We discuss questions related to renormalization group and to nonperturbative aspects of non-Abelian gauge theories with $N=2$ and/or $N=1$ supersymmetry. Results on perturbative and nonperturbative $\beta$ functions of these theories are reviewed, and new mechanisms of confinement and dynamical symmetry breaking recently found in a class of $SU(n_c)$, $USp(2n_c)$ and $SO(n_c)$ theories are discussed.

1 Introduction

We start with a brief review on the NSVZ $\beta$ functions in $N=1$ supersymmetric gauge theories and related issues.

1.1 NSVZ $\beta$ function in $N=1$ supersymmetric gauge theories

The bare Lagrangian of an $N=1$ supersymmetric gauge theory with generic matter content is given by

$$L = \frac{1}{4} \int d^2 \theta \left( \frac{1}{g^2(M)} \right) W^a W^a + h.c. + \int d^4 \theta \sum_i \Phi_i e^{2V_i} \Phi_i \tag{1}$$

where

$$\frac{1}{g^2(M)} = \frac{1}{g^2(M)} + i \frac{\theta(M)}{8\pi^2} \equiv i \frac{\tau(M)}{4\pi} \tag{2}$$

and $g(M)$ and $\theta(M)$ stand for the bare coupling constant and vacuum parameter, $M$ being the ultraviolet cutoff. Note that with this convention the vector fields $A_\mu(x)$ and the gaugino (gluino) fields $\lambda_\alpha(x)$ contain the coupling constant and hence, in accordance with the non Abelian gauge symmetry, are not renormalized.

By a generalized nonrenormalization theorem $^2$ the effective Lagrangian at scale $\mu$ takes the form,

$$L = \frac{1}{4} \int d^2 \theta \left( \frac{1}{g^2(M)} + \frac{b_0}{8\pi^2} \log \frac{M}{\mu} \right) W^a W^a + h.c. + \int d^4 \theta \sum_i Z_i(\mu, M) \Phi_i e^{2V_i} \Phi_i \tag{3}$$

(plus higher dimensional terms). Here

$$b_0 = -3N_c + \sum_i T_{F_i}; \quad T_{F_i} = \frac{1}{2} \quad \text{(quarks)}.$$ \tag{4}

Novikov et. al. then invoked the 1PI effective action to define a “physical” coupling constant for which they obtained the well-known $\beta$ function (Eq.(16) below). \textsuperscript{1}
Recently the derivation of the NVSZ beta function was somewhat streamlined by Arkani-Hamed and Murayama \(^3,^4\). (See also \(^5\).) They obtained the NVSZ beta function in the standard Wilsonian framework, without appealing to the 1PI effective action (hence no subtleties due to zero momentum external lines, such as those leading to apparent violation of nonrenormalization theorem\(^2,^6\)). They insist simply that at each infrared cutoff \(\mu\) the matter kinetic terms be re-normalized so that it resumes the standard canonical form, which is the standard procedure in the Wilsonian renormalization group. But the field rescaling

\[
\Phi_i = Z_i^{-1/2} \Phi_i^{(R)},
\]

introduces necessarily anomalous functional Jacobian \(^7\), and one gets

\[
L = \frac{1}{4} \int d^2 \theta \left( \frac{1}{g^2(M)} + \frac{b_0}{8\pi^2} \log \frac{M}{\mu} - \frac{T_F}{8\pi^2} \log Z_i(\mu, M) \right) W^a W^a + \text{h.c.} \tag{6}
\]

\[
+ \int d^4 \theta \sum_i \Phi_i^{(R)} \Phi_i^{(R)} e^{2\Phi_i^{(R)}} = \frac{1}{4g^2(\mu)} \int d^2 \theta W^a W^a + \text{h.c.} + \int d^4 \theta \sum_i \Phi_i^{(R)} \Phi_i^{(R)} e^{2\Phi_i^{(R)}}. \tag{7}
\]

