We propose a probabilistic two-party communication complexity scenario with a prior nonmaximally entangled state, which results in less communication than that is required with only classical random correlations. A simple all-optical implementation of this protocol is presented and demonstrates our conclusion.

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Quantum mechanics provides novel features to quantum system, extending the capabilities beyond that achievable with system based solely on classical physics. The most prominent examples to date have been quantum computation [1–4], quantum teleportation [5–7], superdense coding [8,9], and quantum cryptography [10–12], all of which have been demonstrated in experiment.

Recently, there has been much interest in using quantum resource to reduce the communication complexity [13–21]. The communication complexity of a function \( f : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{0, 1\} \) is defined as the minimum amount of communication necessary between two parties, conventionally referred to as Alice and Bob, in order for both parties to acquire the value of \( f \). Cleve and Buhrman introduced the first example of the quantum communication complexity scenario [14]. In their model, Alice and Bob have an initial supply of particles in entangled quantum state, such as EPR pairs. They have shown that although entanglement by itself cannot be used to transmit a classical message, it can reduce the communication complexity [22].

In this Letter, an example of a two-party probabilistic communication complexity scenario is presented in the entanglement model [22] which is also realized in an optical system.

Suppose Alice and Bob receive \( x \) and \( y \), respectively, where \( x, y \in U = \{0, 1\}^2 \), and \( x, y \) may be represented in binary notation as \( x_1x_0, y_1y_0 \). The common goal is for each party to learn the value of the Boolean function

\[
 f(x, y) = x_1 \oplus y_1 \oplus (x_0 \land y_0)
\]

after two bits of classical communications with as high probability as possible. If and only if the values determined by Alice and by Bob are both correct, an execution is considered successful.

In the entanglement model, Alice and Bob initially share an entanglement of two qubits,

\[
 |AB\rangle = \alpha |00\rangle + \beta |11\rangle,
\]

where \( \alpha, \beta \) are promised that \( \alpha^2 + \beta^2 = 1 \), and \( |\alpha| > |\beta| \) (assume that \( \alpha, \beta \) are real).

The idea is based on applying CHSH theorem [23] to enable Alice and Bob to obtain bits \( a \) and \( b \) such that \( a \oplus b = x_0 \land y_0 \) is satisfied with certain probability

\[
 \Pr[a \oplus b = x_0 \land y_0] = P(\alpha, \beta).
\]

Bits \( a \) and \( b \) are achieved by the following operations. Suppose \( R(\chi) \) is the rotation by angle \( \chi \) which is represented in the standard basis as

\[
 R(\chi) = \begin{pmatrix} \cos \chi & -\sin \chi \\ \sin \chi & \cos \chi \end{pmatrix},
\]

if \( x_0 = 0 \), Alice applies rotation \( R(\phi_1) \) on qubit \( A \), i.e., her part of the entangled state, else she applies \( R(\phi_2) \) on it, and then measures \( A \) in the standard basis to yield bit \( a \). Similarly, due to the symmetry of entangled states, if \( y_0 = 0 \), Bob applies \( R(\phi_1) \) on the qubit \( B \), else he applies \( R(\phi_2) \), and then measures the qubit \( B \) to yield bit \( b \). Whereas, local rotations \( R(\chi_1) \otimes R(\chi_2) \) applied on the entangled state \( |AB\rangle \) result in the state

\[
 |AB\rangle' = (\alpha \cos \chi_2 \cos \chi_1 + \beta \sin \chi_2 \sin \chi_1) |00\rangle + (\alpha \sin \chi_2 \cos \chi_1 - \beta \cos \chi_2 \sin \chi_1) |01\rangle + (\alpha \cos \chi_2 \sin \chi_1 - \beta \sin \chi_2 \cos \chi_1) |10\rangle + (\alpha \sin \chi_2 \sin \chi_1 + \beta \cos \chi_2 \cos \chi_1) |11\rangle.
\]

After these operations, Alice sends \((a \oplus x_1)\) to Bob, and Bob sends \((b \oplus y_1)\) to Alice, then each party can determine the value

\[
 (a \oplus x_1) \oplus (b \oplus y_1) = x_1 \oplus y_1 \oplus (a \oplus b) = x_1 \oplus y_1 \oplus (x_0 \land y_0)
\]

with probability \( P(\alpha, \beta) \).

The process of the communication is shown in Table I.

