NON-LTE RADIATION PROCESSES:
APPLICATION TO THE SOLAR CORONA

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Abstract. These lectures are intended to present a simple but relatively complete description of the theory needed to understand the formation of lines in non-local thermodynamical equilibrium (NLTE), without appealing to any previous knowledge except a few basics of physics and spectroscopy. After recalling elementary notions of radiation transfer, the chapter is focussed on the computation of the level populations, the source function, the ionization state, and finally the line intensity. An application is made to forbidden coronal lines which were observed during eclipses since decades.

1. Introduction

Astrophysical media, at least for the fraction we can observe, are often very dilute compared to those we are used to on Earth. In the interstellar medium for instance, the number density is in average one atom per cm$^3$, in HII regions and planetary nebulae it is about $10^4$ atoms per cm$^3$ and in the solar corona $10^8$ atoms per cm$^3$, orders of magnitudes less than those one can get in laboratory experiments. Therefore the notion of “thermodynamical equilibrium” does not hold, and quite unusual phenomena are taking place, such as the emission of intense “forbidden lines” never observed on Earth. We will in particular show how to compute the intensity of visible forbidden coronal lines, and show how these lines, and others, can be used to get physical parameters of the solar corona.
2. Basics of Transfer

2.1. PHOTOMETRIC QUANTITIES

We shall first define a few basic photometric quantities. Note that CGS units will generally be used.

- **The specific intensity** $I_\nu$: The energy $dE_\nu$ crossing a surface of area $dA$ in the direction of the normal, in a solid angle $d\Omega$, during a time $dt$, in a frequency interval $d\nu$, is (cf. Fig. 1):

$$dE_\nu = I_\nu \, d\nu \, d\Omega \, dt \, dA.$$  \hspace{1cm} (1)

It can be expressed in Watt m$^{-2}$ ster$^{-1}$ Hz$^{-1}$, but other units are used in the visible or in the X-ray range. It can be also defined per interval of wavelength, and according to the relation $\nu = c$ (speed of light), one gets $I_\lambda = I_\nu c/\lambda^2$.

The specific intensity can be used both for the source of radiation or for the receptor, and it is constant along a path ray in the vacuum.

- **The mean intensity** $J_\nu$:

$$J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega \hspace{1cm} (2)$$

- **The flux** $F_\nu$:

It is the power crossing a unit surface per unit frequency interval, in all directions (cf. Fig. 2):

$$F_\nu = \int I_\nu \cos \theta \, d\Omega \hspace{1cm} (3)$$

$F_\nu$ is generally expressed in Jansky in the radio and far infrared range: 1 J = $10^{-23}$ in CGS, or $10^{-26}$ in MKS (Watts m$^{-2}$ Hz$^{-1}$).
As an application, we can compute the flux from a uniform sphere radiating isotropically (not a very good approximation for a star, actually), cf. Fig. 3):

\[
F_\nu = \int I_\nu \cos \theta \, d\Omega = I_\nu \int_0^{2\pi} d\phi \int_0^{\theta_c} \sin \theta \cos \theta \, d\theta, 
\]

(4)

where \(\theta_c\) is the angle under which the sphere is seen from the observer. If \(R\) is the radius of the star, and \(r\) its distance, one gets \(\sin \theta_c = R/r\), and

\[
F_\nu = \pi I_\nu (R/r)^2. 
\]

(5)

Note that the flux at the surface of the sphere is \(\pi I_\nu\).

2.2. TRANSFER EQUATION

When a light ray does not propagate in the vacuum, the specific intensity is not constant: emission adds energy, and absorption removes energy. There is also diffusion, in which the global luminous energy is not changed, but can be increased in one direction, and decreased in another. We shall not take into account the polarization of the radiation, and the possibility of non stationary phenomena.
2.2.1. Transfer equation in a non diffusive medium

The monochromatic emissivity $\eta_\nu$ is defined as the power emitted per unit solid angle, per unit frequency interval, per unit volume. The monochromatic absorption coefficient $\chi_\nu$ is defined as follows: $\chi_\nu I_\nu$ is the power absorbed per unit solid angle, per unit frequency interval, by a slab of unit length normal to the direction of the propagation. Note that here $\chi_\nu$ is the inverse of a length. Some people use instead the “opacity coefficient”, defined per unit mass, and the emissivity per unit mass and not per unit volume. The variation of $I_\nu$ on a path length $dl$ in the direction of the light ray is therefore:

$$dI_\nu = (-\chi_\nu I_\nu + \eta_\nu)dl. \quad (6)$$

Let us define:

$$d\tau_\nu = -\chi_\nu dl \quad \text{and} \quad \tau_\nu = \int -\chi_\nu dl,$$

\(\tau_\nu\) is the optical depth, which decreases towards the observer.

The transfer equation writes then:

$$\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu,$$  \quad (8)

where $S_\nu = \eta_\nu / \chi_\nu$ is called the source function.

If there are several emission and absorption processes, they have to be all taken into account in the source function and in the absorption coefficient. For instance, if at a frequency $\nu$ a line is superposed onto a continuum, then $S_\nu = S_\nu^{\text{line}} + S_\nu^{\text{cont}}$, and $d\tau_\nu = d\tau_\nu^{\text{line}} + d\tau_\nu^{\text{cont}}$.

2.2.2. Transfer equation in a diffusive medium

If there is a diffusion process, it must also be taken into account in the transfer equation. The diffusion coefficient is defined like the absorption coefficient: $\sigma_\nu I_\nu$ is the power diffused per unit solid angle, per unit frequency interval, by a slab of unit length normal to the direction of the propagation. Let us assume that diffusion is coherent (i.e. without any change of frequency), and isotropic. As a consequence there is a corresponding emission coefficient, which is equal to $\sigma_\nu J_\nu$, and the transfer equation becomes:

$$dI_\nu = [(\chi_\nu - \sigma_\nu)I_\nu + (\sigma_\nu J_\nu + \eta_\nu)]dl. \quad (9)$$

If one defines now an extinction coefficient, $\tau^{\text{tot}}_\nu = \int d\tau^{\text{tot}}_\nu$ with $d\tau^{\text{tot}}_\nu = - (\chi_\nu + \sigma_\nu)dl$, one gets for the transfer equation in presence of diffusion:

$$\frac{dI_\nu}{d\tau^{\text{tot}}_\nu} = I_\nu - S^{\text{tot}}_\nu,$$  \quad (10)
where \( S_{\nu}^{\text{tot}} = (\sigma_{\nu} J_{\nu} + \eta_{\nu})/(\chi_{\nu} + \sigma_{\nu}) \). Although it is formally similar to Eq. 8, it differs in that the intensity appears directly in the second term, so it is an integro-differential equation. Actually, the diffusion process is a probabilistic one, similar to a random walk, and one can show that the distance that a photon will travel before being absorbed is equal to \([\chi_{\nu}(\chi_{\nu} + \sigma_{\nu})]^{-1/2}\), while in the case of pure absorption it is equal to \(\chi_{\nu}^{-1}\).

In a purely diffusing medium, the transfer equation writes:

\[
\frac{dI_{\nu}}{d\tau_{\nu}^{\text{diff}}} = I_{\nu} - J_{\nu},
\]

(11)

In the corona, the diffusion process is due to Thomson scattering by free electrons. As a first approximation it can be considered as a coherent and isotropic process. The diffusion coefficient \(\sigma_{\nu}\) does not depend on frequency, and is equal to \(\sigma_T N_e\), where \(N_e\) is the number of electrons per unit volume, and \(\sigma_T\) is the Thomson cross section, equal to \(6.65 \times 10^{-25} \text{cm}^2\). This process is very important for the continuum in the visible range, as it is responsible of the emission of the K corona. As a consequence one can consider that the corona is a purely diffusive medium for the continuum in the visible range. On the other hand we will show that diffusion is negligible in the transfer of the coronal lines, on which we will focus later on, so we will not consider it in the following sections.