This leads to the beta function (call it \(\beta_h\) to distinguish it from the more commonly used definition):

\[
\beta_h(g) = \mu \frac{d}{d\mu} g = -\frac{g^3}{16\pi^2} \left( 3N_c - \sum_i T_F i(1 - \gamma_i) \right), \tag{8}
\]

where

\[
\gamma_i(g(\mu)) = -\mu \frac{\partial}{\partial \mu} \log Z_i(\mu, M) |_{M, g(\mu)}. \tag{9}
\]

is the anomalous dimension of the \(i\)-th matter field. The same result follows by differentiating (7) with respect to \(M\) with \(\mu\) and \(g(\mu)\) fixed. For SQCD these read

\[
\beta_h(g) = -\frac{g^3}{16\pi^2} (3N_c - N_f (1 - \gamma)), \quad \gamma(g) = -\frac{g^2}{8\pi^2} \frac{N_f^2 - 1}{N_c} + O(g^4), \tag{10}
\]

Eq.(8) and Eq.(10) are the NSVZ \(\beta\) functions \(^1\).

Note that the “holomorphic” coupling constant \(g(\mu)\) is a perfectly good definition of the effective coupling constant: it is finite as \(M \to \infty\); \(\mu = \text{finite}\), and physics below \(\mu\) can be computed in terms of it. Vice versa, the coupling constant defined as the inverse of the coefficient of \(W^a W^a\) in (3), \(\frac{1}{g^2(M)} + \frac{b_0}{8\pi^2} \log \frac{M}{\mu} \)\(^{-1}\), is not a good definition of an effective coupling constant, as long as \(N_f \neq 0\): it is divergent in the limit the ultraviolet cutoff is taken to infinity. In other words, the renormalization of the matter fields (5) is the standard, compulsory step of renormalization, such that the low energy physics is independent of the ultraviolet cutoff, \(M\).
Let us also note that, in spite of its name, the holomorphic coupling constant gets renormalized in a non-holomorphic way, due to the fact that $Z_i(\mu, M)$ is real. Another consequence of the reality of $Z_i(\mu, M)$ is that $\theta$ is not renormalized: this is evident from the same RG equation (8) written in terms of $\tau$ variable,

$$\mu \frac{d}{d\mu} \tau(\mu) = -\frac{i}{2\pi} \left( 3N_c - \sum_i T_{Fi}(1 - \gamma_i) \right), \quad (11)$$

showing that the NSVZ beta function is essentially perturbative. It is interesting to observe that the above procedure parallels nicely the original derivation by Novikov et al. of the beta function by use of some instanton-induced correlation functions. More recently the NSVZ $\beta$ function in $N = 1$ supersymmetric QCD has been rederived by Arnone, Fusi and Yoshida $^5$, by using the method of exact renormalization group.

1.2 Zero of the NVSZ beta function and Seiberg’s duality and CFT in SQCD

For the range of the flavor $\frac{3N_c}{2} < N_f < 3N_c$ (conformal window) Seiberg discovered by using the NVSZ $\beta$ function that the theory at low-energy is at a nontrivial infrared fixed point $^8$. At the zero of the $\beta$ function the anomalous dimension of the matter field is found to be:

$$\gamma(g^*) = \frac{3N_c - N_f}{N_f}. \quad (12)$$

It turns out that this result is in agreement with that determined from the superconformal algebra, which contains the non-anomalous $U_R(1)$ symmetry. This and many other consistency checks allowed Seiberg to conclude that in the conformal window, and at the origin of the moduli space (namely, in the theory where all VEV’s vanish), the theory has a nontrivial infrared fixed point. Such a theory has no particle description, and as such, can be described by more than one type of gauge theory. In fact, in $SU(N_c)$ theory, the theory can be either described as the standard SQCD with $N_f$ flavors, or in terms of a dual theory, which is an $SU(\bar{N}_c)$ gauge theory with $N_f$ sets of dual quarks, plus singlet meson fields, where $\bar{N}_c \equiv N_f - N_c$. They have the same infrared behavior. This is the first example of the $N = 1$ non-Abelian duality, found in many other theories subsequently.

This development enabled Seiberg to complete the picture of dynamical properties of $N = 1$ supersymmetric QCD in all cases. Phase, the low-energy effective degrees of freedom, effective gauge group, etc. are summarized in Table 1 (where the bare quark masses are taken to be zero).

