One crucial problem in string theory is the problem of vacuum selection. It is reasonable to believe that this problem can be solved only in the context of cosmology, by studying the time evolution of generic, inhomogeneous (non-SUSY) string vacua. In this vein, it has been recently found [1,2] that the general solution near a spacelike singularity \((t \to 0)\) of the massless bosonic sector of all superstring models \((D = 10\) IIA, IIB, I, HE, HO), as well as that of \(M\)-theory \((D = 11\) SUGRA), exhibits a never ending oscillatory behaviour of the scale factors and the dilaton at \(t \to 0\). It is convenient to work with the \(10\) field variables \(\beta_{\mu}, \mu = 1, \ldots, 10\), with, in the superstring (Einstein-frame) case, \(\beta^0 = -\ln(a_i)\) \((i = 1, \ldots, 9)\), and \(\beta^{10} = -\varphi\) where \(\varphi\) is the Einstein-frame dilaton. In \(M\)-theory there is no dilaton but \(\mu \equiv i = 1, \ldots, 10\). In the string frame, we define \(\beta_{\mu}^0 \equiv -\ln(\sqrt{g_{\mu\nu}e^{-2\varphi}})\) and label \(\mu = 0, \ldots, 9\).

We shall focus in this letter on the dynamical behaviour of the metric and the string dilaton and recall first the relevant equations from [1,2]. To leading order, the metric (in either the Einstein frame or the string frame) reads \(g_{\mu\nu} dx^\mu dx^\nu = -N^2(dx^\varphi)^2 + \sum_{i=1}^{d} a_i^2 (\omega^i)^2\), where \(d \equiv D-1\) denotes the spatial dimension, and where \(\omega^i(x) = \epsilon^i(x) dx^j\) is a d-bein whose time-dependence is neglected compared to that of the local scale factors \(a_i\). It is convenient to work with the \(10\) field variables \(\beta_{\mu}, \mu = 1, \ldots, 10\), with, in the superstring (Einstein-frame) case, \(\beta^0 = -\ln a_i\) \((i = 1, \ldots, 9)\), and \(\beta^{10} = -\varphi\) where \(\varphi\) is the Einstein-frame dilaton. In \(M\)-theory there is no dilaton but \(\mu \equiv i = 1, \ldots, 10\). In the string frame, we define \(\beta_{\mu}^0 \equiv -\ln(\sqrt{g_{\mu\nu}e^{-2\varphi}})\) and label \(\mu = 0, \ldots, 9\).

We consider the evolution near a past (big-bang) or future (big-crunch) spacelike singularity located at \(t = 0\), where \(t\) is the proper time from the singularity. In the gauge \(N = -\sqrt{g}\) (where \(g\) is the determinant of the Einstein-frame spatial metric), i.e. in terms of the new time variable \(d\tau = -dt/\sqrt{g}\), the action (per unit comoving volume) describing the asymptotic dynamics of \(\beta^\mu\) as \(t \to 0^+\) or \(\tau \to +\infty\) has the form

\[
S = \int d\tau \left[ G_{\mu\nu} \frac{d\beta^\mu}{d\tau} \frac{d\beta^\nu}{d\tau} - V(\beta^\mu) \right],
\]

where

\[
V(\beta) \approx \sum_A C_A e^{-2w_A(\beta)},
\]

In addition, the time reparametrization invariance (i.e. the equation of motion of \(N\) in a general gauge) implies the usual “zero-energy” constraint \(E = G_{\mu\nu}(d\beta^\mu/d\tau)(d\beta^\nu/d\tau) + V(\beta^\mu) = 0\). The metric \(G_{\mu\nu}\) in field-space is a 10-dimensional metric of Lorentzian signature \(-+++\ldots+\). Its explicit expression depends on the model and the choice of variables. In \(M\)-theory,

\[
G^{M}_{\mu\nu} d\beta_{\mu} d\beta_{\nu} = \sum_{\mu=1}^{10} (d\beta_{M}^\mu)^2 - \left(\sum_{\mu=1}^{10} d\beta_{M}^\mu\right)^2,
\]

while in the string models,

\[
G^{S}_{\mu\nu} d\beta_{S} d\beta_{\nu} = \sum_{i=1}^{9} (d\beta_{S}^i)^2 - (d\beta_{S}^9)^2
\]