2.3. APPLICATION TO A PLANE-PARALLEL MEDIUM

A common utilization of the transfer equation concerns a stratified medium, where all physical quantities are constant on infinite parallel planes. It is the usual approximation made for stellar atmospheres (cf. Fig. 4). Often
interstellar clouds or diffuse nebulae are considered also as plane parallel media. The optical depth is then defined in the direction of the normal, and the transfer equation becomes, for a light ray which makes an angle \( \theta \) with the normal to the planes:

\[
\cos \theta \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu,
\]

(12)
The formal solution of this equation is:

\[
\left[ I_\nu \exp \left( -\frac{\tau_\nu}{\mu} \right) \right]^{\tau_\nu=2} = -\int_{\tau_\nu=1}^{\tau_\nu=2} S_\nu \exp \left( -\frac{\tau_\nu}{\mu} \right) \frac{d\tau_\nu}{\mu}
\]

(13)
where \( \mu = \cos \theta \).

We can apply this solution to different cases, according to the boundary conditions.

2.3.1. for a stellar atmosphere
In this case the boundary conditions are:

- no radiation incident on the surface, or \( I_\nu (\mu < 0, \tau_\nu = 0) = 0 \), i.e.:

\[
I_\nu (\mu < 0, \tau_\nu) = \int_0^{\tau_\nu} \frac{S_\nu}{|\mu|} \exp \left( -\frac{(t - \tau_\nu)}{\mu} \right) \, dt,
\]

(14)
- the radiation at \( \tau_\nu = \infty \) remains finite, i.e.:

\[
I_\nu (\mu > 0, \tau_\nu) = \int_{\tau_\nu}^{0} \frac{S_\nu}{|\mu|} \exp \left( -\frac{(t - \tau_\nu)}{\mu} \right) \, dt,
\]

(15)
and the intensity at the surface is:

\[
I_\nu (\tau_\nu = 0) = \int_0^{\infty} S_\nu \exp \left( -\frac{\tau_\nu}{\mu} \right) \frac{d\tau_\nu}{\mu}.
\]

(16)
One gets also the flux emerging from the surface:

\[
F_\nu (\mu > 0) = 2\pi \int_0^1 \mu \, d\mu \int_0^{\infty} S_\nu \exp \left( -\frac{\tau_\nu}{\mu} \right) \frac{d\tau_\nu}{\mu}
\]

(17)
where \( \mathcal{E}_n (x) \) is the order \( n \) integro-exponential:

\[
\mathcal{E}_n (x) = \int_0^{\infty} \exp(-ux) u^{-n} \, du.
\]

Although Eqs. 16 and 17 are only formal solutions which require to know the variation of \( S_\nu \) as a function of depth at all frequencies, they can be of some help to understand intuitively two observations.
First we see that the intensity is approximately equal to the source function at $\tau_\nu = \mu$, and that the internal layers below do not contribute to the radiation. It explains the limb darkening effect at the surface of the Sun. According to the Eddington-Barbier relation, the source function in the photosphere is proportional to $\tau$, so it decreases towards the surface (but not necessarily in the chromosphere, cf. P. Heinzel’s lectures). Thus when we observe the limb ($\mu \to 0$), we are seeing layers close to the “surface” where $S_\nu$ is small, so $I_\nu$ is also small. When we observe the center of the Sun ($\mu = 1$), we are seeing deeper layers where $S_\nu$ is large, and $I_\nu$ is large too. So the center of the disk will appear brighter than the limb.

Second, in the case of stars, we do not observe the intensity, but the flux, which is approximately equal to the source function at $\tau_\nu = 1$. Assuming again that $S_\nu$ decreases with height in the photosphere, we can understand why we see absorption lines in the stellar spectrum. The absorption coefficient $\chi_\nu$ in a line is larger than in the surrounding continuum, so we are seeing in a line the layers close to surface, and in the surrounding continuum the deeper layers: the lines are then in absorption. The effect is inverted if the source function increases with the height, as it is the case for lines formed in the chromosphere, or in extended envelopes of stars; the lines are then in emission.

2.3.2. for a homogeneous slab of finite thickness

One defines an optical thickness increasing towards the observer, $d\tau_\nu = \chi_\nu \, d\ell$. Since the medium is assumed homogeneous, the optical thickness of the slab is $T_\nu = \int \chi_\nu \, d\ell = \chi_\nu \, H$ in the direction of the normal, where $H$ is the geometrical thickness.

The boundary conditions are now:
- incident intensity $I_\nu (\tau_\nu = 0) = I_{\nu 0}$,
- no incident intensity at $\tau_\nu = T_\nu$.

The solution of the transfer equation is thus (for $S_\nu = \text{const.}$):

$$ I_\nu (T_\nu, \mu) = I_{\nu 0} \exp \left( -\frac{T_\nu}{\mu} \right) + S_\nu \left[ 1 - \exp \left( -\frac{T_\nu}{\mu} \right) \right]. \tag{18} $$

If $T_\nu < 1$, the slab is called “optically thin”; if $T_\nu > 1$ it is called “optically thick”.

There are three interesting cases:

- 1. a non emissive layer (a cold cloud in front of an intense source):

$$ I_\nu (T_\nu, \mu) = I_{\nu 0} \exp \left( -\frac{T_\nu}{\mu} \right) \tag{19} $$

$$ I_{\nu 0} \exp \left( -\frac{T_\nu}{\mu} \right) \tag{18} $$
− 2. an optically thin layer with $T_{\nu} \ll 1$ (assuming that $\mu$ is not $\ll 1$):

$$I_{\nu}(T_{\nu}, \mu) = I_{\nu 0} \exp \left( -\frac{T_{\nu}}{\mu} \right) + \frac{S_{\nu}}{\mu} T_{\nu}$$

$$= I_{\nu 0} \exp \left( -\frac{T_{\nu}}{\mu} \right) + \frac{H \eta_{\nu}}{\mu}$$

(20)

− 3. an optically thick layer with $T_{\nu} \gg 1$:

$$I_{\nu}(T_{\nu}, \mu) \sim S_{\nu}.$$  

(21)

For a finite slab with no incident radiation one can show that $F_{\nu} = 2\pi S_{\nu}[0.5 - \varepsilon_{\nu}(T_{\nu})]$, and deduce that:

− 1. in the optically thin case: $F_{\nu} = 2\pi S_{\nu} T_{\nu} = 2\pi \eta_{\nu} H$, or $I_{\nu} = 4\pi \eta_{\nu} \times$

Volume (where $I_{\nu}$ is the total power emitted by the slab),

− 2. in the optically thick case: $F_{\nu} = \pi S_{\nu}$, or $I_{\nu} = \pi S_{\nu} \times$ Surface.

To summarize, in the optically thin case, one “sees” the emissivity, and the power is proportional to the volume. In the optically thick case, one “sees” the source function, and the power is proportional to the surface.

