Behavior of Boundary String Field Theory
Associated with Integrable Massless Flow

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Abstract

We put forward an idea that the boundary entropy associated with integrable massless flow of thermodynamic Bethe ansatz (TBA) is identified with tachyon action of boundary string field theory. We show that the temperature parameterizing a massless flow in the TBA formalism can be identified with tachyon energy for the classical action at least near the ultraviolet fixed point, i.e. the open string vacuum.

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A lot of efforts to reveal tachyon condensation mechanism have been made in an attempt to find a stable vacuum both in bosonic and in supersymmetric string theory. According to Sen’s conjecture, the closed string vacuum is realized after an annihilation mechanism of an open string is completed by the cancellation between tensions of D-branes and energy of the tachyon[1]. In string field theory, main ways to analyze this problem have been through Witten’s cubic string field theory[2] and the boundary string field theory (BSFT)[3, 4].

In the latter context with a special choice of the tachyon profile, some evidence to support Sen’s conjecture has recently been provided in an exact manner [5]-[7]. In this letter we take this latter approach, BSFT. In the Batalin-Vilkovisky formalism used for BSFT[3], the space-time string action \( S \) is conjectured to satisfy the differential equation of the form [5]-[7]

\[
\frac{\partial S}{\partial \lambda_i} = G^{ij} \beta_j.
\] (1)

Here \( \lambda_i \) are the coupling constants of the boundary operators, \( \beta_j \) are the \( \beta \)-functions, and \( G^{ij} \) is the metric in the space of coupling constants.

On the other hand, ground state degeneracy (g-function) [8] of the worldsheet theory with a boundary perturbation is also expected to satisfy a differential equation of the same form as (1). Thus, we expect that string action \( S \) will be identified with \( g \)-function. To investigate this correspondence, we analyze this problem using the boundary sine-Gordon model (BSG)[9] and attendant thermodynamic Bethe ansatz (TBA). TBA is intended to obtain thermodynamic quantities at finite temperature and has also been used to extract information on the \( g \)-function in some of exactly solvable models [10].

We consider the process in which single D25-brane decays into a D24-brane by tachyon condensation. In the context of TBA, temperature in one-dimensional soliton gas is the renormalization group (RG) scale, which is regarded as an order parameter of the tachyon condensation. The sine-Gordon parameter in the boundary term can be identified with the inverse of the compactification radius in BSFT. From this fact, the boundary entropies at two ends of the flow have been shown to give an exact ratio of the brane tensions[5]. In this letter we find that the temperature in TBA can be identified with energy of the classical solution of the tachyon action in BSFT. We provide evidence in support of this correspondence by comparing the behavior of TBA and the behavior of the classical solution in BSFT. This correspondence is confirmed by a numerical calculation, too. We propose that integrable massless flows generated by TBA provide description of the open string action even away from the fixed points.
Massless TBA and $g$-function as string action

To utilize soliton picture, we begin with the action of the sine-Gordon model on a segment $\sigma \in [0, L]$ at finite temperature $\theta$

$$S = \frac{1}{4\pi\alpha'} \int_0^{1/\theta} dt \int_0^L d\sigma \left[ (\partial_{\mu} X(\sigma, t))^2 + G \cos \frac{4\pi}{R} X(\sigma, t) \right] + \zeta \int_0^{1/\theta} dt \cos \frac{2\pi}{R} X(\sigma = 0, t). \quad (2)$$

This system is shown to possess an infinite number of conserved currents and hence is integrable[11]. This action permits the field $X(\sigma, t)$, namely $X_{25}$, to be compactified: $X \sim X + R$. We impose the Dirichlet boundary condition at $\sigma = L$ and pay our attention to the boundary at $\sigma = 0$. We assume that $\lambda = R^2/4\pi^2\alpha' - 1$ is a non-negative integer. The strength $G$ gives mass scale of the soliton/antisoliton and $\zeta$ gets traded with boundary temperature $\theta_B$ [10], which plays a similar role to that of the Kondo temperature in the Kondo problem. Because we are interested in models with conformal invariance in the bulk $\sigma \in (0, L)$, the massless limit $G \to 0$ is taken after TBA formalism is set up.

The free energy of this model in the $L \to \infty$ limit should be

$$\frac{F}{L} = f_{\text{bulk}} - \frac{\theta}{L} \ln g + O(1/L^2), \quad (3)$$

where $g$ is the ground state degeneracy of the system with the boundary at $\sigma = 0$. We focus on this $g$-function.