in the string frame. Each exponential term, labelled by \(A\), in the potential \(V(\beta^\mu)\), Eq. (2), represents the effect, on the evolution of \((g_{\mu\nu}, \varphi)\), of either (i) the spatial curvature of \(g_{ij}\) (“gravitational walls”), (ii) the energy density of some electric-type components of some p-form \(A_{\mu_1 \ldots \mu_p}\) (“electric p-form wall”), or (iii) the energy density of some magnetic-type components of \(A_{\mu_1 \ldots \mu_p}\) (“magnetic p-form wall”). The coefficients \(C_A\) are all found to be positive, so that all the exponential walls in Eq. (2) are repulsive. The \(C_A\)'s vary in space and time, but we neglect their variation compared to the asymptotic effect of \(w_A(\beta)\) discussed below. Each exponent \(-2w_A(\beta)\) appearing in Eq. (2) is a linear form in the
$\beta^\mu : w_A(\beta) = w_{A\mu} \beta^\mu$. The complete list of “wall forms” $w_A(\beta)$, was given in [1] for each string model. The number of walls is enormous, typically of the order of 700.

At this stage, one sees that the $\tau$-time dynamics of the variables $\beta^\mu$ is described by a Toda-like system in a Lorentzian space, with a zero-energy constraint. But it seems daunting to have to deal with $\sim 700$ exponential walls! However, the problem can be greatly simplified because many of the walls turn out to be asymptotically irrelevant. To see this, it is useful to project the motion of the variables $\beta^\mu$ onto the 9-dimensional hyperbolic space $H^9$ (with curvature $-1$). This can be done because the motion of $\beta^\mu$ is always time-like, so that, starting (in our units) from the origin, it will remain within the 10-dimensional Lorentzian light cone of $G_{\mu\nu}$. This follows from the energy constraint and the positivity of $V$. With our definitions, the evolution occurs in the future light-cone. The projection to $H^9$ is performed by decomposing the motion of $\beta^\mu$ into its radial and angular parts (see [4,5] and the generalization [6]). One writes $\beta^\mu = +\rho \gamma^\mu$ with $\rho^2 \equiv -G_{\mu\nu} \beta^\mu \beta^\nu$, $\rho > 0$ and $G_{\mu\nu} \gamma^\mu \gamma^\nu = -1$ (so that $\gamma^\mu$ runs over $H^9$, realized as the future, unit hyperboloid) and one introduces a new evolution parameter:

$$dT = k \, d\tau / \rho^2.$$  

The action (1) becomes

$$S = k \int dT \left[ -\left( \frac{d \ln \rho}{dT} \right)^2 + \left( \frac{d\gamma}{dT} \right)^2 - V_T(\rho, \gamma) \right]$$

where $d\gamma^2 = G_{\mu\nu} \, d\gamma^\mu \, d\gamma^\nu$ is the metric on $H^9$, and where $V_T = -k^{-2} \rho^2 \, V - \sum_A G_{\mu\nu} C_A \rho^2 \exp(2 \rho \, w_A(\gamma))$. When $T \to 0^+$, i.e. $\rho \to +\infty$, the transformed potential $V_T(\rho, \gamma)$ becomes sharper and sharper and reduces in the limit to a set of $\rho$-independent impenetrable walls located at $w_A(\gamma) = 0$ on the unit hyperboloid (i.e. $V_T = 0$ when $w_A(\gamma) > 0$, and $V_T = +\infty$ when $w_A(\gamma) < 0$). In this limit, $d \ln \rho / dT$ becomes constant, and one can choose the constant $k$ so that $d \ln \rho / dT = 1$. The (approximately) linear motion of $\beta^\mu(\tau)$ between two “collisions” with the original multi-exponential potential $V(\beta^\mu)$ is thereby mapped onto a geodesic motion of $\gamma(T)$ on $H^9$, interrupted by specular collisions on sharp hyperplane walls. This motion has unit velocity $(d\gamma / dT)^2 = 1$ because of the energy constraint. In terms of the original variables $\beta^\mu$, the motion is confined to the convex domain (a cone in a 10-dimensional Minkowski space) defined by the intersection of the positive sides of all the wall hyperplanes $w_A(\beta) = 0$ and of the interior of the future light-cone $G_{\mu\nu} \beta^\mu \beta^\nu = 0$.

