another way to do the same is by noticing that the combination of restrictions given by $\phi^{(4)} + 2\lambda \phi^{(1)} + 4e\phi^{(2)}$, is precisely the Noether identity just mentioned, after performing the substitution $l_X = -[L]_X$, $l_c = -[L]_e$ and $\lambda = \dot{e}$. Notice that the Lagrangian generalized symmetry that we started from can be constructed by using this Noether identity.

7 Appendix

Let us establish the conditions for a function $G^c(q, p, \lambda, \dot{\lambda}, ..., t)$ to be a Noether generator for the Lagrangian $L_c$, under the definitions

$$\delta_c q^i = \{q^i, G^c\}, \quad \delta_c p_i = \{p_i, G^c\}, \quad (55)$$

and with $\delta_c \lambda^\mu$ to be determined below.

Compute $\delta_c L_c$,

$$\delta_c L_c = \frac{d}{dt}(p_i \delta_c q^i) - \dot{p}_i \delta_c q^i - \delta_c H_c - \lambda^\mu \delta_c \phi_\mu - (\delta_c \lambda^\mu) \phi_\mu$$

$$= \frac{d}{dt}(p_i \delta_c q^i) - \{H_c, G^c\} - \lambda^\mu \{\phi_\mu, G^c\} - (\delta_c \lambda^\mu) \phi_\mu$$

$$= \frac{d}{dt}(p_i \delta_c q^i) - \frac{dG^c}{dt} + \frac{\partial G^c}{\partial q^i} \frac{\partial}{\partial p_i} - \{H_c, G^c\} - \lambda^\mu \{\phi_\mu, G^c\} - (\delta_c \lambda^\mu) \phi_\mu$$

$$= \frac{d}{dt}(p_i \delta_c q^i - G^c) + \frac{dG^c}{D_t} + \{G^c, H_c\} - \lambda^\mu \{\phi_\mu, G^c\} - (\delta_c \lambda^\mu) \phi_\mu.$$

If we require

$$\mathcal{D} G^c := \frac{dG^c}{D_t} + \{G^c, H_D\} = pc, \quad (56)$$

and if we represent this combination $pc$ of primary constraints as $pc = C^\mu \phi_\mu$, then the definition

$$\delta_c \lambda^\mu = C^\mu, \quad (57)$$

makes $\delta_c L_c = \frac{d}{dt}(p \delta_c q - G^c)$, that is, a Noether transformation for the enlarged formalism. (56) is the result (8) we were looking for.
References