Finally note that in a purely diffusing medium, the same equations hold, replacing $S_{\nu}$ by $J_{\nu}$, and $\tau_{\nu}$ by the diffusion coefficient. So one gets in particular for an optically thin medium (i.e. $T_{\nu}^{\text{diff}} \ll 1$) with no incident radiation on the line of sight:

$$I_{\nu} = \frac{J_{\nu} T_{\nu}^{\text{diff}}}{\mu}.$$  

(22)

This equation applies to the continuum of the solar corona in the visible range (the K corona). For a medium which is absorbing and diffusing, and optically thin both for diffusion and for absorption, but where the diffusion coefficient is negligible compared to the absorption coefficient, the solution of the transfer equation becomes:

$$I_{\nu} = \frac{J_{\nu} T_{\nu}^{\text{diff}}}{\mu} + H \eta_{\nu}.$$  

(23)

This equation applies to the visible lines emitted by the solar corona (cf. later)

3. Local Thermodynamical Equilibrium (LTE)

3.1. Recalling the Laws of Thermodynamical Equilibrium (TE)

The \textit{thermodynamical equilibrium} is the stationary state of an ensemble of interacting particles and photons which should be achieved in an infinitely
thick medium (called a “Black Body”) after an infinite time. Photons and particles have then the most probable energy distribution, which corresponds to microreversibility of all processes. For instance, there are as many radiative (resp. collisional) excitations from the level A to the level B of an atom, as radiative (resp. collisional) deexcitations from the level B to the level A per unit time.

3.1.1. Energy distribution of photons: the Planck law
It writes:
\[ I_\nu \equiv B_\nu = \frac{2\hbar \nu^3}{c^2} \left[ \exp \left( \frac{h \nu}{kT} \right) - 1 \right]^{-1}, \tag{24} \]

- \( h \): Planck constant = 6.6262 \times 10^{-27} \text{ erg s},
- \( k \): Boltzmann constant = 1.3806 \times 10^{-16} \text{ erg K}^{-1}
- \( T \): temperature.

Caution: it can also be expressed in units of wavelength, and it writes then:
\[ B_\lambda = \frac{2 \hbar c^2}{\lambda^5} \left[ \exp \left( \frac{hc}{\lambda kT} \right) - 1 \right]^{-1}, \tag{25} \]

which has different shape and position of the maximum.

The integration over \( \nu \) or \( \lambda \) gives:
\[ B = \int B_\nu \, d\nu = \frac{\sigma T^4}{\pi}, \tag{26} \]

where \( \sigma \) is the Stefan constant = 5.6696 \times 10^{-5} \text{ erg cm}^{-2} \text{ s}^{-1} \text{ K}^{-4}.

There are two limiting cases:
- For \( h \nu \ll kT \), the Rayleigh-Jeans law (used in radio-astronomy):
  \[ B_\nu = \frac{2h \nu^3}{c^2} \frac{kT}{h \nu} = \frac{2kT}{\lambda^2}; \tag{27} \]
- For \( h \nu \gg kT \), the Wien law (used in the X-ray range):
  \[ B_\nu = \frac{2h \nu^3}{c^2} \exp \left( \frac{h \nu}{kT} \right). \tag{28} \]

3.1.2. Energy distribution of particles in non quantified levels: the Maxwell law
It gives the number of particles per unit volume whose velocity projected on an axis \( z \) is between \( v_z \) and \( v_z + dv_z \):
\[ dN_z = N \left( \frac{M}{2\pi kT} \right)^{1/2} \exp \left( -\frac{Mv_z^2}{2kT} \right) dv_z, \tag{29} \]
where $N$ is the number of particles per unit volume, and $M$ is the mass of the particles. This expression is used e.g. to compute the profile of spectral lines broadened by thermal Doppler effect (cf. later).

Since at TE the velocity distribution is isotropic, the number of particles per unit volume whose absolute velocity is between $v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$ and $v + dv$ is equal to:

$$dN = N \left( \frac{M}{2\pi kT} \right)^{3/2} \exp \left( -\frac{M (v_x^2 + v_y^2 + v_z^2)}{2kT} \right) dv_x \, dv_y \, dv_z$$

and the number of particles per unit volume with energy between $E = Mv^2/2$ and $E + dE$ is:

$$dN = N \left( \frac{1}{\pi kT} \right)^{3/2} \exp \left( -\frac{E}{kT} \right) 2\pi v^2 \, dv.$$

This expression is used for instance to compute the rate of collisional excitations (cf. later).

Note that the Maxwellian distribution is normalized ($\int dN = N$).

3.1.3. Energy distribution of particles in quantified levels: the Boltzmann law
Consider two levels of energy $E_m$ and $E_n$ ($E_m > E_n$), with $E_m - E_n = E_{nm} = h\nu_{nm}$, where $\nu_{nm}$ is the frequency of the corresponding atomic transition. The number of particles per unit volume in each level (called the “level population”) is given by:

$$\frac{N_m}{N_n} = \frac{g_m}{g_n} \exp \left( -\frac{E_{nm}}{kT} \right)$$

where $g_m$ and $g_n$ are the “statistical weights” of the levels $m$ and $n$ (they are given by spectroscopic tables, as well as the energies of the levels).

3.1.4. Distribution of particles in different ionization states: the Saha law
It gives the distribution of particles of a given species in a given ionization state:

$$\frac{N_{i+1}^1 N_i}{N_i^1} = \frac{2g_{i+1}^1}{g_i^1} \left( \frac{2\pi mkT}{h^2} \right)^{3/2} \exp \left( -\frac{\chi_i}{kT} \right) ;$$

where $N_{i+1}^1$ and $N_i^1$ are the numbers of ions on the fundamental level of states ionized $i + 1$ and $i$ times, per unit volume,
- \( N_e \) is number of electrons, per unit volume,
- \( g_{i+1} \) and \( g_i \) are the statistical weights of the fundamental level of states \( i + 1 \) and \( i \),
- \( \chi_i \) is the ionization potential of state \( i \) (i.e. the energy needed to extract an electron from the fundamental level),
- \( m \) is the electron mass \((9.1096 \times 10^{-28} \text{ g})\).

Using Boltzmann law, one gets:

\[
\frac{N_{i+1}N_e}{N_i} = \frac{2U_{i+1}}{U_i} \left( \frac{2\pi mkT}{\hbar^2} \right)^{3/2} \exp \left( -\frac{\chi_i}{kT} \right) ; \tag{34}
\]

- \( N_i \) and \( N_{i+1} \) are the total numbers of ions in states ionized \( i \) and \( i + 1 \) times, per unit volume,
- \( U_i \) and \( U_{i+1} \) are called the “partition functions”:

\[
U_i(T) = \sum g_m \exp \left( -\frac{E_{1m}}{kT} \right) . \tag{35}
\]

For small \( T \) \((E_{1m}/kT > 1)\), \( U_i(T) = g_1 \). For large \( T \), the summation must be made on a large number of levels (depending on the density).

One can show (cf. books on statistical mechanics) that micro reversibility is equivalent to Maxwell + Boltzmann + Saha equations.

At TE the optical thickness is infinite and the intensity is isotropic, so:

\[
B_\nu(T) = I_\nu(\tau_\nu \to \infty) = S_\nu \tag{36}
\]

which writes:

\[
\eta_\nu = \chi_\nu B_\nu(T). \tag{37}
\]

It is the well known Kirchhoff law, which is used to compute the emissivity when the absorption coefficient is known (or inversely).

3.2. FROM TE TO LTE

At TE, \( I_\nu \) is equal to \( B_\nu(T) \) and since \( T \) is constant, \( dI_\nu/d\tau_\nu = 0 \) and there is no transfer of radiation.