A further, useful simplification is obtained by quotienting the dynamics of $\beta^\mu$ by the natural permutation symmetries inherent in the problem, which correspond to “large diffeomorphisms” exchanging the various proper directions of expansion and the corresponding scale factors. The natural configuration space is therefore $R^d / S_d$, which can be parametrized by the ordered multiplets $3^\mu \leq 2^\mu \leq \cdots \leq d^\mu$. This quotienting is standard in most investigations of the BKL oscillations [3] and can be implemented in $R^d$ by introducing further sharp walls located at $\beta^\mu = \beta^\mu + 1$. Note that the natural permutation symmetry group is different in $M$-theory (where it is $S_{10}$), and in the $D = 10$ string models ($S_9$), and would be still smaller in the successive dimensional reductions of these theories. However, there is a natural consistency in quotienting each model by its natural permutation symmetry. Indeed, one finds that, upon dimensional reduction, there arise new (exponential) walls, which replace the missing permutation symmetries in lower dimensions [7]. Finally the dynamics of the models is equivalent, at each spatial point, to a hyperbolic billiard problem. The specific shape of this model-dependent billiard is determined by the original walls and the permutation walls. Only the “innermost” walls (those which are not “hidden” behind others) are relevant.

We have determined the set of innermost walls for all string models. The analysis is straightforward [7] and we report here only the final results, which are remarkably simple. Instead of the $O(700)$ original walls we find, in all cases, that there are only 10 relevant walls. In fact, the seven string theories M, IIA, IIB, I, HO, HE and the closed bosonic string in $D = 10$ [8], split into three separate blocks of theories, corresponding to three distinct billiards. The first block (with 2 SYSy’s in $D = 10$ is $B_2 = \{M, IIA, IIB\}$ and its ten walls are (in the natural variables of $M$-theory $\beta^\mu = \beta^\mu_M$),

$$B_2 : w_i[2](\beta) = - \beta^i + \beta^{i+1}(i = 1, \ldots, 9),$$

$$w_0[2](\beta) = \beta^1 + \beta^2 + \beta^3.$$  

The second block is $B_1 = \{I, HO, HE\}$ and its ten walls read (when written in terms of the string-frame variables of the heterotic theory $\alpha^i = \beta^i_S$, $\alpha^0 = \beta^0_S$)

$$B_1 : w_i[1](\alpha) = \alpha^i, \quad w_i[1](\alpha) = - \alpha^{i-1} + \alpha^i(i = 2, \ldots, 9),$$

$$w_0[1](\alpha) = \alpha^0 - \alpha^7 - \alpha^8 - \alpha^9.$$  

The third block is simply $B_0 = \{D = 10\}$ closed bosonic and its ten walls read (in string variables)

$$B_0 : w_i[0](\alpha) = \alpha^1 + \alpha^2, \quad w_i[0](\alpha) = - \alpha^{i-1} + \alpha^i(i = 2, \ldots, 9),$$

$$w_0[0](\alpha) = \alpha^0 - \alpha^7 - \alpha^8 - \alpha^9.$$  

In all cases, these walls define a simplex of $H^9$ which is non-compact but of finite volume, and which has remarkable symmetry properties.

The most economical way to describe the geometry of the simplices is through their Coxeter diagrams. This diagram encodes the angles between the faces and is obtained by computing the Gram matrix of the scalar products between the unit normals to the faces, say $G^{[n]}_{ij} = \tilde{w}_i \cdot \tilde{w}_j$, where $\tilde{w}_i = w_i / \sqrt{w_i \cdot w_i}$, $i = 1, \ldots, 10$ labels the forms defining the (hyperplanar) faces of a simplex, and the dot denotes the scalar product (between co-vectors) induced by the metric $G_{\mu\nu} : w_i \cdot w_j = G^{\mu\nu} w_i^\mu w_j^\nu$. For $w_i(\beta) = w_{i\mu} \beta^\mu$. This Gram matrix does not depend
on the normalization of the forms $w_i$, but actually, all
the wall forms $w_i$ listed above are normalized in a na-
tural way, i.e. have a natural length. This is clear for
the forms which are directly associated with dynamical
walls in $D = 10$ or $11$, but this can also be extended
to all the permutation-symmetry walls because they ap-
pear as dynamical walls after dimensional reduction [7].
When the wall forms are normalized accordingly (i.e.
such that $V_{\text{dynamical}}^n \propto \exp(-2 w_i(\beta))$, they all have a
squared length $w_i^n \cdot w_i^n = 2$, except for $w_i^{[1]}$ if $w_i^{[1]} = 1$
in the $B_1$ block. We can then compute the “Cartan ma-
trix”, $a_{ij}^{[n]} = 2 w_i^n \cdot w_j^n / w_i^n \cdot w_i^n$, and the corresponding
Dynkin diagram. One finds the diagrams given in Fig. 1.

![Dynkin diagrams](image)

FIG. 1. Dynkin diagrams defined (for each $n = 2, 1, 0$) by
the ten wall forms $w_i^n(\beta^n)$, $i = 1, \ldots, 10$ that determine the
billiard dynamics, near a cosmological singularity, of the three
blocks of theories $B_2 = \{M, \text{IIA}, \text{IIB}\}$, $B_1 = \{\text{I, HO, HE}\}$ and
$B_0 = \{D = 10$ closed bosonic $\}$ where the node labels $1, \ldots, 10$
correspond to the form label $i$ used in the text.