1.3 Meaning of the pole of the NVSZ beta function

The status of the denominator of the so-called NSVZ $\beta$ function is subtler. Although it is not necessary, one might wish to make a further finite renormalization in Eq.(7) to get the canonical form of gauge kinetic terms, $-F_{\mu\nu}F^{\mu\nu}/4 + i\lambda\sigma_\mu D^\mu \bar{\lambda}$. The redefinition needed is

$$A_\mu = g_c A_{c\mu}, \quad \lambda = g_c \bar{\lambda}_c. \quad (13)$$
Table 1. Phases of $N = 1$ supersymmetric $SU(N_c)$ gauge theory with $N_f$ flavors. $M_{ij} = \tilde{Q}_i Q_j$ and $B = \epsilon_{a_1 a_2 \ldots a_{N_c}} q^{a_1} q^{a_2} \ldots q^{a_{N_c}}$ ($\tilde{B}$ is constructed similarly from the antiquarks, $\tilde{Q}$'s) stand for the meson and baryon like supermultiplets. “Unbroken” means that the full chiral symmetry $G_F = SU_L(N_f) \times SU_R(N_f) \times U(1)$ is realized linearly at low energies. Actually, for $N_f > N_c$ continuous vacuum degeneracy of the theory survives quantum effects, and the entities in the table refers to a representative vacuum at the origin of the quantum moduli space (QMS).

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>Deg. Freed.</th>
<th>Eff. Gauge Group</th>
<th>Phase</th>
<th>Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (SYM)</td>
<td>-</td>
<td>-</td>
<td>Confinement</td>
<td>-</td>
</tr>
<tr>
<td>$1 \leq N_f &lt; N_c$</td>
<td>$M, B, B$</td>
<td>-</td>
<td>Confinement</td>
<td>$U(N_f)$</td>
</tr>
<tr>
<td>$N_c + 1$</td>
<td>$M, B, B$</td>
<td>-</td>
<td>Confinement</td>
<td>Unbroken</td>
</tr>
<tr>
<td>$N_c &lt; N_f &lt; \frac{3N_c}{2}$</td>
<td>$q, \tilde{q}, M$</td>
<td>$SU(N_f)$</td>
<td>Free-magnetic</td>
<td>Unbroken</td>
</tr>
<tr>
<td>$\frac{3N_c}{2} &lt; N_f &lt; 3N_c$</td>
<td>$q, \tilde{q}, M$ or $Q, \tilde{Q}$</td>
<td>$SU(N_f)$ or $SU(N_c)$</td>
<td>SCFT</td>
<td>Unbroken</td>
</tr>
<tr>
<td>$N_f = 3N_c$</td>
<td>$Q, Q$</td>
<td>$SU(N_c)$</td>
<td>SCFT (finite)</td>
<td>Unbroken</td>
</tr>
<tr>
<td>$N_f &gt; 3N_c$</td>
<td>$Q, \tilde{Q}$</td>
<td>$SU(N_f)$</td>
<td>Free Electric</td>
<td>Unbroken</td>
</tr>
</tbody>
</table>

This introduces as functional–integral Jacobian an extra factor

$$\exp \frac{1}{4} \int d^4x \int d^2\theta \frac{N_c \log g_c^2}{8\pi^2} W^a W^a + \text{h.c.} \quad (14)$$

and as a consequence, leads to the change of the coupling constant

$$\frac{1}{g^2} = \frac{1}{g_c^2} + \frac{N_c}{8\pi^2} \log g_c^2, \quad (15)$$

introduced earlier by Shifman and Vainshtein as a way to reconcile holomorphy and renormalization. The well-known NSVZ beta function of the form

$$\beta(g_c) = -\frac{g_c^3}{16\pi^2} \frac{3N_c - \sum_i T_{Fi}(1 - \gamma_i)}{1 - N_c g_c^2 / 8\pi^2}. \quad (16)$$

follows then from (15) and (8). In the case of $N = 1$ pure Yang-Mills theory such a pole of the beta function has led to an interesting conjecture by Kogan and Shifman that there is another, dual phase of the theory in which the coupling constant $g_c$ grows in the ultraviolet.