Let us now consider a state where \( T \) varies in the medium, but Maxwell, Boltzmann and Saha equations are satisfied locally (cf. Fig. 5). In such a state micro reversibility is achieved. It is called Local Thermodynamical Equilibrium (LTE). We will show later that this state is reached if the density and/or the optical thickness are large. The source function is then given by the Planck law (but not \( I_\nu \), which varies in the medium and is the solution of the transfer equation).
4. Source function of a spectral line in non-LTE (NLTE)

A spectral line corresponds to the radiative transition between two bound levels of a given atom. If LTE is not reached, there is no microreversibility between the transitions but simply a stationary equilibrium in which the number of ALL processes populating a level is equal to the number of ALL processes depopulating this level. The main direct and inverse atomic processes populating and depopulating a bound level are:

- collisional excitations and deexcitations
- radiative excitations and deexcitations
- collisional ionizations and recombinations
- radiative ionizations and recombinations.

Several other processes can be important, such as dielectronic recombinations and auto ionizations, Auger processes, charge exchanges, but we will not consider them here.

4.1. RADIATIVE EXCITATIONS AND DEEXCITATIONS

Consider again the two bound levels $m$ and $n$ with populations $N_m$ and $N_n$, energies $E_m > E_n$, $E_{nm} = E_m - E_n$.

4.1.1. Einstein probabilities

The Einstein probabilities $A_{mn}$, $B_{mn}$ and $B_{nm}$ are defined as follows:

- the number of radiative spontaneous transitions from level $m$ to level $n$ per unit volume per second is $N_m A_{mn}$. Each corresponds to the emission of a photon $h\nu = E_{nm}$.
- the number of radiative transitions from level $n$ to level $m$ per unit volume per second, corresponding to the absorption of a photon $h\nu = E_{nm}$, is $N_n J_\nu B_{nm}$.
- the number of radiative transitions from level $m$ to level $n$, induced by a photon $h\nu = E_{nm}$, per unit volume per second and leading to the emission of a second photon $h\nu = E_{nm}$, is $N_m J_\nu B_{mn}$. It is called
“induced emission” (discovered by Einstein). Note that $J_\nu$ is the mean intensity at frequency $\nu = E_{nm}/\hbar$.

One can determine the relations between these coefficients in TE. Microreversibility between radiative excitation and deexcitation gives:

$$N_m(A_{mn} + B_{mn} J_\nu) = N_n B_{nm} J_\nu$$

Boltzmann equation between levels $m$ and $n$, and the Planck law $J_\nu \equiv B_\nu$ lead to:

$$J_\nu = \left( \frac{A_{mn}}{B_{mn}} \right) \frac{g_m B_{mn}}{g_n B_{nm}} \left[ \exp \left( \frac{E_{nm}}{kT} \right) - 1 \right]^{-1}$$

Identifying the terms depending and not depending on $T$ (as these relations should be verified for any $T$), one gets:

$$g_n B_{nm} = g_m B_{mn}, \text{ and } \frac{A_{mn}}{B_{mn}} = \frac{2\hbar\nu^3}{c^2}. \quad (40)$$

These relations are equally valid in non-LTE, since $A_{mn}$, $B_{nm}$ and $B_{nm}$ are atomic parameters and do not depend on the physical conditions of the medium. The $A_{mn}$ are given in spectroscopic tables (or equivalently the oscillator strength $f_{nm}$, cf. later). Note that a different definition of the Einstein probabilities is sometimes used, with the density of radiation instead of the intensity in Eq. 38.

#### 4.1.2. Relations of the Einstein probabilities with the absorption coefficient, the emissivity, and the source function

Spectral lines are broadened by several processes, corresponding to an absorption profile $\Phi(\nu)$ and an emission profile $\Psi(\nu)$, normalized so that $\int \Phi(\nu)d\nu = 1$ and $\int \Psi(\nu)d\nu = 1$. The emissivity then writes:

$$\eta_\nu = \frac{h\nu}{4\pi} N_m A_{mn} \Psi(\nu), \quad (41)$$

and the absorption coefficient:

$$\chi_\nu = \frac{h\nu}{4\pi} N_n B_{nm} \Phi(\nu) \left( 1 - \frac{N_m B_{mn}}{N_n B_{nm}} \right). \quad (42)$$

It contains a negative term of induced emission, $(N_m B_{mn})/(N_n B_{nm}) = (N_m g_n)/(N_n g_m)$, which can be larger than unity in the radio range (because the exponential term in the Boltzmann equation is close to unity). In this case the medium becomes an amplifier instead of an absorbant (MASER effect).
If $\Psi(\nu) = \Phi(\nu)$, one gets:

$$S = \frac{\eta_\nu}{\chi_\nu} = \left( \frac{N_m A_{mn}}{N_n B_{nm}} \right) \left( 1 - \frac{N_m B_{mn}}{N_n B_{nm}} \right)^{-1}$$

(43)

and the source function is constant along the line profile.

The absorption and emission profiles are generally the convolution of a Gaussian and a Lorentzian function, the first being due to thermal and turbulent broadening, the latter due to radiative and collisional damping. Such a convolution is called “Voigt profile”. The emission and absorption profiles are identical in the case of coherent scattering, or if on the contrary there is complete redistribution of frequencies in the absorption-reemission process. It is the case for a majority of lines, in particular for the coronal lines considered later in this chapter. Since they correspond to forbidden transitions, radiative damping is negligible, and since they are formed in a dilute medium, collisional damping is also negligible. So the emission/absorption profile reduces to a gaussian function. Let us rapidly recall the theory.

If an atom with a transition $\hbar \nu_0 = E_{nm}$ has a velocity projected on the line of sight equal to $v_z$, it absorbs or emits photons propagating towards the observer, shifted by Doppler effect to $\nu = \nu_0 (1 + v_z/c)$. For an assembly of atoms with a Maxwellian distribution of velocities, the number of atoms able to absorb or emit at a frequency $\nu$, in a unit frequency interval, is therefore, according to Eq. 29:

$$\frac{dN_z}{dv_z} = \frac{c}{\nu_0} N_n \left( \frac{M}{2\pi kT} \right)^{3/2} \exp \left( -\frac{Mv_z^2}{2kT} \right),$$

(44)

and one deduces for the absorption/emission profile:

$$\Phi(\nu) = \frac{1}{\sqrt{\pi} \Delta \nu_D} \exp \left[ -\left( \frac{\nu - \nu_0}{\Delta \nu_D} \right)^2 \right],$$

(45)

where

$$\Delta \nu_D = \frac{\nu_0}{c} \left( \frac{2kT}{M} \right)^{1/2}$$

(46)

is called the “Doppler width” of the line. If the gas has also a micro-turbulent velocity $V_{turb}$, it should be taken into account in the Doppler width:

$$\Delta \nu_D = \frac{\nu_0}{c} \left( \frac{2kT}{M} + V_{turb}^2 \right)^{1/2}$$

(47)

as turbulence corresponds also to a Gaussian distribution of velocities.
The absorption coefficient at a frequency $\nu$ thus writes:

$$\chi_\nu = \frac{\hbar \nu_0}{4\pi} N_n B_{nm} \frac{1}{\sqrt{\pi} \Delta \nu D} \exp \left[ - \left( \frac{\nu - \nu_0}{\Delta \nu D} \right)^2 \right] \left( 1 - \frac{N_m}{N_n} \frac{B_{mn}}{B_{nm}} \right)$$

(48)

and the absorption coefficient at the line center is:

$$\chi_\nu = \frac{\hbar \nu_0}{4\pi} N_n B_{nm} \frac{1}{\sqrt{\pi} \Delta \nu D} \left( 1 - \frac{N_m}{N_n} \frac{B_{mn}}{B_{nm}} \right) .$$

