Radiation from a uniformly accelerated charge in the outskirts of a wormhole throat

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Using traversable wormholes as theoretical background, we revisit a deep question of general relativity: Does a uniformly accelerated charged particle radiate? We particularize to the recently proposed gravitational Čerenkov radiation, that happens when the spatial part of the Ricci tensor is negative. If $R_{+}^{(3+1)} < 0$, the matter threading the gravitational field violates the weak energy condition. In this case, the effective refractive index for light is bigger than 1, i.e., particles propagate, in that medium, faster than photons. This leads to a violation of the equivalence principle.

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Soon after the classic by Morris and Thorne [1], “flaring-out condition” became the nick-name in vogue for any marginally anti-trapped surface. An anti-trapped surface is a closed two-dimensional spatial hypersurface such that one of the two future-directed null geodesic congruences orthogonal to it is just beginning to diverge. Stated mathematically, the expansion $\theta$ of one of the two orthogonal null geodesic congruences vanishes on the surface: $\theta^+ = 0$ or $\theta^- = 0$, and the rate-of-change of the expansion along the same null direction $(u_\pm)$ is positive-semi-definite at the surface $d\theta^+/du_\pm > 0$ [2].

These “exotic” hypersurfaces, popularly known as wormhole throats, would require probably unrealistic amounts of negative energy. Therefore, it is far from clear whether stable macroscopic wormholes can naturally exist in the universe [3]. As far as we are aware, the first observational proposal to search for natural wormholes was presented by Cramer et al. [4]. They suggested that gravitational lensing effects of these exotic objects can be monitored from Earth. As some of us discussed elsewhere [5], wormhole lensing effects upon the light of high redshifted active galactic nuclei would yield temporal profiles quite similar to some detected by the Burst and Transient Source Experiment. However, no conclusive observational evidence has been found yet [6]. More recently, propagation of electromagnetic waves through a wormhole throat and mimicking systems were suggested as a possible testable arena [7,8].

In this article we shall discuss new unusual properties happening in the surroundings of a wormhole throat. To begin with let us briefly consider one of the most enduring questions of General Relativity: Does a uniformly accelerated charge radiate? (For a recent account on this issue, together with historical comments, see the paper by Pauri and Vallisneri [9] and reference therein.) The uniformly accelerated motion of a particle can conveniently be described by the orbits of the Rindler spacetime (see Appendix A for details on the main properties of this space-time). As may be seen by looking at Fig. 1, no matter how long an accelerated observer waits, he will never receive any information from about half of the space-time. Because he is asymptotically approaching the speed of light, one quarter of the space-time is everywhere space-like, whereas another quarter can receive signals from the observer but cannot send signals to him. The metric in region I is static (the co-accelerated observer sees no change with respect to his time $\tau$). However, $\tau$ gives the observer position along the orbit $(z, t)$, i.e., the hyperbola in Fig. 1. At each different time, the observer has a different velocity. Physically, this means that by making a successive series of Lorentz boosts one can follow an accelerated particle.

It is often (mistakenly) thought that a charged particle at rest in a static field cannot radiate, and hence, that a uniformly accelerated particle cannot radiate either. Because of the equivalence principle, a uniformly accelerated frame must be indistinguishable from a gravitational field. However, as shown by Boulware [10], radiation does exist in this case. Freely falling observers measure the standard radiation of an accelerated charge, whereas co-accelerating observers measure no radiation at all, not because it is not produced, but because all the radiation goes into the region of space-time in-accessible to the co-accelerating observer [10]. Specifically the co-accelerated observer has an event horizon with respect to the world line of the particle. From the co-accelerated coordinate system, the field at any point may be regarded either as the Coulomb field + outgoing radiation field of the charge at the intersection of its world line with the backward light cone, or as a Coulomb field + incoming radiation field of the charge at the intersection of its world line with the forward light cone of the field point (again, for details, see [10]). If one defines the radiation field as the semi-difference between the retarded and the advanced fields, the observer will not be able to decide whether there is any radiation or not. Thus, the co-accelerated observer only detects a Coulomb field, with no radiation at all. By contrast, one cannot argue that accelerated observers find no radiation. If one calculates the field in one of these systems, one again finds that it is a Coulomb field + outgoing radiation which cannot be interpreted as incoming radiation because the observer is
charged particle through a curved space-time. This may be emitted through the Čerenkov process by a sufficiently fast, charged, non-gravitating particle (see Ref. [14] for a survey and bibliography on the subject) implies that photons cannot be coupled to the gravitational field of our galaxy more strongly than relativistic charged baryons, to an accuracy of at least one part in $10^{14}$ [13].