However, the very way the denominator arises from the functional change of variables, (13), (15), reveals the meaning of such a behavior of the beta function. In fact, the right hand side of (15) has a minimum at $g_c^2 = 8\pi^2 / N_c$, precisely corresponding to the pole of the NSVZ beta function, where it takes the value

$$\frac{N_c}{8\pi^2} \log \frac{8\pi^2 e}{N_c} > 0 \quad (17)$$

(for $N_c < 215$). On the contrary, the left hand side of (15) evolves down to zero if the beta function has no zero ($N_f < 3N_c/2$). Thus for large values of $g$ ($g > 8\pi^2 / N_c \log(8\pi^2 e / N_c)$) the redefinition (15), with a real “canonical coupling.
constant”, is not possible. In other words, the change of the functional variables involved is not a proper one, the new variable $A_{c\mu}$ being complex.

The runnings of the holomorphic and canonical coupling constants are compared in Fig. 1 for the case of $N_f = 0$, from which one sees that both the absence of evolution below the scale $(8\pi^2 e / N_c)^{1/3}\Lambda$ and the apparent new phase of the theory are artifacts caused by the improper change of the variable (15). The pole of the beta function signals the failure of $g_c$ as a coupling constant (and $A_{c\mu}(x)$ as a functional variable).

This however means that if one starts in the UV by using the standard “canonical” coupling constant and studies the RG evolution towards the low energies, one must switch to the “holomorphic” description at certain point (in any case, before the “critical” value $\alpha_c = 2\pi / N_c$ is reached), in order to describe physics smoothly down to $\mu = \Lambda$. The impossibility of writing a low energy effective Lagrangian with canonically normalized gauge kinetic terms, does not represent any inconsistency, since the low-energy physical degrees of freedom are not gauge (and quark) fields themselves. This last statement does not apply for $N_f > 3N_c$, but there is no obstruction in using the canonical variables in these cases, since the theory is infrared-free.

The success of the NSVZ beta function in the case of SQCD in the conformal window ($3N_c/2 < N_f < 3N_c$) discussed already, especially the determination of the anomalous dimension of the matter fields at the infrared fixed point, does not require the use of the canonical coupling constant. This is important because the anomalous dimensions at the IR fixed point are physically significant numbers.

1.4 Exactness of the NSVZ beta function

One might wonder how “exact” all this is. It is clear that the diagrammatic proof of the generalized nonrenormalization theorem of used in Eq.(3) is valid only within perturbation theory.

It was argued on the other hand in that due to the existence of an anomalous $U_R(1)$ symmetry the beta functions are purely perturbative, hence the NSVZ beta
function is exact perturbatively and nonperturbatively, at least for pure $N = 1$ Yang–Mills theory. In fact, the (holomorphic) coupling constant at scale $\mu$ must satisfy

$$\tau(\mu) = \tau(M) + f(\tau(M), \mu/M), \quad (18)$$

where $f$ is a holomorphic function of $\tau$. It follows that $\beta(\tau) = \mu(1/\mu)\tau(\mu)$ shares the same property. Together with the periodicity in $\theta$ with period $2\pi$, one finds that

$$\beta(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}, \quad (19)$$

where $a_n$ is the $n$-instanton contribution. If the right hand side is independent of $\theta$ it must consist only of the perturbative term, $n = 0$. That leads back to the NSVZ $\beta$ function.

This argument is however only valid in theories in which the anomalous $U_R(1)$ symmetry is not spontaneously broken by the VEVs of some scalar field. Examples are the pure $N = 1$ Yang–Mills theory or the $N = 1$ SQCD at the origin of the space of vacua (with all scalar vevs vanishing): there the argument of 4 is valid and the NSVZ $\beta$ function is exact perturbatively and nonperturbatively.

Vive versa, in a generic point of the space of vacua of $N = 1$ SQCD, or at any point of a $N = 2$ supersymmetric Yang–Mills theory (a $N = 1$ supersymmetric gauge theory with a matter chiral multiplet in the adjoint representation), for example, $U_R(1)$ invariance is spontaneously broken, i.e., by anomaly as well as by the VEVs of certain gauge invariant composite fields. There is a nontrivial $\theta$ dependence. By holomorphic dependence of $\beta$ on $\tau = \frac{\theta}{2\pi} + \frac{4\pi}{g^2}$ this implies that the beta function gets necessarily instanton corrections.