(49)

Instead of the Einstein probability, one often uses the oscillator strength $f_{nm}$ which is related to $B_{nm}$ by:

$$B_{nm} = \frac{\pi e^2}{m c} f_{nm} \frac{4\pi}{\hbar \nu}$$

(50)

leading to:

$$\chi_\nu = N_n \frac{\pi e^2}{m c} f_{nm} \frac{1}{\sqrt{\pi} \Delta \nu D} \left( 1 - \frac{N_m}{N_n} \frac{B_{mn}}{B_{nm}} \right) .$$

(51)

Numerically, $\pi e^2 / m c = 0.027$ in CGS, $g_n f_{nm} = 1.5 \times 10^{-8} g_m A_{nm} \lambda^2$ (\(\lambda\) in micron, $A_{nm}$ in s$^{-1}$), so $f$ is of the order of unity for permitted lines, and $\ll 1$ for forbidden lines. As an example, for hydrogen:

$$f_{nm} = \frac{2^6}{3\sqrt{3\pi}} g_n \left( \frac{1}{n^2} - \frac{1}{m^2} \right)^{-3} \frac{1}{n^3} \frac{1}{m^3} g$$

(52)

where $g$ is the so-called “Gaunt factor”, of the order of unity.

4.2. COLLISIONAL EXCITATIONS AND DEEXCITATIONS

In passing close to an atom, a perturber $P$, having a kinetic energy $E_{kin}$, can loose kinetic energy in exciting an atom from the bound level $n$ to the bound level $m$. The kinetic energy $E_{kin}'$ of the perturber after the collision is equal to $E_{kin} - E_{nm}$. In the inverse process, the perturber gains the energy $E_{nm}$ in deexciting the atom from the level $m$ to the level $n$.

One defines the “collisional excitation rate” $C_{nm}$ and the “collisional deexcitation rate” $C_{mn}$ as follows:

- the number of collisional excitations per second per unit volume is $N_P N_n C_{nm}$
- the number of collisional deexcitations per second per unit volume is $N_P N_m C_{mn}$,

where $N_P$ is the number density of the perturbers.

The perturbers are most often electrons, except in cold interstellar medium, where they can be hydrogen atoms or molecules. Let us call $\sigma_{nm}(\nu)$
\( \sigma_{nm}(v) \) the cross section for a collisional deexcitation (resp. a collisional excitation), \( v \) being the relative velocity of the interacting particles. Generally the mean kinetic energy of different types of particles (in other words their temperature) is the same. The velocity of the electrons is thus much larger than the velocity of the (more massive) ions. If the perturbers are electrons, the relative velocity \( v \) reduces therefore to the velocity of the electrons. According to the definition of \( C_{nm} \) and \( C_{mn} \), one gets:

\[
C_{mn} (\text{respt.} C_{nm}) = \langle \sigma_{mn} v \rangle \quad (\text{respt.} \langle \sigma_{nm} v \rangle) \text{s}^{-1} \text{cm}^3 ,
\]

where the average is taken over the velocity distribution of the perturbers, i.e. the Maxwellian distribution of the electrons:

\[
C_{mn}(T_e) = \left( \frac{m}{2\pi kT_e} \right)^{3/2} \int_0^\infty \sigma_{mn}(v) \exp \left( -\frac{mv^2}{2kT_e} \right) 4\pi v^3 dv , \tag{53}
\]

\[
C_{nm}(T_e) = \left( \frac{m}{2\pi kT_e} \right)^{3/2} \int_{E_{mn}}^\infty \sigma_{mn}(v) \exp \left( -\frac{mv^2}{2kT_e} \right) 4\pi v^3 dv . \tag{54}
\]

where \( T_e \) is the electron temperature (generally equal to the ion temperature). One should note that for \( C_{nm}(T_e) \) the integration is performed from \( E_{mn} \) to \( \infty \), since the perturber must have a kinetic energy larger than \( E_{mn} \) to be able to excite the transition. The cross sections are determined experimentally and/or theoretically as functions of \( v \), then integrated over the Maxwellian distribution, and \( C_{mn}(T_e) \) is tabulated.

As for the radiative rates, one can determine the relations between \( C_{nm} \) and \( C_{mn} \) in TE:

- 1. Microreversibility between collisional excitation and deexcitation gives:

\[
N_m N_e C_{nm} = N_m N_e C_{mn} , \tag{55}
\]

- 2. Boltzmann equation between levels \( m \) and \( n \) gives therefore:

\[
C_{nm} = \frac{g_m}{g_n} C_{mn} \exp \left( -\frac{E_{mn}}{kT_e} \right) \tag{56}
\]

where \( T_e \) is the electron temperature and \( N_e \) the electron number density. This relation is also valid in non-LTE.

Numerically one has:

\[
C_{mn} = \frac{8.63 \times 10^{-6}}{T_e^{1/2}} \frac{\Omega_{mn}}{g_m} \text{s}^{-1} \text{cm}^3 \tag{57}
\]

where \( \Omega_{mn} \) is a dimensionless “collision strength”, given in different tables and articles.
4.3. PHOTOIONIZATIONS AND RADIATIVE RECOMBINATIONS

A photon with an energy $h\nu$ larger than the energy required to extract an electron from a bound level $n$, $\chi_n$, can ionize an atom in the state $n$, ejecting an electron with a kinetic energy $E_{\text{kin}}$. This is a “bound-free” process, because the energy of the ejected electron is not quantified, while the excitation between two bound levels is a “bound-bound” process. It is called photoionization and corresponds to the absorption of the photon $h\nu$. Photoionization can take place either from outer valence shells, or from inner shells, when the ionizing photons have X-ray energies. The inverse process, corresponding to the emission of a photon $h\nu$, is the radiative recombination. Finally, as for bound-bound processes, there is also stimulated recombination induced by an incident photon.

Note: we do not discuss here dielectronic recombinations and the inverse process, autoionizations. Autoionizations are generally unimportant.

One defines the rate of photoionization per unit volume per second from level $n$ as $N_n P_n$, and the rate of radiative recombinations onto level $n$ (which can include dielectronic recombinations) as $N_n N_{n+1} \alpha_n$. $\alpha_n$ is called the recombination coefficient.

The absorption coefficient corresponding to a photoionization from the level $n$ can be computed from quantum mechanics, or determined experimentally. For hydrogenic ions it writes:

$$\chi_{vn} = \frac{2^4 e^2 c^2 R^2 Z^4 g}{3^{3/2} m c} \frac{1}{n^5 \nu^3} \left[ 1 - \exp \left( \frac{-h\nu}{kT_e} \right) \right]$$

$$= 2.815 \times 10^{29} N_n Z^4 \frac{1}{n^5 \nu^3} \left[ 1 - \exp \left( -\frac{h\nu}{kT_e} \right) \right] \text{ cm}^{-1} , \quad (58)$$

where $g$ is a mean “Gaunt factor”, of order unity (for example it is equal 0.9 for photoionization from level 2). Note that the absorption coefficient is proportional to $\nu^{-3}$. The dependence in $\nu$ is generally different for non hydrogenic ions. The factor $\left[ 1 - \exp \left( -\frac{h\nu}{kT_e} \right) \right]$ takes into account induced recombinations (for the demonstration, see Mihalas, 1978).

