Energy and momentum of the elementary excitations become independent variables in medium: energy and momentum statistical distributions are not identical. The momentum distribution and not the energy distribution is relevant for barrier penetration. The deviations of the momentum distribution from the Maxwell-Boltzmann energy distribution can be expressed in terms of the imaginary part of the self-energy of the quasi-particle. It is possible to obtain an effective Tsallis' distribution for the kinetic energy. These effects are different from static or dynamical screening and can have important consequences for reaction rates in stars.

1 Introduction

Theoretical calculations of nuclear rates as function of temperature, density and composition are fundamental ingredients of our understanding of stellar structure. Solar standard models (SSMs) are based on rates of the nuclear reactions inside the Sun calculated according to what is the actual experimental and theoretical state-of-the-art 1.

The solar structure is quite robust and even large changes of nuclear rates yield non standard solar models 2 whose structure is very similar to the standard one: only recent precise measurements of helioseismic frequencies can discriminate between standard and non standard models 3. In addition, there exist changes in chemical composition, such as those produced by He\(^3\) mixing 4, that cannot be detected even by seismic measurements.

The flux and energy spectrum of neutrinos from the Sun are observables quite sensitive to some of the reaction rates, but the possibility of oscillations weakens the link between the observed fluxes and the rates inside the Sun 5.

It is, therefore, of the outmost importance that nuclear rates inside the Sun, and more in general inside the stars, be accurately calculated and all possible effects be taken into account.

Cross sections are fundamental for the determination of the rates: the astrophysical S-factors used in stellar model calculations are often extrapola-
tions from experimental data at higher energies than those relevant for stellar interiors; some reactions, e.g., the $p + p$ reaction, cannot be measured. Although the underlying theory is generally robust, new calculations are still necessary.

As an example, a recent and more accurate calculation of the $\text{He}^3 + p$ reaction predicts a S-factor five times larger than the one used in the SSM. Even if this reaction remains not important for the basic structure of the Sun, it becomes not negligible for the high-energy part of the neutrino spectrum. In particular, it could be important in the interpretation of the possible excess of solar-neutrino events above 13 MeV, even if the latest experimental data seem to suggest that the excess was a statistical fluctuation. Nevertheless, there exist a vacuum oscillation solution to solar neutrino problem that reproduces the apparent seasonal variation of the temporal series of the GALLEX and Homestake data and that would imply a high energy distortion of the spectrum of the recoil electrons. The most distinct signature of this solution is a semiannual seasonal variation of the $^7\text{Be}$ neutrino flux with maximal amplitude, future detectors (BOREXINO, LENS and probably GNO) will be able to test it: the accurate calculation of the $\text{He}^3 + p$ reaction will be very important also for the theoretical interpretation of these results.

However, nuclear rates depend not only on the reaction cross sections but also on the properties of the nuclear plasma. Nuclear plasmas, i.e., neutral systems of charged particles (ions and electrons), are complicated many-body systems: charged are screened, but the screening depends in general on the energy of the particle and the remaining interaction is long range and nonlocal. The resulting spatial and temporal correlations between ions have large effects especially on fusion reactions, which occur between those high-energy ions that can tunnel the Coulomb barrier.

An accurate theoretical determination of these rates requires a good understanding of all possible plasma effects; at the same time, reactions that select ions in the tail of the energy distribution are probes of the dynamics of the plasma itself.

In this contribution we only want to discuss one such plasma effect: the possibility that the effective momentum distribution of the fusing ions could deviate from the standard Maxwell-Boltzmann (MB) distribution. In particular, we want to report on our work in progress that tries and link the deviation from the MB distribution to the imaginary part of the self-energy of the quasi-particle states.

While this effect is potentially important for many astrophysical phenomena (the solar neutrino problem, brown dwarves, dark matter distribution), we shall focus on parameters relevant to the solar core.
2 Non ideal plasmas

The solar core is a weakly-nonideal plasma where: 1) the mean Coulomb energy potential is of the order of the thermal kinetic energy; 2) the Debye screening length $R_D \approx a$ (interparticle distance): Debye-Hückel conditions are only approximately verified; 3) it is not possible to separate individual and collective degrees of freedom; 4) the inverse solar plasma frequency ($t_{pl} = \omega_{pl}^{-1} = \sqrt{m/4\pi ne^2} \approx 10^{-17}$) is of the same order of magnitude of the collision time $t_{coll} = f^{-1} = \langle n\sigma v \rangle$; 5) particles loose memory of the initial state only after many collisions: the scattering process cannot be considered Markovian; 6) the time needed to build up again the screening, after hard collisions, is not negligible $^8$. 