The corresponding Coxeter diagrams are obtained from
the Dynkin diagrams by forgetting about the norms
of the wall forms, i.e., by deleting the arrow in $BE_{10}$. As
can be seen from the figure, the Dynkin diagrams asso-
ciated with the billiards turn out to be the Dynkin dia-
grams of the following rank-10 hyperbolic Kac-Moody
algebras (see [9]): $E_{10}$, $BE_{10}$ and $DE_{10}$ (for $B_2$, $B_1$ and
$B_0$, respectively). It is remarkable that the three bil-
liards exhaust the only three possible simplex Coxeter
diagrams on $H^9$ with discrete associated Coxeter group
(and this is the highest dimension where such simplices exist) [10]. The analysis suggests to identify the 10 wall
forms $w_i^n(\beta)$, $i = 1, \ldots, 10$ of the billiards $B_2$, $B_1$ and
$B_0$ with a basis of simple roots of the hyperbolic Kac-
Moody algebras $E_{10}$, $BE_{10}$ and $DE_{10}$, while the 10 dy-
namical variables $\beta^{\mu}$, $\mu = 1, \ldots, 10$, can be considered as
parametrizing a generic vector in the Cartan subalgebra
of these algebras. It was conjectured some time ago [11]
that $E_{10}$ should be, in some sense, the symmetry group of
SUGRA$_{11}$ reduced to one dimension. Our results, which
indeed concern the one-dimensional reduction, à la BKL,
of M/string theories exhibit a clear sense in which $E_{10}$
lies behind the one-dimensional evolution of the block $B_2$
of theories: their asymptotic cosmological evolution as
$t \to 0$ is a billiard motion, and the group of reflections
in the walls of this billiard is nothing else than the Weyl
group of $E_{10}$ (i.e. the group of reflections in the hyper-
planes corresponding to the roots of $E_{10}$, which can be
generated by the 10 simple roots of its Dynkin diagram).
It is intriguing – and, to our knowledge, unanticipated –
that the cosmological evolution of the second block of theories, $B_1 = \{I, \text{HO, HE}\}$, be described by another remarka-
able billiard, whose group of reflections is the Weyl
group of $BE_{10}$. The root lattices of $E_{10}$ and $BE_{10}$ ex-
haust the only two possible unimodular even and odd
Lorentzian 10-dimensional lattices [9].

A first consequence of the exceptional properties of the
billiards concerns the nature of the cosmological oscilla-
tory behaviour. They lead to a direct technical proof
that these oscillations, for all three blocks, are chaotic in
a mathematically well-defined sense. This is done by re-
formulating, in a standard manner, the billiard dynamics
as an equivalent collision-free geodesic motion on a hyper-
bolic, finite-volume manifold (without boundary) $\mathcal{M}$
obtained by quotienting $H^9$ by an appropriate torsion-free
discrete group. These geodesic motions define the “most
chaotic” type of dynamical systems. They are Anosov
flows [12], which imply, in particular, that they are “mix-
ing”. In principle, one could (at least numerically) com-
pute their largest, positive Lyapunov exponent, say $\lambda$,
and their (positive) Kolmogorov-Sinai entropy, say $h$.
As we work on a manifold with curvature normalized to
$-1$, and walls given in terms of equations containing only
numbers of order unity, these quantities will also be of
order unity. Furthermore, the two Coxeter groups of $E_{10}$
and $BE_{10}$ are the only reflective arithmetic groups in $H^9$
[10] so that the chaotic motion in the fundamental sim-
plexes of $E_{10}$ and $BE_{10}$ will be of the exceptional “arith-
metical” type [13]. We therefore expect that the quan-
tum motion on these two billiards, and in particular the
spectrum of the Laplacian operator, exhibits exceptional
features (Poisson statistics of level-spacing, ...), linked
to the existence of a Hecke algebra of mutually commuting,
conserved operators. Another (related) remarkable fea-
ture of the billiard motions for all these blocks is their
link, pointed out above, with Toda systems. This fact
is probably quite significant, both classically and quan-
tum mechanically, because Toda systems whose walls are
given in terms of the simple roots of a Lie algebra enjoy
remarkable properties. We leave to future work a study
of our Toda systems which involve infinite-dimensional
hyperbolic Lie algebras.