More recently, a different approach of the gravitational Čerenkov effect was suggested [14]. In this new framework, photons and charged particles have the same coupling constants but the wave equations are different (i.e., photons couple to the Ricci tensor whereas fermions couple to the scalar curvature), so in some special metrics it may be possible for fermions to travel faster than photons. For photons, the wave equation is,

$$g^\mu\nu \nabla_\mu \nabla_\nu A^\alpha - R^\alpha_\mu A^\mu = 0,$$

where $A^\mu$ is the four vector potential. For fermions, instead, the wave equation is,

$$g^\mu\nu p_\mu p_\nu + \frac{1}{4}R + m^2 = 0,$$

where $p_\mu$ is the four momentum [14].

In Appendix B we discuss some features of Čerenkov radiation. At this stage it is worthwhile to analyze the nature of the matter that generates a gravitational field suitable for Čerenkov radiation. We shall discuss this with the help of the energy conditions.

The (point-like) energy conditions state that various linear combinations of the components of the stress-energy tensor (at any specified point in the space-time) should be positive, or at least non-negative [16]. Over the years, there have been much discussion on how fundamental the energy conditions really are. In particular, outside the backward light cone of the charge. The radiation is certainly present and may be identified by any of the standard methods.

We shall discuss some particular cases in which the space-time manifold is warped in such a way that both the charged particle and the co-accelerated observer experience an effective faster-than-light travel out of the event horizon, and gain access to the information stored there. As a consequence, the observer would start measuring the particle’s radiation, yielding a violation of the equivalence principle.

The possibility that an external gravitational field acts as an effective refractive index for light has a long history. As far as we are aware, it was first suggested by Beall [11], that a sufficiently fast, charged, non-gravitating particle would radiate strongly in a classical gravitational field. In the non-geodesic system, this radiation may be interpreted intuitively as the result of the gravitational field “slowing down” light waves, in analogy to the effect of the refractive medium in the case of Čerenkov radiation. However, if the equivalence principle is valid, no radiation can be emitted through the Čerenkov process by a charged particle through a curved space-time. This may be easily inferred from the previous discussion. Intuitively, if radiation is generated only by the existence of a particular gravitational field, and the co-moving observer does see it, the equivalence principle is violated.

Greek indices run from 0 to 3 and refer to the space-time; Latin indices from the middle of the alphabet ($i, j, k, \ldots$) run from 1 to 3 and refer to space; Greek indices from the beginning of the alphabet ($a, b, c, \ldots$) will run from 1 to 2 and will be used to refer to the wormhole throat and directions parallel to it. Hats refer to the orthonormal frame [1], and $\text{tr}(X)$ denotes $g^{\alpha\beta}X_{\alpha\beta}$.
it has become increasingly obvious that there are subtle quantum effects capable of violating all the energy conditions [17]. It has also become clear that there are quite reasonable classical systems, field theories that are compatible with all known experimental data and that are very natural from a quantum field theory point of view, which violate all these conditions too [18].

The aforementioned possibility that a background gravitational field has an effective refractive index greater than 1 is accompanied by unavoidable violations of one of these conditions, namely, the weak energy condition (WEC). WEC is satisfied, if and only if, for all future directed time-like vector $\xi^\mu$, $T^\mu_\nu \xi^\mu \xi^\nu \geq 0$. In terms of the density $\rho$ and principal pressures $p_i$, WEC $\iff$ $\rho \geq 0$ and $\forall j$, $\rho + p_j \geq 0$. To check for WEC violation we decompose the static metric in block diagonal form,

$$ds^2 = g_{\mu \nu}dx^\mu dx^\nu = -e^{2\phi}dt^2 + g_{ij}dx^i dx^j, \quad (3)$$