$\chi_{vn}$ is equal to zero for $h\nu < \chi_n$, which is called the photoionization edge. One uses currently also $a_{vn} = \chi_{vn}/N_n$, called the photoionization cross section. Its value at the edge, $a_{\nu_0 n}$, writes for hydrogenic ions (not taking into account induced recombinations):

$$a_{\nu_0 n} = 7.906 \times 10^{-18} \frac{n g}{Z^2} \text{ cm}^2 . \quad (59)$$

This coefficient is sometimes extended to non hydrogenic ions by defining an “effective” $Z$, $Z + s$, $s$ being of the order of unity, but it is better to use results from quantum-mechanical computations now available in several
databases, such as obtained by the OPACITY project or the CHIANTI database.

Using the Kirchhoff law \( \eta_\nu = \chi_\nu B_\nu(T) \) and the laws of TE one can compute the emissivity corresponding to the recombination process. For hydrogen, it gives:
\[
\eta_\nu = N_e N_H + \frac{C}{T^{3/2}} \frac{g}{n^3} \frac{h}{4\pi} \exp \left( \frac{\chi_n - h\nu}{kT} \right) \tag{60}
\]
where \( C = 3.26 \times 10^{-6} \) in CGS.

We can now write the photoionization rate from a level \( n \)
\[
P_n = \int_{\nu_n}^{\infty} \chi_{\nu n} \left[ 1 - \exp \left( -\frac{h\nu}{kT} \right) \right] \frac{4\pi}{h} J_\nu d\nu \tag{61}
\]
which gives, for hydrogen,
\[
P_n = 2.815 \times 10^{29} N_n \frac{4\pi}{h} \frac{g}{n^3} \int_{\nu_n}^{\infty} \left[ 1 - \exp \left( -\frac{h\nu}{kT} \right) \right] \frac{J_\nu}{\nu^3} d\nu \text{ cm}^{-3} \text{s}^{-1}, \tag{62}
\]
The recombination rate onto level \( n \) is:
\[
N_e N_{i+1} \alpha_n = \frac{4\pi}{h} \int_{\nu_n}^{\infty} \chi_{\nu n} \nu \frac{J_\nu}{\nu} d\nu \text{ cm}^{-3} \text{s}^{-1}, \tag{63}
\]
which gives for hydrogen:
\[
\alpha_n = \frac{C}{T^{3/2}} \frac{g}{n^3} \exp \left( \frac{\chi_n}{kT} \right) \mathcal{E}_1 \left( \frac{\chi_n}{kT} \right) \tag{64}
\]
where \( \mathcal{E}_1 \) is the order 1 integro-exponential. The total recombination coefficient is \( \alpha_H = \sum \alpha_n \). As shown by Eq. 64 it depends only weakly on the temperature. It is roughly given by \( \alpha_H = 2.10^{-11} T^{-1/2} \Phi \text{ cm}^{-3} \text{s}^{-1} \), where \( \Phi \) varies from 1 to 3 for \( T \) varying from \( 10^5 \) to \( 10^3 \) K. These expressions are valid for hydrogenic ions with a multiplication by \( Z^2 \). For other ions, published tables give recombination coefficients as functions of the temperature, or absorption coefficients as functions of the photon energy.

4.4. COLLISIONAL IONIZATIONS AND RECOMBINATIONS

An electron can also be ejected from an atom in the energy state \( n \), when another free electron is passing close to it (or another perturber, but electrons are generally more important, as for collisional excitations); this is called a collisional ionization. After the collision the first electron has lost an amount of kinetic energy equal to the kinetic energy of the second electron plus the potential energy \( \chi_n \). The conservation of energy writes therefore:
\[
E_{\text{kin,1}} = E_{\text{kin,1}}' + E_{\text{kin,2}} + \chi_n, \tag{65}
\]
where $E_{\text{kin}}$ and $E'_{\text{kin}}$ are the kinetic energies of the perturber before and after the collision, and $E_{\text{kin}}$ is the kinetic energy of the ejected electron. In the inverse process, a free electron induces the recombination of a second electron on the state $n$; this process is called collisional recombination, or three body recombination (because three particles are interacting).

The rate of collisional ionizations from a level $n$, per unit volume and time, and the rate of collisional recombinations on level $n$, are written respectively $N_n N_e C_{n_e}$ and $N^2_e N_{n+1} C_{n_e}$. Note that with this definition $C_{n_e}$ and $C_{n_n}$ do not have the same dimension! Like for collisional excitations, the ionization rate is equal to $< \sigma_n v >$, where the average is taken on the velocity distribution integrated from the ionization edge, and $\sigma_{n_e}(v)$ is a cross section either measured or computed. A general expression valid for a Maxwellian distribution and for $kT_e < \chi_i$ is:

$$C_{1n} \approx 10^{-8} \sigma_i \frac{T_e^{3/2}}{\lambda_i(eV)} \exp \left( -\frac{\chi_i}{kT_e} \right)$$

where $\sigma_i$ is here the number of optical electrons and $\chi_i(eV)$ is expressed in eV.

As usual one finds the relation between the direct and inverse coefficients using the TE laws:

$$C_{n1} = C_{1n} \left[ \frac{N_i}{N_e N_{n+1}} \right]_{\text{Saha}}$$

4.5. STATINARY EQUILIBRIUM OF AN ATOM (STATISTICAL EQUILIBRIUM EQUATIONS)

For each level $n$, one can write an equation corresponding to the equality of all processes depopulating and populating the level $n$:

$$\sum_{m \neq n} N_m \left( B_{mn} \int J_\nu \Phi_\nu d\nu + A_{mn} + N_e C_{mn}(T_e) \right)$$

$$+ N_n N_{n+1} \alpha_n(T_e) + N^2_e N_{n+1} C_{n_e}(T_e)$$

$$= \sum_{m \neq n} N_n \left( B_{mn} \int J_\nu \Phi_\nu d\nu + A_{nm} + N_e C_{nm}(T_e) + P_n + N_e C_{n_e}(T_e) \right)$$

where $A_{mn} = 0$ for $m < n$.

These equations can be simplified using the relations between the inverse processes shown above. This is generally done replacing also the level populations by their ratios to the LTE populations, $b_n$, defined as:

$$N_n = b_n N_{n+1} N_e \frac{g_n}{2} \left( \frac{2\pi m kT}{h^2} \right)^{-3/2} \exp \left( \frac{\chi_n}{kT} \right)$$

where $\chi_n$ is the ionization potential from level $n$. 

This set of equations forms a linear system of \( n \) equations and \( n \) unknown (the level populations), when \( T_e, N_e, N_{i+1} \) and \( \int J_\nu \Phi_\nu \, d\nu \) are known. If the medium is relatively highly ionized, and since it is electrically neutral and made mainly of hydrogen, the electrons are provided mainly by hydrogen, and \( N_e \) is equal roughly to the number of hydrogen nuclei, so \( N_e \approx 1.2 N_{\mathrm{H}^+} \approx 1.2 N_{\mathrm{H}} \) (the correction of 20% being due to the contribution from helium and heavy elements). If the medium is weakly ionized, the electrons are provided by elements having a low ionization potential, like \( \mathrm{C}^+ \) and metals, and one cannot estimate a priori the value of \( N_e \). We will see later how the \( N_{i+1} \) are determined. \( T_e \) is known through an energy balance equilibrium, and \( \int J_\nu \Phi_\nu \, d\nu \) requires the solution of the transfer equation. So the problem is very complex, and involves the sophisticated numerical methods described in P. Heinzel’s lectures.