At the thermal equilibrium reacting ions are usually described as quasi free particles with Maxwell-Boltzmann (MB) velocity distribution. But many-body effects inside the plasma could cause deviations from a pure Maxwell-Boltzmann statistics for the effective degrees of freedom. Because reacting ions belong to the high momentum tail of the distribution, at least for fusion reactions between charged ions, even tiny deviation from the MB tail can cause large modifications (enhancement or depletion) of the rates.

The value of the collision frequency $f$ determines the possibility of two different effects that produce important deviations from the Maxwellian distribution $F_M(p)$ at high momenta:

1. Quantum uncertainty effect. When the Coulomb collisional frequency is large ($hf > kT$) the ions cannot be considered as quasi-free particles: the energy and momentum distributions are different and one must decide which one is relevant for the reaction rates. The fact that the two distributions are not equivalent is related to the finite life-time of the quasi-particles and to the quantum uncertainty. Since nuclear rates should be evaluated averaging the quasi-classical cross section $\sigma(p)$ over the momentum distribution, rather than the energy distribution, even if the energy distribution is Maxwellian, the effective distribution can acquire a non-Maxwellian tail $^{10,11}$.

2. Weak nonextensivity effect. Tsallis statistics $^{12}$ with entropic parameter $q$ can describe systems that are not extensive due to long-range interactions or non-Markovian memory effects; the energy distribution itself deviate from the standard free-particle statistics. When deviations are small ($q \approx 1$) the correction (enhanced or depleted tail) can be described by the factor $\exp[-\frac{1}{q} \left(\frac{\epsilon}{kT}\right)^2]$. 

Submitted to World Scientific on November 30, 2000
Deviation from the Maxwellian tail due to either $Q$ or $q$ effect (or both) may lead to strong increase or decrease of the nuclear rates in the solar core (non standard solar models due to large changes of the nuclear rates and their implications for the solar neutrino problem are described in Ref. 2).

In this contribution we discuss only the $Q$ effect, i.e., the deviation of the momentum distribution relative to the energy distribution. As we shall see, this effect leads only to an enhanced tail.

### 3 Quasi-particle momentum distribution

Many properties of interacting systems can often be described by weakly interacting excitations or quasi-particles. The energy-momentum dispersion relation (position of the pole of the one-particle Green’s function) of these excitations is found by solving

$$\omega = \frac{p^2}{2m} + \Sigma(\omega, p^2),$$

where $\Sigma(\omega, p^2) = \Sigma_R + i\Sigma_I$ is the self-energy of the one-particle propagator.

In the approximation of a constant real part of the self-energy $\Sigma_R$, we obtain the shift of the energy due to the static mean-field (in plasma it produces non-dynamical screening).

The $p^2$ dependence of $\Sigma_R$ reflects the spatial nonlocality of the effective interaction and may be understood qualitatively by considering the nonlocality of the exchange term of the Hartree-Fock potential, while the $\omega$ dependence reflects the nonlocality of $\Sigma$ in time.

As long as the imaginary part $\Sigma_I$ is zero, there exist a one-to-one correspondence between the energy $\omega$ and momentum $p$ (or kinetic energy $p^2/(2m)$). In real systems the imaginary part of the self-energy of the quasi-particle is not zero and energy and momentum become independent variables; however, they are still strongly correlated when $\Sigma_I$ is small (only if $\Sigma_I$ is small the concept of quasi-particle is useful).

Barrier penetration is determined by the momentum of the (quasi-)particle and not by its energy (when they do not coincide).

In this preliminary presentation, we shall restrict ourselves to the case of an energy distribution that is Maxwellian: $P(E) \sim \exp(-\beta E)$.