where $\phi$ is the redshift function (for traversable wormholes $\phi$ must be finite throughout the space-time to ensure the absence of event horizons). Being static, $t$ defines the direction of a Killing vector, thus, the space-time geometry may be analyzed in terms of the three geometry of the space. We conveniently adopt the natural time coordinate to separate the space-time into space + time. Now, using the Gauss-Codazzi and Gauss-Weingarten equations, straightforward computations decompose the $(3+1)$-dimensional space-time curvature tensor in terms of the three dimensional spatial curvature tensor and the gravitational potential. Using this decomposition it is possible to show that [19] $^{(3+1)}G_{ii} = 1/2 \cdot R^i_i$. If $^{(3+1)}R^i_i < 0$, then the WEC must be violated. If $^{(3)}R < 0$ then WEC is straightforwardly violated. On the other hand, if $^{(3)}R > 0$ and $^{(3+1)}R^i_i < 0$, it is easily seen that also $^{(3+1)}R < 0$. Then, since $^{(3+1)}R$ has to preserve its value under coordinate transformations, one can always find a coordinate system in which the observer measures a negative energy density. In other words, since negative energy densities implies the defocusing of null geodesic congruences, this inherent property of the space-time cannot depend on the coordinate system. Consequently, if in the same region of the space an observer measures negative energy, and another observer a positive one, the latter must measure an enormous tension in the $i$ direction such that $\rho + p_i < 0$, so that the surface could produce the defocusing of the null geodesic congruence. Then, if the matter threading the gravitational field violates WEC, $^{(3+1)}R^i_i < 0$.

Let us turn now to the analysis of possible “Čerenkov” radiation in the surroundings of a static wormhole throat. In an appropriate Gaussian normal coordinate system $x^i = (x^0 : \ell)$, where the anti-trapped surface $\Sigma$ is taken to be at $\ell = 0$ we get,

$$^{(3)}g_{ij}dx^i dx^j = (2)g_{ab}dx^a dx^b + d\ell^2. \quad (4)$$

Now, define the extrinsic curvature $K_{ab} = -1/2 \partial g_{ab}/\partial \ell$, and compute the variation in the area of $\Sigma$ obtained by pushing the surface at $\ell = 0$ out to $\ell = \delta \ell(x)$ [19],

$$\delta A(\Sigma) = \int \sqrt{g} \operatorname{tr}(K) \delta \ell(x) \, d^3x. \quad (5)$$

Since this expression must vanish for arbitrary $\delta \ell(x)$, the condition for the area to be extremal is simply $\operatorname{tr}(K) = 0$. For the area to be minimal the additional constraint $\delta^2 A(\Sigma) \geq 0$ is required. Equivalently, $\partial \operatorname{tr}(K)/\partial \ell < 0$. After a bit of more algebra, one can decompose the 3-dimensional spatial curvature tensor in terms of the two dimensional curvature tensor and the extrinsic curvature of the throat as an embedded hypersurface in the 3-geometry so that,

$$^{(3)}R = ^{(2)}R + 2 \frac{\partial \operatorname{tr}K}{\partial \ell} - \operatorname{tr}(K^2). \quad (6)$$

The third term in Eq. (6) is negative semi-definite by inspection, the second term must be negative semi-definite at the throat in order to flare outward the wormhole. Going into a little more detail, the Gauss curvature $^{(2)}R$ may be expressed in terms of the genus of the surface $g$ by means of the Euler characteristic $\chi = 2(1 - g)$ using the relation [20],

$$\frac{1}{4\pi} \int d^2x \sqrt{g}^{(2)}R = \chi. \quad (7)$$

Note that if $g < 1$, there must be places on the throat such that $^{(3)}R < 0$. Thus, ultra-static wormholes throats ($\phi$ = constant) of high genus will always have regions with $\rho < 0$ guaranteeing $n^2 \gg 1$. Recall that this last consequence is obtained because for ultra-static cases $^{(3)}R = ^{(3+1)}R^i_i$, and is negative because of the field equations. We can then argue that traversable wormholes provide an appropriate environment for a possible Čerenkov process.