1.5 Other developments

One interesting development involves the existence in many models of infrared fixed lines, as can be shown by using the explicit formula for the supersymmetric anomaly multiplet. In other words in these models there are exactly marginal operators. Another very interesting development deals with the possible $c-$ (or $a-$) theorem in four dimensional supersymmetric gauge theories. There remain some longstanding problems such as the "$4/5$ puzzle" in the computation of the gaugino condensate in the super Yang Mills theory, which is made more acute after some recent analysis.

2 $N = 2$ gauge theories

2.1 $\beta$ function

The (bare) Lagrangian of the pure $N = 2$ $SU(2)$ Yang–Mills theory reads

$$\frac{1}{4\pi} \text{Im} \tau cl \left[ \int d^4 \theta \Phi^1 e^V \Phi + \int d^2 \theta \frac{1}{2} WW \right], \quad (20)$$
\[ \tau_{cl} = \frac{\theta_0}{2\pi} + \frac{4\pi i}{g_0^2}, \quad (21) \]

where \( \Phi = \phi + \sqrt{2} \theta \psi + \ldots \), and \( W_\alpha = -i \lambda + \frac{i}{2} (\sigma^\mu \bar{\sigma}^\nu)^\alpha_\beta F_{\mu \nu} \theta \beta + \ldots \) are both in the adjoint representation of the gauge group. \( N = 2 \) supersymmetry restricts the form of the low-energy effective action to be

\[ L_{\text{eff}} = \frac{1}{4\pi} \text{Im} \left[ \int d^4 \theta \frac{\partial F(A)}{\partial A} \bar{A} + \int d^2 \theta \frac{1}{2} \frac{\partial^2 F(A)}{\partial A^2} WW \right], \quad (22) \]

where \( F(A) \), holomorphic in \( A \), is called prepotential. As seen from this expression (the VEV of)

\[ \tau(a) = \frac{\partial^2 F(a)}{\partial a^2} = \frac{\theta_{\text{eff}}}{2\pi} + 4\pi i g_2^2 \]

Perturbative and nonperturbative (instanton) corrections lead to the general form

\[ F(a) = \frac{i}{2\pi} A^2 \log \frac{A^2}{\Lambda^2} + F^{\text{inst}}, \quad F^{\text{inst}} = \sum_{k=1}^\infty c_k (\Lambda/A)^{4k} A^2, \quad (23) \]

is the contribution from multistantion effects. The prepotential \( F \) has been found by Seiberg and Witten \(^{15}\) through the introduction of an auxiliary curve (torus), which in the simplest case of the pure \( SU(2) \) \( N = 2 \) Yang-Mills theory reads,

\[ y^2 = (x^2 - \Lambda^4) (x - u), \quad u = \langle \text{Tr} \Phi^2 \rangle, \quad (24) \]

and \( da_D/du \) and \( da/du \) are given by the period integrals

\[ da_D/du = \text{const} \int_a \frac{dx}{y}; \quad da/du = \text{const} \int_a \frac{dx}{y}. \quad (25) \]

Such a construction has then been generalized to \( SU(n_c) \), \( USp(2n_c) \), and \( SO(n_c) \) gauge groups with an arbitrary number of flavors \(^{16}\).

Due to the holomorphic nature of Wilsonian effective action the RG equation can be cast into the form \(^{17}\)

\[ \beta(\tau) \equiv \mu \frac{d \tau}{d \mu} = \frac{2i}{\pi} (1 + c_1 e^{2\pi i \tau} + c_2 e^{4\pi i \tau} + \ldots) \quad (26) \]

for \( \text{Im} \tau \gg 1 \) (or \( g^2 \ll 1 \)) where

\[ \tau = \frac{\theta}{2\pi} + \frac{4\pi i g^2}{g^2}, \quad (27) \]

and \( \mu \) is the scale. Recently several papers discussed the calculation of the exact, nonperturbative \( \beta \) function.\(^{18}\) All of these “\( \beta \) functions” have the correct UV behavior by construction, but none of them has the correct IR behavior. The correct behavior at \( \tau \sim 0 \) can be found by noting that in the theory near \( u = \Lambda^2 \) the IR cutoff is given by \( \mu_{IR} = \sqrt{2} |a_D| \). The leading behavior is that of a dual QED with a single monopole, and transformed back to the electric description by a duality transformation it gives

\[ \beta(\tau) \sim \frac{1}{4\pi \tau^2_D} = -\frac{i}{\pi} \tau^2 \quad \text{as} \quad \tau \to 0, \quad (28) \]
Figure 2. The $\beta$ function of the theory for $u$ real and $u \geq \Lambda^2$.