If we are given the relation between $E$ and $p^2$ in the form $F(E, p^2)$, the momentum distribution is obtained

$$P(p^2) = \int_0^\infty dE e^{-\beta E} F(E, p^2).$$
For free particles \( F(E, p^2) = \delta \left( E - p^2 / (2m) \right) \) and, therefore,

\[
P(p^2) = e^{-\beta \frac{p^2}{2m}} .
\]  

(3)

If \( \Sigma_I = 0 \) and \( \Sigma_R \) is constant, then \( F(E, p^2) = \delta \left( E - p^2 / (2m) - \Sigma_R \right) \) and the distribution is still Maxwellian

\[
P(p^2) = e^{-\beta \left( \frac{p^2}{2m} + \Sigma_R \right)} .
\]  

(4)

If \( \Sigma_R \) is not constant but can be expanded in the region of interest

\[
\Sigma_R(\omega, p^2) = \Sigma_R + \frac{\partial \Sigma_R}{\partial p^2}(p^2 - p_0^2) + \frac{\partial \Sigma_R}{\partial \omega}(\omega - \omega_0)
\]  

(5)

the result is

\[
P(p^2) = e^{-\beta \left( \frac{p^2}{2m^*} + \Sigma_R \right)} ,
\]  

(6)

where

\[
m^* = m \left( 1 + 2m \frac{\partial \Sigma_R}{\partial p^2} \right)^{-1} \left( 1 - \frac{\partial \Sigma_R}{\partial \omega} \right) .
\]  

(7)

In all the above cases the distribution for the variable \( p^2 \) is still Maxwellian (in general it follows the energy distribution), even if the shift in energy or the effective mass can have important phenomenological consequences (screening, level densities, etc.).

When the imaginary part of the self-energy cannot be disregarded \( (\Sigma_I > 0) \), there appear deviations from the Maxwellian distribution. For the sake of discussion let us consider the following relation between \( \omega \) and \( \epsilon_p \equiv p^2 / (2m^*) + \Sigma_R \):

\[
F_\nu(\omega, p^2) = \frac{1}{\Gamma(1/\nu)} \int_0^\infty dE F_\nu(E, p^2) = \frac{1}{\Gamma(1/\nu)} \int_0^\infty dxx^{1/\nu-1}e^{-x} = 1 ,
\]  

(9)

and in the limit \( \nu \to 0^+ \)

\[
\log(F_\nu) = \frac{1}{\nu} \left[ 1 - \frac{E}{\epsilon_p} + \log \frac{E}{\epsilon_p} \right] - \frac{1}{2} \log(2\pi \nu E^2) + O(\nu)
\]  

(10)
The function between bracket \( f(x) = 1 - x + \log(x) \leq 0 \) and is zero only for \( x = 1 \); therefore, if \( E \neq \epsilon_p \), \( \lim_{\nu \to 0^+} \log(F_\nu) = -\infty \) and \( \lim_{\nu \to 0^+} F_\nu = 0 \). If \( E = \epsilon_p \), \( \lim_{\nu \to 0^+} F_\nu = \frac{1}{\sqrt{2\pi \nu E}} = \infty \). In summary

\[
\lim_{\nu \to 0^+} F_\nu(E, p^2) = \delta(E - \epsilon_p) \quad (11)
\]

If \( \nu \) is small but not zero, we try the Ansatz \( E/\epsilon_p = 1 + a \sqrt{\nu} \) into the equation (10) and find

\[
\log(F_\nu) = -\frac{a^2}{2} - \frac{1}{2 \log(2\pi \nu E^2)} + O(\nu) \quad (12)
\]

where the terms that have been dropped are really of order \( \nu \) as long as \( a = (E/\epsilon_p - 1)/\sqrt{\nu} \) remains of order one, i.e., as long as \( E/\epsilon_p - 1 \) does not becomes large compared to \( \sqrt{\nu} \) (remains of order \( \sqrt{\nu} \)).

From the definition \( a^2 = (E/\epsilon_p - 1)^2/\nu \) we can rewrite Eq. (12)

\[
\lim_{\nu \to 0^+} F_\nu = \frac{1}{\sqrt{2\pi \nu E}} e^{-\frac{1}{2} \left( \frac{E - \epsilon_p}{\sqrt{\nu} \epsilon_p} \right)^2} = \frac{1}{\sqrt{2\pi \nu \epsilon_p}} e^{-\frac{1}{2} \left( \frac{E - \epsilon_p}{\sqrt{\nu} \epsilon_p} \right)^2} \quad (13)
\]

Since \( \sqrt{\nu} \epsilon_p \) is the width of the distribution and the imaginary part of the self-energy \( \Sigma_I \) is also proportional to the width of the distribution (when it is in Lorenzian form), it is plausible that

\[
\nu = C \left( \frac{\Sigma_I}{\epsilon_p} \right)^2 \quad (14)
\]

at least in the limit \( \Sigma_I \ll \epsilon_p \); the constant \( C \) depends on the precise definition of the limit of the quasi-particle (we are working on the microscopical derivation of such kind of relation and of \( C \)).