Wormhole throats with the topology of the sphere not always have regions where $\rho < 0$. In particular, the ultra-static metric with spherical symmetry is given by,

$$dx^i dx^j = \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (8)$$

where $b(r)$ is the shape function. For the space-time to be asymptotically flat, far from the throat $b(r)/r \to 0$. In addition, $b(r)/r \leq 1$ with the equality holding at throat so as to flare outward the wormhole. Denoting with prime derivatives with respect to $r$ for this special case we get,

$$^{(3)}R = \frac{b'(r)}{r^2}. \quad (9)$$

Then, it is easily seen that the large flare out from the throat required to obtain $n^2 \gg 1$ will confine the exotic matter to an arbitrarily small neck region. A possible shape function with this properties was introduced in [1],
\[ b(r) = b_0 \left( 1 - \frac{r - b_0}{a_0} \right)^2 \text{ if } b_0 \leq r \leq b_0 + a_0, \quad (10) \]
\[ b(r) = 0 \text{ if } r > b_0 + a_0, \quad (11) \]

where \( b_0 \) is the throat radius and \( a_0 \) is a cutoff in the energy density. As stated above if \( n_\gamma^2 \gg 1 \) (and the equivalence principle is not valid) it would be possible to search for wormholes topologies via its output in Čerenkov radiation. Then from Eq. (18) we can easily estimate the region where \( n_\gamma^2 > 1 \), and hence of Čerenkov emission.

Take as an example \((3+1)R^\gamma_i/k_0\) to be of order 1, such as to obtain \( n_\gamma^2 = 2 \). For a proton energy of 10 GeV, the required flaring outward for \( b_0 = 1 \) m yields \( a_0 = 10^{-37} \) m. Certainly, no Čerenkov radiation is meaningful, as expected, the exotic matter region was confined to a slab of small thickness (worse is the situation for bigger values of \( n \)). As we have seen, the WEC violation region may be relaxed if the throat has the topology of a torus. In this kind of wormholes, usually called “ring-holes” [21], the strong requirement for the slope (almost vertical) of the shape function in Eq. (9) disappears, then one could expect the region appropriate for Čerenkov radiation to be long enough to produce observational consequences. This situation can be further improved increasing the genus of the throat.

Using traversable wormholes as theoretical background, we have revisited a still open question of general relativity: Does a uniformly accelerated charged particle radiate? Over more than forty years, it was a matter of controversy whether radiation would be emitted from a uniformly accelerated charge. In the last decades, however, a general consensus favoring the existence of radiation seems to have been reached. The discussion has now moved to another insightful question: Do co-accelerated observers measure (see) the charged particle radiation? A quite consistent picture was presented by Boulware [10], who argued that the radiation remains hidden behind the observer’s event horizon. However, Parrott [22] asserted that the definition of energy in Ref. [10] is erroneous, and that the adoption of the correct one makes purely local experiments to distinguish a stationary charged particle in a static gravitational field from an accelerated particle in Minkowskian space. If this is the case, the equivalence principle is not valid for charged particles.

In this article we have noticed that if Einstein gravity is valid, \((3+1)R^\gamma_i < 0\), and then WEC is not sustained, the effective refractive index for light is greater than 1, i.e. particles propagates, in that medium, faster than photons. This yields to Čerenkov radiation, produced only because of the existence of a particular gravitational field. This radiation can be seen by the comoving observer [14], and then he could tell that he has entered in a gravitational field. This leads to a violation of the equivalence principle. It is important to stress that substantial evidence in favor of the violation of the equivalence principle by quantum systems (some of them associated to negative energy densities) has been recently put forward [25].

There are, then, two distinct ways in which the photon velocity can be lower than the charged fermion velocity. These two ways are 1) a violation of the equivalence principle, as in Ref. [18], and 2) when the source energy-momentum tensor of the metric violates WEC. Condition 2 operates within Einstein’s gravity. Condition 1 is valid for theories which admit violation of the equivalence principle beforehand.

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APPENDIX A

The Rindler’s line element is given by,
\[ ds^2 = -g^2 Z^2 dx^2 + dZ^2 + dy^2 + dx^2. \quad (12) \]

The main properties of this metric are:

1. The curvature vanishes, showing that it must simply be a portion of Minkowski space-time. Indeed, if \( t, z, y, x \) denote the usual Minkowski coordinates, then the identification is (for \( z > t \), region I of Fig. 1):
\[ z = Z \cosh(\nu \tau), \quad t = Z \sinh(\nu \tau). \quad (13) \]

Similar transformations are valid for the other regions. Note that \( x^2 - t^2 = Z^2 \), and that for fixed \( Z \), the world line is a hyperbola.