For CP invariant cases ($\theta = 0$) this means the behavior

$$\beta(g) \sim -\frac{2}{g},$$

(29)

at large $g$. See Fig. 2. Actually, the "$\beta$ function" computed by these authors is equal to $2u \frac{d\tau_{\text{eff}}}{du}$, where both $\tau_{\text{eff}}$ and $u$ are renormalization-group invariants. It corresponds to the variation of the low-energy effective coupling constant within the QMS, not to the standard $\beta$ function.

2.2 Nonrenormalization of $a$ and renormalization of $\theta$ in pure $N = 2$ Yang-Mills theory

Even though the exact, nonperturbative $\beta$ function still eludes us, the exact Seiberg-Witten solution yields a result which amounts to the integral of the renormalization group equation. Namely, in each theory (i.e., at each point of QMS) characterised by $u$, both the bare and the corresponding low-energy $\theta$ parameters are known exactly. The key relation is

$$a = a^{(cl)}.$$  

(30)

This relation has the following meaning: on the left hand side, one has $a \equiv \langle A \rangle$, where $A$ is the low-energy scalar supermultiplet, $N = 2$ superpartners of the photonlike gauge multiplet $W$. The right hand side is the classical VEV of the adjoint field,

$$\langle \phi \rangle = \frac{1}{2} \begin{pmatrix} a^{(cl)} & 0 \\ 0 & -a^{(cl)} \end{pmatrix}.$$  

(31)

To show (30), note that because of the exact mass formula,

$$M_{nm, nc} = \sqrt{2} |n_m a_D + n_c a|,$$  

(32)

$|a|$ represents the physical mass of a $(0, 1)$ particle ("$W^{\pm}$" bosons or their fermion partners). Classically, it can be read off from the Lagrangian

$$\sqrt{2} \, \text{Tr} \, \phi^* [\lambda, \psi],$$  

(33)
and is indeed equal to $\sqrt{2} \, a^{(cl)}$. As $A$ is a $N = 2$ superpartner of the Yang-Mills field $A_\mu$ ($gA_\mu$ in the canonical definition) it is not renormalized perturbatively. In order to see whether $a$ gets nontrivial instanton contributions one must compute, e.g.,

$$\langle \lambda^+(x)\psi^-(y) - \psi^+(x)\lambda^-(y) \rangle. \quad (34)$$

This Green function obviously gets the classical contribution proportional to

$$\sqrt{2} \, a^{(cl)} \int d^4 z S_F(x - z)S_F(y - z). \quad (35)$$

By studying the possible instanton corrections to (34) it can be shown that there are no instanton corrections whatsoever to this result, to any instanton number. The reason is that because of the symmetry of the classical Lagrangian, there are four "supersymmetry zero modes" (two for $\lambda$, two for $\psi$) to any instanton number: only two of them are eliminated in the Green function (34) hence the functional integration over the other zero modes yields a vanishing result. Eq. (30) is thus exact perturbatively and nonperturbatively.

This is to be contrasted with the case of the one point function, $u \equiv \langle \text{Tr} \phi^2 \rangle$, or the Green function

$$\langle \lambda(x)\lambda(y)\psi(z)\psi(w) \rangle, \quad (36)$$

in which case nonvanishing contributions are found to all instanton numbers.

Let us note that the relation Eq. (30) is implicitly assumed in all direct instanton checks of the Seiberg-Witten curves $^{19}$.

An immediate consequence of (30) is that the bare and renormalized $\theta$ can be computed for each $u$:

$$\theta_{UV} = \theta_{bare} = 4 \, \text{Arg} \, a^{(cl)} = 4 \, \text{Arg} \, a;$$
$$\theta_{IR} = \theta_{eff} = 2\pi \, \text{Re} \, \tau_{eff} = 2\pi \, \text{Re} \, \frac{da_D}{da}. \quad (37)$$

This instanton-induced renormalization effect is illustrated for several representative points of QMS in Table 2. Thus in a generic point of QMS in $N = 2$ supersymmetric pure Yang-Mills theory $\theta$ grows in the infrared. This is opposite to what was found in a model with soft supersymmetry breaking where CP violation was found to be suppressed in the infrared by the instanton effects $^{20}$.

