Approximate Decoherence of Histories
and ’t Hooft’s Deterministic Quantum Theory

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ABSTRACT: In the decoherent histories approach to quantum theory, sets of histories are said to be decoherent when the decoherence functional, measuring interference between pairs of histories, is exactly diagonal. In realistic situations, however, only approximate diagonality is ever achieved, raising the question of what approximate decoherence actually means and how it is related to exact decoherence. This paper explores the possibility that an exactly decoherent set of histories may be constructed from an approximate set by small distortions of the operators characterizing the histories. In particular, for the case of histories of positions and momenta, this is achieved by doubling the set of operators and then finding, amongst this enlarged set, new position and momentum operators which commute, so decohere exactly, and which are “close” to the original operators. Two derivations are given, one in terms of the decoherence functional, the second in terms of Wigner functions. The enlarged, exactly decoherent, theory has the same classical dynamics as the original one, and coincides with the so-called deterministic quantum theories of the type recently studied by ’t Hooft. These results suggest that the comparison of standard and deterministic quantum theories may provide an alternative method of characterizing emergent classicality. A side-product is the surprising result that histories of momenta in the quantum Brownian motion model (for the free particle in the high-temperature limit) are exactly decoherent.
1. INTRODUCTION

How close to classical mechanics can quantum mechanics be? One of the main aims of the decoherent histories approach is to demonstrate the emergence of classical mechanics as an effective theory, starting from the assumption that quantum mechanics is the exact underlying theory [1,2,3,4,5]. In such studies, the effective classical theory almost always emerges in an approximate way, rarely exact. The main reason for this is that decoherence, the destruction of quantum interference, is almost always approximate. What does approximate decoherence mean? What is the nature of the histories that approximately decoherent histories are an approximation to?

The aim of this paper is to explore the idea that approximate decoherence of histories can be turned into exact decoherence by suitable “small” modifications of the operators characterizing the histories. In particular, histories characterized by fixed values of coordinates and momenta \( x, p \) are rendered exactly decoherent by replacing \( x, p \) with new coordinates and momenta \( X, P \) which commute. This replacement, we show, is a valid approximation provided that the original histories are approximately decoherent. The new theory in terms of the commuting variables \( X, P \) has the same form as the so-called deterministic quantum theories of the type recently studied by ’t Hooft [6].

To set up the problem in more detail, we briefly review the decoherent histories approach [1,2,3,4,7,8]. In the decoherent histories approach to quantum theory, probabilities are assigned to histories of a closed system via the formula,

\[
p(\alpha_1, \alpha_2, \cdots, \alpha_n) = \text{Tr} \left( P_{\alpha_n}(t_n) \cdots P_{\alpha_1}(t_1) \rho P_{\alpha_1}(t_1) \cdots P_{\alpha_n}(t_n) \right) \tag{1.1}
\]

The projection operators \( P_\alpha \) characterized the different alternatives describing the histories at each moment of time. The projectors satisfy

\[
\sum_\alpha P_\alpha = 1, \quad P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha \tag{1.2}
\]

and the projectors appearing in (1.1) are in the Heisenberg picture,

\[
P_{\alpha_k}(t_k) = e^{iH(t_k-t_0)} P_{\alpha_k} e^{-iH(t_k-t_0)} \tag{1.3}
\]

Probabilities can be assigned to histories if and only if all histories in the set obey the condition of consistency, which is that

\[
\text{Re} D(\alpha, \alpha') = 0 \tag{1.4}
\]
The formula (3.13) bears a close resemblance to Eq.(2.13), the probabilities for histories in the exactly decoherent DQT. There are, however, three differences. First, (3.13) has dissipation in the equations of motion but (2.13) does not, but this is easily fixed by the trivial generalization of (2.13) to the case of the dissipative action (1.17). Second, (2.13) has a delta-function peak about the equation of motion, whilst (3.13) has only a Gaussian peak, due to the thermal fluctuations. This Gaussian peak becomes sharper as the mass of the particle increases. Moreover, the difference between the two types of peak will not be noticed if the width of the projections in (3.13) are much greater than the width of the Gaussian. Third, (3.13) has a (not necessarily positive) Wigner function weighting its initial conditions, whilst (2.13) has a positive weight function. But given that the fluctuations tend to smear \( W \) so as to be positive anyway (as will be discussed at greater length below), for a wide variety of initial states it ought to be possible to choose an initial state in (2.13) to give essentially the same results as (3.13).

Of the above differences, the most important one is the delta-function versus Gaussian peak. We therefore conclude that as long as the particle is sufficiently massive to substantially resist the effects of thermal fluctuations, the exactly decoherent DQT of Section 2 approximately reproduces the probabilities of the approximately decoherent histories of standard quantum theory described above. This is our first result on the closeness of DQT and standard quantum theory.

The above result applies, however, only to the case when the mass of the particle is sufficiently large to resist thermal fluctuations. It does not apply to the case where there is approximate decoherence but the fluctuations about classical deterministic behaviour are not small, as in the case of small mass. The most general effective theories emerging from an underlying quantum theory are classical stochastic theories, perhaps with large fluctuations. We therefore need to generalize our comparison of DQT and standard quantum theory to this case, and this turns out to be somewhat more complicated. It requires comparing the quantum Brownian motion model of Section 3 to a DQT including an environment to provide fluctuations.

We have seen for a simple linear system with action \( S[x] \), a closely related DQT may be constructed using the action \( S = S[x] - S[y] \) and by focusing on the variable \( X = x + y \). The coupling to an environment, as in Eq.(3.1), requires a reconsideration of the question of how to construct the related DQT. On the basis of what we have seen so far – that the DQT is obtained by doubling what we already have – it seems natural to double up both the system and the environment. Whilst this in fact turns out to be correct, one might