The TBA procedure gives us an equation for the $g$-function in terms of the hole energy functions $\epsilon_r (r = 1, 2, \ldots, \lambda + 1)[10]$:

$$\ln g = \sum_{r=1}^{\lambda+1} \int_{-\infty}^{\infty} \frac{dv}{2\pi} \kappa_r(v - \ln(\theta/\theta_B)) \ln(1 + e^{-\epsilon_r(v)}), \quad (4)$$

where $\kappa_r$ are the kernels whose Fourier transforms are

$$\tilde{\kappa}_n(y) = \frac{\sinh y}{2\sinh y \cosh \lambda y},$$

$$\tilde{\kappa}_{\lambda}(y) = \frac{\sinh(\lambda - 1)y}{2\sinh 2y \cosh \lambda y}, \quad \tilde{\kappa}_{\lambda+1}(y) = \tilde{\kappa}_\lambda(y) + \frac{1}{2 \cosh y}, \quad (5)$$

$$\tilde{\kappa}_r(y) = \int_{-\infty}^{\infty} \frac{dv}{2\pi} e^{2\lambda y/\pi} K(v).$$

The hole energies $\epsilon_r(v)$ satisfy the following TBA equation

$$\epsilon_r(v) = \sum_{s=1}^{\lambda+1} \alpha_{rs} \int_{-\infty}^{\infty} \frac{dv'}{2\pi} \frac{1}{\cosh(v - v')} \ln(1 + e^{\epsilon_s(v')}), \quad (6)$$

1Strictly speaking, we are going to consider just the difference of the $g$-functions at two different temperatures.
where \( a_{rs} \) is the incidence matrix of \( D_{\lambda+1} \)-type Dynkin diagram;

\[
\begin{align*}
    a_{ij} &= \delta_{i,j+1} + \delta_{i,j-1} \quad (i, j = 1, 2, \ldots, \lambda - 1), \\
    a_{\lambda,j} &= \delta_{j,\lambda - 1}, \quad a_{\lambda+1,j} = \delta_{j,\lambda - 1}. \\
\end{align*}
\] (7)

In the two limits, \( \theta/\theta_B = 0 \) and \( \theta/\theta_B = \infty \), the above TBA equation can be solved analytically\[10\]. We call the former limit infrared (IR) and the latter ultraviolet (UV). The difference of the boundary entropies in these two limit is

\[
\frac{g_{\text{UV}}}{g_{\text{IR}}} = \frac{R^2}{2\pi \sqrt{\alpha'}}. 
\] (8)

As is conjectured by the \( g \)-theorem\[8\], \( g \) decreases along the RG flow from UV to IR if \( R > 2\pi \sqrt{\alpha'} \), i.e. the boundary perturbation is relevant. \( g \)'s in the two limits have been identified with the respective values of the tachyon actions\[5, 12\]. We can compare the tensions of D25- and D24-branes, \( \tau_{25} \) and \( \tau_{24} \) respectively. In this view, we should set \( g_{\text{UV}} = \tau_{25} R \) and \( g_{\text{IR}} = \tau_{24} \). Thus, we get the well-known relation \( \tau_{24} = 2\pi \sqrt{\alpha'} \tau_{25} \).

Even at general \( \theta \), \( g \) is obtained numerically by means of TBA. We expect that this \( g \) will give the string action even in an intermediate region between the open string vacuum and the closed string one. The quantity \( \ln(\theta/\theta_B) \) is identified with the RG scale. As an example, let us calculate the \( g \)-function for \( \lambda = 2 \) (i.e. \( R = 2\sqrt{3\alpha'} \pi \)) case explicitly. The plot \( \ln(\theta/\theta_B) - \ln g \) is shown in Figure 1. As is seen in Figure 1, \( g \) becomes stationary both as \( \ln(\theta/\theta_B) \to -\infty \) and as \( \ln(\theta/\theta_B) \to +\infty \). Having this aspect of the \( g \)-function in mind, we would like to gain more insight into the RG behavior of the tachyon condensation.

Let us consider the behavior of the \( g \)-function at large \( \theta/\theta_B \) in field theory analysis \[8\]. Let the dimension of the boundary term be \( \Delta \). We see that the \( \beta \)-function of the coupling \( \zeta \) is

\[
\beta(\zeta) = \frac{d \zeta}{d \ln |x|} = (1 - \Delta) \zeta + O(\zeta^2),
\] (9)
where $|x|$ is the inverse of the momentum cut-off at the boundary and equal to the ratio, $(\theta/\theta_B)^{-1}$. Thus, we see the relation $\zeta \sim (\theta/\theta_B)^{-(1-\Delta)}$ for large $\theta/\theta_B$. Upon taking (1) into account, we obtain the asymptotic behavior

$$g \sim g_{\text{UV}} - c_0(\theta/\theta_B)^{-2(1-\Delta)}$$

(10)

for small $\zeta$, where $c_0$ is a constant. We will compare this with the tachyon action later.