Table I: The input of \( x_0y_0 \), the corresponding local rotations, the component of \( |AB\rangle' \) for successful communication and the result of Boolean function.
According to Table I and Eq. (5), the total success probability of the communication is

\[
P(\alpha, \beta) = \frac{1}{2} + \frac{1}{4} \cos 2\phi_1 \cos 2\phi_2 + \frac{1}{2} \alpha \beta \sin 2\phi_1 \sin 2\phi_2
\]

(7)

Then we can yield the maximum probability \(P_{\text{max}}\)

\[
P_{\text{max}}(\alpha, \beta) = \frac{1}{2} + \frac{1}{4} \sqrt{1 + 4\alpha^2 \beta^2}
\]

(8)

if and only if

\[
\phi_1 = -\frac{1}{4} \arccos \left(\frac{1 + 2\alpha \beta - T}{1 - 2\alpha \beta}\right)
\]

(9)

\[
\phi_2 = \frac{1}{4} \arccos \left(\frac{1 + 2\alpha \beta + T}{1 - 2\alpha \beta}\right)
\]

(10)

where \(T = \frac{4\alpha \beta}{\sqrt{1 + 4\alpha^2 \beta^2}}\).

If the two parties have previously shared an EPR pair,

\[
|\Phi^-\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)
\]

(11)

i.e., \(2\alpha \beta = -1\), the success probability of communication \(P_{\text{max}}(\alpha, \beta) = \frac{1}{2} + \frac{\sqrt{2}}{4} = \cos^2 \left(\frac{\pi}{8}\right)\), and \(\phi_1 = -\frac{\pi}{16}\), \(\phi_2 = \frac{3\pi}{16}\), which is just the result of Ref. [22].

This process may also be completed in another way. As we know, the nonmaximally entangled state \(|AB\rangle = \alpha |00\rangle + \beta |11\rangle\) (here \(\beta < 0\)) can be concentrated to EPR state \(|\Phi^-\rangle\) with probability \(2\beta^2\) [24,25]. If the concentration fails we obtain a product state which is similar to a classical state. Whereas, with shared random correlation instead of entanglement, the success probability cannot exceed \(\frac{3}{4}\) (see below for detailed description). The communication is accomplished after the concentration, therefore the total success probability of the communication is

\[
P = \frac{3}{4} + \left(2 \cos^2 \frac{\pi}{8} - \frac{3}{2}\right) \beta^2.
\]

(12)

This function \(P\) has a similar curve with \(P_{\text{max}}\) (shown in Fig. 2), but, obviously, the latter has higher probability, that is the protocol presented here is superior to the protocol using entanglement concentration, the reason lies in that some entanglement is wasted during the unitary-reduction process of concentration.

Buhrman et al. prove that in this two-bit protocol with a shared random correlation instead of entanglement pair, the probability is no more than \(\frac{3}{4}\) with method of “protocol tree” which represents any two-bit protocol as a binary tree of depth two with non-leaf nodes labelled A and B [22]. In our case, from Eq. (8), we can verify this result with “limit analysis method”. If \(\alpha \beta\) tends to 0, \(|AB\rangle\) tends to classical two-bit correlation, \(|00\rangle\) or \(|11\rangle\). It is obvious that the probability \(P_{\text{max}}(\alpha, \beta)\) cannot exceed \(\frac{3}{4}\).

Experimental realization of this quantum communication complexity scenario necessitates both production and manipulation of nonmaximally entangled states. Thus far there are only a few experimental techniques by which one can prepare nonmaximally entangled states [26–30].

States with a fixed degree of entanglement, \(\varepsilon \approx \frac{1}{3}\), have been deterministically generated in ion traps [26], and via postselection, nonmaximally entangled states were controllably generated in several optical experiments [27,28]. White et al. use a new kind of ultrabright source of polarization-entangled photons to produce nearly pure entangled states without postselection [30], and we use the same method in this experiment. The \(\sim 1.7\)-mm-long pump beam at 351.1 nm (100 mW, single frequency) was produced by an Ar+ laser (Sabra, COHERENT), and directed to the two BBO crystals (5.0 \(\times\) 5.0 \(\times\) 0.59 mm), which are aligned so that their optic axes lie in planes perpendicular to each other. Polarization-entangled photons are produced with type I spontaneous down-conversion process. The twin photons at 702.2 nm are emitted into a cone of half-opening angle 3.0°. Both maximally and nonmaximally entangled states are produced simply by rotating the pump polarization. For a polarization angle of \(\chi\) with respect to the vertical, the output state is \(|\Psi\rangle = (|HH\rangle + \varepsilon e^{i\phi} |VV\rangle) / \sqrt{1 + \varepsilon^2}\), where \(H\) and \(V\), respectively, represent the horizontal and vertical polarizations of two separated photons, and \(\Phi\) is adjusted via the tilttable zero-order ultraviolet quarter-wave plate (UV QWP), and the degree of entanglement, \(\varepsilon = |\tan \chi|\) [30].

In this experimental realization, an entangled pair of photons is produced in the nonmaximally entangled state \(|AB\rangle = \alpha |HH\rangle + e^{i\Phi} \beta |VV\rangle\), where we replace the state \(|00\rangle + |\beta |11\rangle\) in Eq. (2) by \(|HH\rangle + e^{i\Phi} \beta |VV\rangle\), i.e., we choose horizontal and vertical polarizations as the basis, and \(\alpha = \cos \chi, \beta = \sin \chi\). By tilting the UV QWP, the phase angle \(\Phi\) is tuned to 0. To show that our scenario is universal for a complete range, a set of nonmaximally entangled states has to be achieved by varying \(\chi\) from \(-45^\circ\) to \(0^\circ\) (adjusting the UV HWP). Any local rotation of the polarization (such as \(R(\phi_1), R(\phi_2)\)) is realized with two HWPs, whose axes are properly oriented, in the corresponding down-conversion beam.