To simplify the discussion we will assume that the atom is reduced to two bound levels \( n \) and \( m \). This picture is correct for some forbidden lines, such as the HI 21 cm transition, for the forbidden coronal lines considered later, and generally when two levels in the ground configuration can be decoupled from the resonant lines and from the continuum, due to a large difference in potential energies. A most frequent case is also that of 3 levels in the ground configuration, which are decoupled from the other levels and from the continuum. For a two level atom the equilibrium equation reduces then to:

\[
N_m \left( B_{mn} \int J_\nu \Phi_\nu \, d\nu + A_{mn} + N_e C_{mn}(T_e) \right) =
N_n \left( B_{nm} \int J_\nu \Phi_\nu \, d\nu + N_e C_{nm}(T_e) \right) .
\tag{70}
\]

Using Eq. (43) for the source function, one gets:

\[
S(\text{line}) = \frac{J_\nu \Phi_\nu \, d\nu + \epsilon B_\nu(T_e)}{1 + \epsilon} \quad \text{with} \quad \epsilon = \frac{N_e C_{mn}}{A_{mn}} \left[ 1 - \exp \left( - \frac{E_{nm}}{kT_e} \right) \right] .
\tag{71}
\]

\( \int J_\nu \Phi_\nu \, d\nu \) is a diffusion term, as it does not correspond to any emission or absorption of radiation. From this expression we immediately see that if the density is high, \( \epsilon \) dominates and \( S(\text{line}) = B_\nu(T_e) \). In other words, the transition is at LTE. We can also note that LTE is reached more easily when \( \int J_\nu \Phi_\nu \, d\nu \) is small (if the medium is optically thin, for instance). The expression of \( S \) is actually more complicated in general because \( \Phi(\nu) \neq \Psi(\nu) \).

In the case of a multi-level atom, it is possible to write the source function as:

\[
S(\text{line}) = \frac{J_\nu \Phi_\nu \, d\nu + \epsilon B_\nu(T_e) + \epsilon'}{1 + \epsilon + \epsilon''},
\tag{72}
\]
where $\epsilon'$ and $\epsilon''$ contain the other transitions, and to solve the system by iteration. This method was used in the past for a limited number of levels, but more sophisticated methods converging much more rapidly are used presently (cf. P. Heinzel’s lectures).

4.6. REMARKS CONCERNING THE NOTION OF TEMPERATURES IN A NON-LTE MEDIUM

In LTE a unique temperature is sufficient to describe the medium. This is not the case in non-LTE, so it is common to use several temperatures, according to the process considered. The most important temperatures which are introduced are:

- The “electron temperature” $T_e$. It is the temperature given by the mean value of the kinetic energy, assuming a Maxwellian distribution; Eq. 31 gives thus:
  \[
  T_e = \frac{m \langle v^2 \rangle}{3k}
  \]  
  (73)

  where $\langle v^2 \rangle$ is the root mean square velocity. This is the temperature intervening in the collisional rates $C_{mn}$ and $C_{nm}$, $C_{en}$ and $C_{ne}$.

- The “excitation temperature” $T_{exc}$, defined by:
  \[
  \frac{N_m}{N_n} = \frac{g_m}{g_n} \exp \left( -\frac{E_{mn}}{kT_{exc}} \right),
  \]  
  (74)

  which gives:
  \[
  S_\nu = \left( \frac{N_m A_{mn}}{N_n B_{nm}} \right) \left( 1 - \frac{N_m B_{mn}}{N_n B_{nm}} \right)^{-1} = B_\nu(T_{exc}).
  \]  
  (75)

  According to this equation, the absorption coefficient becomes:
  \[
  \chi_\nu = \frac{h \nu}{4\pi} N_n B_{nm} \Phi(\nu) \left[ 1 - \exp \left( -\frac{E_{mn}}{kT_{exc}} \right) \right].
  \]  
  (76)

  A negative excitation temperature correspond to a negative absorption coefficient, i.e. to a maser or a laser effect.

- The “radiation temperature” $T_r$, defined by:
  \[
  I_\nu = B_\nu(T_r).
  \]  
  (77)

  In the Rayleigh-Jeans regime, it leads to:
  \[
  T_r = \frac{\lambda^2}{2k} I_\nu
  \]  
  (78)
and the solution of the transfer equation for a finite layer becomes:

\[ T_r = T_r(0) \exp\left(-\frac{T_r}{\mu}\right) + T_{\text{exc}} \left[1 - \exp\left(-\frac{T_r}{\mu}\right)\right]. \quad (79) \]

This equation is commonly used in the radio range.

- The “effective temperature” \( T_{\text{eff}} \), defined by:

\[ F = \int F_\nu \, d\nu = \pi B(T_{\text{eff}}) = \sigma T_{\text{eff}}^4. \quad (80) \]

- The “color temperature” \( T_c \), defined by:

\[ \frac{dI_\nu}{d\nu} = \frac{dB_\nu(T_c)}{d\nu}. \quad (81) \]

For instance in a nebula ionized by a hot star, \( T_c \) is constant in the visible and UV range, and equal to the surface temperature of the star. Also the color temperature of the solar corona is that of the photosphere in the visible range (owing to diffusion).

5. Application to the solar corona

5.1. Ionization Equilibrium in Non-LTE

To solve the statistical equilibrium equations we need to know the number of ions in a given ionization state. If for each level the rate of collisional ionizations is equal to the rate of collisional recombinations, Saha’s law applies. One can show that it occurs for large values of the density \( N_e \geq 10^{16} \text{ cm}^{-3} \) which are not reached in the corona and in nebular media. These media are therefore in non-LTE for ionization equilibrium, which is given by the stationarity equation:

\[
\text{rate of (photoionizations + collisional ionizations) from all levels} = \text{rate of (radiative + collisional recombinations) on all levels.}
\]

Actually the problem is simplified by the fact that in these media the populations of excited levels are very small, so only ionizations from the ground level should be taken into account.

Other simplifications take place in nebular media and in the corona.

a. Nebular media:

HI I regions and planetary nebulae are dilute media photoionized by hot stars. Their kinetic or electron temperature is relatively low \( (10^4 \text{ K}) \). One can show that in these conditions radiative ionizations dominate over collisional ionizations, and radiative recombinations dominate over collisional recombinations in the ionization equilibrium, which reduces to:
photoionizations from the ground level 
= radiative recombinations on all levels.

The photoionization rate depends on the distance, on the temperature, and on the luminosity of the hot star. For more details on this subject, one can refer to Osterbrock’s book (1989).

b. Solar corona:

In the corona $T_e$ is of the order of a few $10^6$ K, while $T_r$ is only 5800 K. Therefore photoionizations are negligible with respect to collisional ionizations. On the other hand the density is relatively low ($N_e \leq 10^{10} \text{ cm}^{-3}$), and collisional recombinations are negligible with respect to radiative recombinations. The ionization equilibrium thus writes:

collisional ionizations from the ground level 
= radiative recombinations on all levels

or:

$$N_e N_i C_{i^+}(T_e) = N_e N_{i+1} \alpha_i(T)$$

where $\alpha_i(T_e) = \sum_{n=1}^{\infty} \alpha_n(T_e)$, and $C_{i^+}(T_e)$ is the rate of ionizations from the ground level.

It can be written approximately:

$$\frac{N_{i+1}}{N_i} = 2 \times 10^4 \frac{P_i(T_e)}{n} \left( \frac{\chi_H}{\chi_i} \right)^2 \exp \left( -\frac{\chi_i}{kT_e} \right) \left( \frac{\chi_i}{kT_e} \right)^{-1}$$

where $P_i(T_e)$ is a tabulated function, of the order of unity, $n$ is the principal quantum number, and $\chi_H, \chi_i$ are the ionization potentials of hydrogen and of the ion $i$.