**Tachyon field and its energy**

Let us consider the tachyon field with the co-dimension one *i.e.* the case in which the tachyon field depends on just one coordinate $X^{25}(= x)$. The action obtained in [6, 7] is

$$S = \tau_{25} V_{25} \int_{-\infty}^{+\infty} \left[ \alpha' e^{-T\dot{T}^2} + V(T) \right] dx,$$

(11)

where $V(T) = e^{-T}(T + 1)$ is the tachyon potential, $\dot{T} = dT/dx$, and $V_{25}$ is the volume of 25-dimensional space-time. Here we have ignored the higher derivative corrections. Let us set $\tau_{25} V_{25}$ to be $1/2\pi$ for simplicity and $\alpha' = 1$. Let us consider classical solutions of this action (11). The equation of motion obtained from (11) is

$$2\ddot{T} - \dot{T}^2 - e^T V'(T) = 0,$$

(12)

which can be integrated once to give

$$\dot{T} = \pm e^{T/2} \sqrt{E + V(T)}.$$

(13)

Here $E$ is a constant that can be regarded as energy. If $-1 \leq E < 0$, the tachyon field $T(x)$ looks like a classical lump. That is, $T(x)$ oscillates between $T_i$ and $T_f$. The constants $T_i$ and $T_f$ are respectively the negative and positive solutions of the equation

$$E + V(T) = 0$$

(14)

and we set $T(0) = T_i$ (see Figure 2). For example, setting $E = -1$, we see $T_i = T_f = 0$, thus, $T(x) = 0$ for all $x \in (-\infty, +\infty)$, which is on the UV fixed point and regarded as the tachyonic open string vacuum. Another example is $E = 0$ which has $T_i = -1$ and $T_f = +\infty$. In this case, (13) is easily integrated to give

$$T(x) = -1 + x^2/4.$$

(15)

This form is already used in [7].

Let us evaluate (11) for given energy $(-1 \leq E \leq 0)$. Because (11) naively diverges, it is regularized by introducing the cut-off like

$$S(E, R) = \int_{-R/2}^{R/2} \frac{dx}{2\pi} \left[ e^{-T\dot{T}^2} + V(T) \right].$$

(16)
This cut-off $R$ is identified with that in (2). Using (13), we see

$$S(E, R) = -\frac{RE}{2\pi} + 4 \int_{T_i}^{T_f} \frac{dT}{2\pi} e^{-T/2} \sqrt{E + V(T)},$$

where we have set $T(R/2) = T(-R/2) = T_R$. From the form of (17), we conclude $S(-1, R) = R/2\pi$ and $S(0, R) = e/\sqrt{\pi} + O(e^{-R^2}/R)$. We should note that the precise value of $S(0, \infty)$ should be 1 as is seen in (8). In order to get this precise value, a more careful treatment to include the higher derivative terms, which we have ignored in (11), is necessary.[7]

Let us concentrate on the region near the UV fixed point, namely the region where the condensation is forming. Let $E = -1 + \epsilon^2$ \quad (0 < \epsilon \ll 1). Because $|T_i|, |T_R|, T_f \ll 1$ there, we approximate $V(T) \cong 1 - T^2/2$. Thus, we see $-T_i \cong T_f \cong \sqrt{2}\epsilon$. After some elementary calculation, we obtain, for example,

$$S(-1 + \epsilon^2, R) = R/2\pi - \epsilon^2 \cdot \sqrt{2} \sin(R/\sqrt{2}) + O(\epsilon^3) \quad \text{for} \quad 2\sqrt{2}\pi < R < 3\sqrt{2}\pi. \quad (18)$$

Let us compare the action (18) with the $g$-function (10) of BSG model with TBA. Comparing the scaling of $g$ with $\theta/\theta_B$ and the scaling of $S$ with $\epsilon$, we find

$$\epsilon \propto (\theta/\theta_B)^{-(1-\Delta)}. \quad (19)$$

Then we expect $S(-1 + \epsilon^2, R) = g(\theta/\theta_B, R)$ after fixing the constant $c_0$ in (10) appropriately. Let us consider the case where the cut-off is $R = 2\sqrt{3}\pi$. We have already shown the flow of the $g$-function in Figure 1. The string action $S$ can also be calculated numerically by using (14) and (17). Because, in this case, the boundary interaction has the dimension, $\Delta = 1/3$, the two scaling parameters $E$ and $\theta/\theta_B$ should be related as $E + 1 \sim (\theta/\theta_B)^{-4/3}$. The plots are shown in Figure 3, where we put $\epsilon \cong 5.4(\theta/\theta_B)^{-2/3}$. Figure 3 indicates the numerical agreement of the scalings for $g$ and $S$ for $\sqrt{E + 1} \lesssim 0.2$. When approaching the IR fixed point, $\theta/\theta_B \to 0$ and $E \to 0$, the higher derivative correction for (11) must become important in order that the relation between $S$ and $g$ holds in this region as well.

![Figure 2: Classical solution](image-url)
Figure 3: \( g \)-function and tachyon action for \( R = 2\sqrt{3}\pi \)

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References