The present investigation a priori concerned only the
“low-energy” ($E \ll (\alpha')^{-1/2}$), classical cosmological
behaviour of string theories. In fact, if (when going to-
ward the singularity) one starts at some “initial” time
$t_0 \sim (d\beta/dt)^{-1}$ and insists on limiting the application
of our results to time scales $|t| > (\alpha')^{1/2} \equiv t_s$, the
total number of “oscillations”, i.e. the number of collis-
sions on the walls of our billiard will be finite, and will
not be very large. The results above show that the number of collisions between $t_0$ and $t \to 0$ is of order $N_{\text{coll}} \sim \ln \tau \sim \ln(t_0/t)$. This is only $N_{\text{coll}} \sim 5$ if $t_0$ corresponds to the present Hubble scale and $t$ to the string scale $t_s$. However, the strongly mixing properties of geodesic motion on hyperbolic spaces make it large enough for churning up the fabric of spacetime and transforming any, non-particularly homogeneous at time $t_0$, patch of space into a turbulent foam at $t = t_s$. Indeed, the mere fact that the walls associated with the spatial curvature and the form fields repeatedly rise up (during the collisions) to the same level as the “time” curvature terms $\sim t^{-2}$, means that the spatial inhomogeneities at $t \sim t_s$ will also be of order $t_s^{-2}$, corresponding to a string scale foam.

Our results on the $B_2$ theories probably involve a deep (and not a priori evident) connection with those of Ref. [14] on the structure of the moduli space of $M$-theory compactified on the ten torus $T^{10}$, with vanishing 3-form potential. In both cases the Weyl group of $E_{10}$ appears. In our case it is (partly) dynamically realized as reflections in the walls of a billiard, while in Ref. [14] it is kinematically realized as a symmetry group of the moduli space of compactifications preserving the maximal number of supersymmetries. In particular, the crucial $E$-type node of the Dynkin diagram of $E_{10}$ (Fig. 1) comes, in our study and in the case of $M$-theory, from the wall form $w_{10}^{(2)}(\beta) = \beta_1^0 + \beta_2^0 + \beta_3^0$ associated with the electric energy of the 3-form. By contrast, in [14] the 3-form is set to zero, and the reflection in $w_{10}^{(2)}$ comes from the 2/5 duality transformation (which is a double $T$ duality in type II theories), which exchanges (in $M$-theory) the 2-brane and the 5-brane. As we emphasized above, dimensional reduction transforms kinematical (permutation) walls into dynamical ones. This suggests that there is no difference of nature between our walls, and that, viewed from a higher standpoint (12-dimension ?), they would all look kinematical, as they are in [14]. By analogy, our findings for the $B_1$ theories suggest that the Weyl group of $BE_{10}$ is a symmetry group of the moduli space of $T^3$ compactifications of $\{I, HO, HE\}$.

Perhaps the most interesting aspect of the above analysis is to provide hints for a scenario of vacuum selection in string cosmology. If we heuristically extend our classical, low-energy, tree-level) results to the quantum, stringy ($t \sim t_s$) and/or strongly coupled ($g_s \sim 1$) regime, we are led to conjecture that the initial state of the universe is equivalent to the quantum motion in a certain finite volume chaotic billiard. This billiard is (as in a hall of mirrors game) the fundamental polytope of a discrete symmetry group which contains, as subgroups, the Weyl groups of both $E_{10}$ and $BE_{10}$ [15]. We are here assuming that there is (for finite spatial volume universes) a non-zero transition amplitude between the moduli spaces of the two blocks of superstring “theories” (viewed as “states” of an underlying theory). If we had a description of the resulting combined moduli space (orbifolded by its discrete symmetry group) we might even consider as most probable initial state of the universe the fundamental mode of the combined billiard, though this does not seem crucial for vacuum selection purposes. This picture is a generalization of the picture of Ref. [17] and, like the latter, might solve the problem of cosmological vacuum selection in allowing the initial state to have a finite probability of exploring the subregions of moduli space which have a chance of inflating and evolving into our present universe.

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[8] We include for the sake of comparison the bosonic string model in $D = 10$. Though $D = 10$ is not the critical dimension of the quantum bosonic string, it is the critical dimension above which the never ending oscillations in its cosmological evolution disappear [2].
[15] With $F$-theory [16] in mind, it is tempting to look for a Kac-Moody algebra with ultra-hyperbolic signature $\ldots -+ + \cdot \cdot \cdot$ containing both $E_{10}$ and $BE_{10}$. According to V. Kac (private communication) the smallest such algebra is the rank-20 algebra whose Dynkin diagram is obtained by connecting, by a simple line, the $w_{10}^{(2)}$ and $w_9^{(1)}$ nodes in Fig. 1.