For each nonmaximally entangled state, there are four
kinds of classical inputs $x_0y_0 = 00, 01, 10, 11,$ with equal probability $\frac{1}{4}$. According to the input of $x_0y_0$, the state $|AB\rangle$ is rotated by adjusting the HWP’s in each down-conversion beam (reffering to Table I). Then with the polarizing beam splitters (PBS), the resulting state $|AB\rangle'$ can be measured to yield the bits $a$ and $b$, where $a(b) = 0$ or $1$, corresponding to the horizontal or vertical polarization of each photon. The bit $a$ is detected with detector 1 and 2 $(D_1$ and $D_2)$; whereas, the bit $b$ is detected with $D_3$ and $D_4$. Each detector assembly comprises an iris and a narrow band interference filter $(702 nm \pm 2 nm)$, to reduce background and select (nearly) degenerate photons; a $40 \times$ lens to collect the photons; and a single-photon counter (EG&G SPCM-AQR-16-FC), with efficiency of $\sim 70\%$ and dark count rates no more than $25 \text{ s}^{-1}$. The detector outputs are recorded singly, and in coincidence using a time to amplitude converter (TAC) and a signal-channel analyzer (SCA). A coincidence window of 5 ns was sufficient to capture true coincidences. Compared to the typical true coincidences of $30 \text{ s}^{-1}$, the “accidental” coincidence rate is negligible ($< 0.01 \text{ s}^{-1}$).

In this experiment, for each classical input of a nonmaximally entangled state, we detect four coincidences of which two are corresponding to the process of successful communications. According to Table I, if $x_0y_0 = 00$, 01, or 10, it is successful communication when we detect a $D_1D_3$ coincidence (between detectors 1 and 3) or a $D_2D_4$ coincidence (between detectors 2 and 4); whereas, if $x_0y_0 = 11$, it is a successful communication when we detect a $D_1D_4$ or $D_2D_3$ coincidence. Then we can obtain the total success probability of communication for every nonmaximally entangled state.

With this source, we attain probabilities of better than 98%, when the photons are created in the maximally entangled state. As Fig. 2 shows, across a wide range of entanglement there is good agreement between the experimental result of success probabilities and the theoretical predictions of Eq. (8). The error is about $\pm 2.8\%$.

According to Table I, for the classical input $x_0y_0 = 11$, the output for successful communication i.e., $|01\rangle$ or $|10\rangle$, is different from that $(|00\rangle$ or $|11\rangle$) for the classical input $x_0y_0 = 00$, 01 or 10. If the previously shared entanglement is EPR state, the success probabilities for the four kinds of input are identical, i.e., equal to $\cos^2 \left( \frac{\pi}{8} \right)$. However, with the entanglement decrease to 0, the success probability for $x_0y_0 = 11$ decrease gradually to 0, whereas, the success probabilities for $x_0y_0 = 00, 01$ or 10 increase gradually to 1, consequently, the overall success probability tends to $\frac{3}{4}$. This kind of symmetry-broken to reduce the communication complexity is based entirely on quantum nonlocality and also been testified in our experiment.

From Eq. (8), the success probability of communication is a function of $|\alpha\beta\rangle$, and every probability corresponds to a nonmaximally entangled pure state. The larger $|\alpha\beta\rangle$, the higher probability. From this point of view the monotonicity of the probability of communication in the present protocol may be regarded as a kind of entanglement monotone [31] of a single copy of arbitrary nonmaximally entangled pure state. We expect that quantum communication complexity may be help to measure the entanglement in multipartitie quantum systems.

In summary, a probabilistic two-party communication complexity scenario is proposed and demonstrated in experiment. We showed that quantum entanglement resulted in less communication than is required with only classical random correlations. These results are a noteworthy contrast to actually simulate communication among remote parties.

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\textbf{Figure captions:}

\textbf{Figure 1.} Experimental setup (top view). The pump beam is polarization filtered via a polarizing beam splitter (UV PBS). The polarization of the pump beam is set by a half-wave plate (UV HWP). The local rotation on the entangled state is completed via two UV HWPs.

\textbf{Figure 2.} The success probabilities of communication for a spectrum of nonmaximally entangled states. \textit{Points:} Experimentally determined success probabilities, with uncertainties of $\pm 2.8\%$ (counted over 100 s); \textit{Curves:} The solid line represents predicted settings for the success probabilities in our protocol; see text for details. The dotted line represents the success probabilities yielded by entanglement concentration.