The key relation Eq.(30) furthermore allows to give a precise meaning to the statement that classical and quantum moduli space are equal in the pure $N = 2$ Yang-Mills theory. Namely, if the fundamental region $a$ space is defined as the image $\mathcal{A}$ of the upper half plane of $u$ (Fig.3), then the exact statement is

$$(CMS)_\mathcal{A} = QMS. \quad (38)$$

Note that other regions of CMS ($a^{(cl)}$ outside $\mathcal{A}$) do not represent inequivalent vacua: each such theory is equivalent to some of the theory in $\mathcal{A}$ as they are related by an $SL(2, \mathbb{Z})$ transformation (monodromy transformations in QMS).
Table 2. Instanton-induced renormalization of $\theta$

<table>
<thead>
<tr>
<th>$u/\Lambda^4$</th>
<th>$\theta_{UV}$</th>
<th>$\theta_{IR}$</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>CP</td>
</tr>
<tr>
<td>$-2 + 0.0001i$</td>
<td>$\sim 2\pi$</td>
<td>$\sim 2\pi$</td>
<td>$\sim$ CP</td>
</tr>
<tr>
<td>$i$</td>
<td>$\pi$</td>
<td>$\pi$</td>
<td>CP</td>
</tr>
<tr>
<td>$1 + 0.01i$</td>
<td>0.3305</td>
<td>0.8554</td>
<td>strong coupling</td>
</tr>
<tr>
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<td>0.2560</td>
<td>0.4968</td>
<td>strong coupling</td>
</tr>
<tr>
<td>$0.9 + 0.01i$</td>
<td>0.4027</td>
<td>1.3310</td>
<td>strong coupling</td>
</tr>
<tr>
<td>$0.5 + 0.01i$</td>
<td>1.1966</td>
<td>2.3789</td>
<td></td>
</tr>
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<td>$4 + 4i$</td>
<td>1.5786</td>
<td>1.5903</td>
<td>semi-classical</td>
</tr>
</tbody>
</table>

Figure 3. Fundamental region of the classical space of vacua (CMS)

2.3 The zero of the $\beta$ function at the infrared fixed points (SCFT) of $N = 2$ susy gauge theories

A very interesting development was the discovery of the large classes of nontrivial superconformal theories within the context of $N = 2$ supersymmetric theories. They occur at some special points of QMS (space of vacua) and/or for particular value of the parameters (such as bare masses), for different gauge groups and with different matter contents, but fall into various universality classes. They have been classified by Eguchi et. al.22

3 New mechanisms of confinement/dynamical symmetry breaking in $SU(n_c), USp(2n_c)$ and $SO(n_c)$ gauge theories

Recently the questions such as: i) the mechanism of confinement; ii) the mechanism of flavor (chiral) symmetry breaking; and the relation between the two; iii) the
existence of other phases (CFT, oblique confinement, etc.), have been studied in
detail in a large class of models based on $SU(n_c), USp(2n_c)$ and $SO(n_c)$ gauge
groups and with $N = 2$ supersymmetry, where the adjoint scalar mass term breaks
the supersymmetry to $N = 1$.

The most striking results of our analysis, summarized in Table 3 and Table 4
for $SU(n_c)$ and $USp(2n_c)$ theories, are the following.

<table>
<thead>
<tr>
<th>Deg.Freed.</th>
<th>Eff. Gauge Group</th>
<th>Phase</th>
<th>Global Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>monopoles</td>
<td>$U(1)^{n_c-1}$</td>
<td>Confinement</td>
<td>$U(n_f)$</td>
</tr>
<tr>
<td>monopoles</td>
<td>$U(1)^{n_c-1}$</td>
<td>Confinement</td>
<td>$U(n_f-1) \times U(1)$</td>
</tr>
<tr>
<td>dual quarks</td>
<td>$SU(r) \times U(1)^{n_c-r}$</td>
<td>Confinement</td>
<td>$U(n_f-r) \times U(r)$</td>
</tr>
<tr>
<td>rel. nonloc.</td>
<td>-</td>
<td>Almost SCFT</td>
<td>$U(n_f/2) \times U(n_f/2)$</td>
</tr>
<tr>
<td>dual quarks</td>
<td>$SU(n_c) \times U(1)^{n_c-n_c}$</td>
<td>Free Magnetic</td>
<td>$U(n_f)$</td>
</tr>
</tbody>
</table>