If we instead calculate the distribution of \( \epsilon_p \) from the relation \( F_\nu(\omega, p^2) \) between energy and momentum for general \( \nu \) (without expansion for small \( \nu \)), we find

\[
\int_0^\infty d\omega e^{-\beta \omega} F(\omega, p^2) = \int_0^\infty d\omega e^{-\beta \omega} \frac{1}{\Gamma(1/\nu)} \frac{1}{\omega} \left( \frac{\omega}{\nu \epsilon_p} \right)^{1/\nu} e^{-\frac{\omega}{\nu \epsilon_p}}
\]

\[
= (1 + \nu/\beta \epsilon_p)^{-1/\nu} \quad (15)
\]

which is the Tsallis’ distribution with \( \nu = q - 1 \geq 0 \).

It is not possible to obtain a Tsallis’ distribution with \( \nu = q - 1 < 0 \) by this kind of effect, since the broadening of dispersion relation between energy and momentum physically has the effect of increasing the tail of the distribution: it cannot “cut” the tail.
At least in the limit of small $\Sigma_I$, we have given a phenomenological interpretation of the parameter $q$ in the Tsallis’ distribution

$$q - 1 = C \left( \frac{\Sigma_I}{\epsilon_p} \right)^2 .$$

(16)

4 Width comparison

We are studying a more general approach that would allow to compare different relations between $p^2$ and $E$, at least in the asymptotic limit that each relation $F(E, p^2)$ tends to a $\delta$-function, $F(E, p^2) \to \delta(E - \epsilon_p)$. As long as we consider one-parameter generalizations of the $\delta$-function, there should be some mapping from one parameterization to the other. We have already seen how the relation in Eq. (8) and a Gaussian relation coincide in the limit of small width.

An other interesting case is the Lorenzian, which also becomes a $\delta$-function in the limit of vanishing width. In fact, the quasi-particles dispersion relation becomes in dense media (at least in the limit of large life-time)\textsuperscript{13}

$$\delta_g(\epsilon) = \frac{1}{\pi} \frac{g(\epsilon, p)}{[(\epsilon - \epsilon_p - \Delta(\epsilon, \epsilon_p))^2 + g(\epsilon, \epsilon_p)^2]} ,$$

(17)

where $\epsilon_p = p^2/2m$, $\Delta(\epsilon, \epsilon_p)$ and $g(\epsilon, p)$ are the real and imaginary parts of the one-particle retarded Green’s function self-energy.

For weakly non ideal plasmas one can show that $\Delta \approx kT \Gamma / 2$, where $\Gamma$ is the plasma parameter ($\Gamma = e^2 / R_D kT$) and $g \propto hf$. At non zero value of $g$, a nonexponential tail appears in the distribution function $F_Q(p)$. For large momenta, it has been found\textsuperscript{14,15} that

$$F_Q(p) = F_M(p) + \frac{hf}{2\pi} \frac{kT}{\epsilon_p^2} e^{\mu/kT} ,$$

(18)

where $\mu$ is the chemical potential and $F_M(p)$ the Maxwellian distribution.

5 Conclusions

We have shown that even when ions have a MB energy distribution the finite life-time of the quasi-particles in the plasma can produce a non-Maxwellian momentum distribution. Since the tunneling probability between charged ions must be evaluated using the momentum distribution, the reaction rates are effectively obtained by using distributions that depart from the MB one. This departure from the MB distributions can be calculated if one knows the spectral dispersion relation between energy and momentum.
In particular, we have shown that it is possible to have relations between energy and momentum that yield momentum distributions of Tsallis type with entropic parameter $q > 1$, which corresponds to an enhanced tail.

In this framework, we have suggested a possible interpretation of the parameter $q$ in terms of the imaginary part of the quasi-particle self-energy, at least in the limit of $q \rightarrow 1$ (small deviations from MB).

Acknowledgments

It is a pleasure to thank the Organizers for the very interesting and fruitful Meeting. This work was supported by “Confinanziamento MURST-PRIN”.

References

5. G. Fiorentini et al, Phys. Rev. D 49, 6298 (1994);
   G. Fiorentini et al, Phys. Lett. B 324, 425 (1994);
6. L. E. Marucci et al, Phys. Rev. Lett. 84, 5959 (2000);
   L. E. Marucci in these Proceedings.
    see also http://tsallis.cat.cbpf.br/biblio.htm
13. V. M. Galitskii, V. V. Yakimets, JETP 24, 637 (1967).