2. If \( Z \) is a constant, \( Z = Z_0 \), the four velocity in Minkowski space-time is given by (for fixed \( x, y \))
\[ u^\nu = \frac{d\nu}{d\lambda} = gZ_0 \left( \cosh(\nu \tau), \sinh(\nu \tau) \right) \frac{d\tau}{d\lambda} \quad (14) \]

where \( \lambda \) is the proper time of the particle. We can immediately see that, because of \( u^2 = -1 \), \( d\tau/d\lambda = 1/gZ_0 \).

3. The four acceleration in Minkowski space-time is
\[ a^\nu = \frac{da^\nu}{d\lambda} = \frac{1}{Z_0} \left( \sinh(\nu \tau), \cosh(\nu \tau) \right), \quad (15) \]

and its square is the constant \( 1/Z_0^2 \).
4. Temporal translations in Eq. (1) are equivalent to Lorentz boosts in Minkowski space-time. Under \( \tau \rightarrow \tau + \alpha \), the transformed plane coordinates are, 
\[
z' \pm t' = (z \pm t)e^{\pm \alpha} = \frac{1 \pm v}{\sqrt{1 - v^2}} ,
\]
resulting in Lorentz transformations with velocity \( v \).

5. As seen by the uniformly accelerated observer, a world line crosses the boundaries of region I at \( \tau = \pm \infty \). There is a future event horizon at \( Z = 0 \).

APPENDIX B

Čerenkov radiation occurs when a fast particle moves through a medium at a constant velocity \( v \), which is greater than the velocity of light in that medium. Because of the superluminal motion of the particle, a shock wave is created and this yields to a loss of energy. The wavefront of the radiation propagates at a fixed angle
\[
\cos \theta = \frac{v_{\text{phase}}}{v} = \frac{c/n(\nu)}{v} , \tag{17}
\]
where \( \nu \) is the photon frequency and \( n \) is the refractive index. Only in this direction do the wavefronts add up coherently. The value \( \cos \theta = 1 \) corresponds to the threshold for emission. It is clear that \( \cos \theta < 1 \) cannot be satisfied (and then will be no radiation) if \( n < 1 \) (because \( v \) always is less than \( c \)). See, for example, Refs. [23,24].

It was shown in Ref. [14], that a static gravitational field (with metric \( g_{\mu \nu} \)) has an effective refractive index given by
\[
n^2(\nu_0) = |g^{00}| \left( 1 - \frac{\left(3+1\right)_{R^1_i}}{|g^{00}k_0^2|} \right) . \tag{18}\]
Here, \( R^1_i \) stands for the sum on the spatial indices of the Ricci tensor \( R^\mu_\nu \), and \( \nu_0 \) is the frequency of the emitted photon \( \gamma \). Recall that hats refer to the orthonormal frame. The crucial point, in order that the Čerenkov radiation may kinematically allowed, is that \( 3+1\gamma_{R^1_i} < 0 \), so that \( n^2(\nu_0) > 1 \).

The energy radiated by Čerenkov process by a charged particle moving in a background gravitational field is given by (for details, see [14])
\[
\frac{dE}{dt} = \frac{Q^2 \alpha_{\text{em}}}{4\pi p_0} \int_{k_0}^{k_0^2} d(k_0) = \int_{k_0}^{k_0^2} d(k_0) \left( p_0(p_0 - k_0) - \frac{1}{2} k_0^2 \right) k_0^2 \frac{n^2 - 1}{n^2} . \tag{19}\]
where \( Q \) is the charge of the fermion emitting the photon, \( \alpha_{\text{em}} \) is the electromagnetic coupling constant, \( p_0 \) is the energy of the fermion, and \( k_{01}, k_{02} \) are the allowed range of frequencies where radiation can occur.

The interesting result is obtained for \( n^2 \gg 1 \), since the spectrum of energy radiated by a charged particle, Eq. (19), assumes the form
\[
\frac{dE}{d\omega} \frac{d\omega}{d\tau} = \frac{Q^2 \alpha_{\text{em}}}{4\pi} \left[ 1 - \frac{k_0}{p_0} - \frac{k_0^2}{2p_0} \right] k_0 , \tag{20}\]
which differs in a substantial way from the thermal or synchrotron emission.

celó and M. Visser, Phys. Lett. B 466, 127 (1999); [gr-qc/0003025].