Table 3. Phases of $SU(n_c)$ gauge theory with $n_f$ flavors. The label $r$ in the third row runs for $r = 2, 3, \ldots, \left\lfloor \frac{n_f}{2} \right\rfloor$. “rel. nonloc.” means that relatively nonlocal monopoles and dyons coexist as low-energy effective degrees of freedom. “Confinement” and “Free Magnetic” refer to phases with $\mu \neq 0$. “Almost SCFT” means that the theory is a non-trivial superconformal one for $\mu = 0$ but confines with $\mu \neq 0$. $n_c \equiv n_f - n_c$.

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<td>$USp(2n_c) \times U(1)^{n_c-n_c}$</td>
<td>Free Magnetic</td>
<td>$SO(2n_f)$</td>
</tr>
</tbody>
</table>

Table 4. Phases of $USp(2n_c)$ gauge theory with $n_f$ flavors with $m_i \to 0$. $n_c \equiv n_f - n_c - 2$.

The ’t Hooft - Mandelstam picture of confinement, caused by the condensation of
monopoles in the maximal Abelian subgroup $U(1)^k$, ($k =$ Rank of the gauge
group), is in fact realized but in some of the vacua. In a more “typical” vacuum of
$SU(n_c)$ gauge theory, the effective, infrared degrees of freedom involve a set of
dual quarks, interacting with low-energy effective non-Abelian $SU(r)$ gauge fields.
The condensation of these magnetic quarks as well as of certain Abelian monopoles
also present in the theory, upon $\mu$ perturbation, lead to confinement and dynamical
symmetry breaking. The semi-classical monopoles may be interpreted as baryonic
composites made of these magnetic quarks and monopoles, which break up into
their constituents before they become massless, as we move from the semiclassical
region of the space of $N = 2$ vacua (parametrized by a set of gauge invariant VEVS)
towards the relevant singularity. These theories are essentially infrared-free.

The second most interesting result is that the special vacua ($r = n_f/2$ in Table
3) in $SU(n_c)$ theory as well as the entire first group of vacua in $USp(2n_c)$ or
$SO(n_c)$ theory, correspond to various nontrivial infrared fixed points (SCFT). The
low-energy effective degrees of freedom in general contain relatively nonlocal states
and there is no local effective Lagrangian description of these theories, though
the symmetry breaking pattern can be found from the analysis at large adjoint
mass $\mu$. The symmetry breaking pattern of these vacua is $SU(n_f) \rightarrow SU(n_f/2)$,
$USp(2n_f) \rightarrow U(n_f)$ and $SO(2n_f) \rightarrow U(n_f)$ in $SU(n_c), USp(2n_c)$ and $SO(n_c)$
gauge theories, respectively.

Finally, in both type of gauge theories, for large number of flavor, there is a
second group of vacua in free-magnetic phase, with no confinement and no spontaneous flavor symmetry breaking. In these vacua the low energy degrees of freedom
are weakly interacting non-Abelian dual quarks and gauge particles, as well as some monopoles of products of $U(1)$ groups. In $SO(n_c)$ theories, the situation is qualitatively similar to $USp(2n_c)$ cases; however, the effective gauge group
and the unbroken global group in the vacua in free-magnetic phase are given by
$SO(\tilde{n}_c) = SO(2n_f - n_c + 4)$ and $USp(2n_f)$, respectively, in these theories.

To summarise, the picture of confinement due to condensation of monopoles in
the maximally Abelian gauge subgroup (à la ’t Hooft-Mandelstam) is realized in
few of the vacua only. Two other mechanisms of confinement/dynamical symmetry
breaking have been discovered. One is the condensation of dual quarks with effective (dual) non-Abelian gauge interactions; the other is based on nontrivial SCFT
(condensing entities involve relatively nonlocal dyons). In both these cases, the
maximally Abelian monopoles of $U(1)^R$ theory ($R=\text{rank of the gauge group}$) do
not represent the correct low-energy degrees of freedom.

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