This equilibrium is called coronal equilibrium and holds not only in the solar corona, but also in the hot interstellar medium, in the hot gas of galaxy clusters, in supernovae remnants, etc. Its particularity is that $N_i/\sum N_i$ is a function of only $T_e$. As an example we show the ionization equilibrium of iron on Fig. 6. We see that for the kinetic temperature of the corona, $T_e \sim 10^6$ K, iron should be in the form of highly ionized species, Fe$^{+9}$ to Fe$^{+13}$ in particular. This figure shows also that the ionic fraction is a rapidly varying function of the temperature, implying that if it can be obtained from the observations, the corresponding temperature will be accurately determined.

5.2. THE VISIBLE CORONAL LINES

Since the beginning of the century many lines of unknown origin have been observed in the visible during eclipses. They were attributed to a new element “the coronium” (remember the “nebulium” of planetary nebulae!). Grotian (1939) understood that the intense red line at 6374Å is due to a forbidden transition in the ground configuration of FeX (cf. Fig. 7).
Many other lines of highly ionized elements were then identified in the visible coronal spectrum, the most intense being identified by Edlen in 1942 to the forbidden transition in the 3s fundamental configuration of Fe\textsuperscript{+13}. These observations lead to the discovery of the high temperature of the corona. These lines are only visible on the limb, during eclipses, or using a coronagraph, as the disk is much too bright in the visible range (cf. Fig. 8).

It is not the same for extreme UV and X-ray lines of highly ionized atoms, which are emitted only by the corona, and can therefore be seen in emission on the disk itself because there is practically no underlying continuum in these bands. When the extreme ultraviolet and the X-ray bands were opened to observation with the launch of rockets and satellites, a number of permitted coronal lines due to resonant transitions were discovered. As an example Fig. 9 displays an image of the Sun obtained with EIT on board of SOHO in the FeX/XI lines at 171Å. It shows clearly strong vari-

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**Figure 6.** Coronal equilibrium of iron.

**Figure 7.** Term diagram of FeX showing 4 resonance lines in the EUV range and the forbidden visible line at 6374 Å.
But here we shall focus on forbidden visible lines, because they are more simple to handle than resonant lines, and as an example we will determine the intensity of the FeXIV 5303Å line emitted by the corona. The theory of these lines is established since decades, in particular with the pioneering work of Pottasch (1963).

The transition between levels 2 and 1 in the ground configuration of Fe$^{+13}$ producing this line is represented on Fig. 10. One has first to check that it can be decoupled from the rest of the atom and from the continuum. This requires to show that the rates of radiative and collisional deexcitations from all upper levels onto level 2, and the rates of radiative and collisional
recombinations onto level 2, are negligible compared to $N_1 B_{12} \int J_\nu \Phi_\nu d\nu$ and $N_1 N_e C_{12}$ (populating level 2), and that the rates of photoionization and collisional ionization from level 2, and the rates of induced radiative and collisional excitations to all upper levels from level 2, are negligible compared to $N_2 B_{21} \int J_\nu \Phi_\nu d\nu$, $N_2 A_{21}$ and $N_2 N_e C_{21}$ (depopulating level 2). For instance collisional and induced radiative deexcitations from excited levels are not important because these levels are not highly populated, owing to their large energies, and collisional and radiative excitations to excited levels are negligible compared to excitations between the levels in the ground configuration, owing to the exponential term in the excitation rates. Collisional recombinations are negligible because of the low density. Actually this is not true for all visible coronal lines, and in several cases cascades from highly excited levels, are important in populating the upper level of the forbidden transition.

Let us consider thus a two-level atom in an ionized medium (i.e. perturbers are electrons), and compare the different terms of Eq. 70. The atomic data we need are: the Einstein coefficient $A_{21}$ of the transition, equal to 60 s$^{-1}$, the statistical weights $g_1$ and $g_2$, respectively 2 and 4, and the “collision strength” $\Omega_{21}$ (cf. Eq. 57 and Fig. 10), of the order of unity (it decreases from 3 at $T = 10^5$ K to 0.3 at $T = 10^7$ K).

To compute the rate of radiative excitation, we need $J_\nu$. We will not determine the optical thickness in the continuum at 5303 Å, as it would require to know the absorption coefficient in the continuum at this wavelength, due to free-free process that we have not discussed. But we know from observations that the corona is optically thin in the optical band, as we see the underlying disk without any absorption. We infer that the whole corona is illuminated by the photospheric radiation field, close to a black body of temperature $T_0 = 5800$K. Since the line is narrow ($\Delta \nu_D \ll \nu_0$), $J_\nu$ is approximately constant over the line profile. The corona being illuminated only from one side, one gets:

$$\int J_\nu \Phi_\nu d\nu = J_\nu = B_\nu (T_\nu) = \frac{B_\nu (T_0)}{2} \left( \frac{R_{\text{sun}}}{z + R_{\text{sun}}} \right)^2$$

where $R_{\text{sun}}$ is the radius of the sun and $z$ the height in the corona, and one deduces the radiative excitation rate (using the relations between the
Einstein coefficients):

$$N_1 B_{12} J_\nu = N_1 A_{21} \left[ \exp \left( \frac{E_{12}}{kT_0} \right) - 1 \right]^{-1} \frac{R_{\text{sum}}}{(z + R_{\text{sum}})^2}.$$  (85)

For the basis of the corona it is equal to 0.9 $N_1$ s$^{-1}$ cm$^{-3}$, and the induced deexcitation rate, $N_2 B_{23} J_\nu$, is equal to 0.45 $N_2$ s$^{-1}$ cm$^{-3}$, therefore negligible with respect to the spontaneous deexcitation rate.

The exponential term in the collisional excitation rate $\exp \left( -\frac{E_{12}}{kT_0} \right)$ is roughly equal to unity, as $E_{12} \ll kT_e$, so $g_1 C_{12} = g_2 C_{21}$. According to Eq. 57 one gets thus for the rate of collisional deexcitation $N_2 N_e C_{21} \sim 2 \times 10^{-9} N_2 N_e$ s$^{-1}$ cm$^{-3}$ for $T_e = 10^6$ K, and the dependence on $T_e$ is small. Comparing this with the radiative deexcitation rate $N_2 A_{21}$, we see that above a critical density $N_e \sim 10^{11}$ cm$^{-3}$, deexcitation would be dominated by collisional processes. It is a very high density, and one can admit that deexcitation is always dominated by radiative spontaneous transitions. For the excitation the critical density is much smaller, $\sim 10^9$ cm$^{-3}$ at the basis of the corona, and it decreases with height, as the radiative rate is proportional to $R_{\text{sun}}^2 / (z + R_{\text{sun}})^2$. So excitation is sometimes dominated by radiative transitions, sometimes by collisional transitions.

It is important to note here that we have used 3 different temperatures, namely the temperature of the underlying blackbody $T_0$, the radiation temperature $T_r$, and the electron temperature $T_e$. All are different from the excitation temperature $T_{\text{exc}}$ (cf. below).

As a rough approximation we can consider the corona outside the disk as a finite slab of thickness $2 R_{\text{sun}} \sqrt{1 - \left(1 - \frac{R}{R_{\text{sun}}} \right)^2} \sim 2 \sqrt{H R_{\text{sun}}}$, where $H$ is the scale height of the corona ($< R_{\text{sun}}$), with no incident intensity on the line of sight. The source function in the line is given by Eq. 71. But we do not need this source function to determine the line flux, as we can check that the corona is optically thin at the center of the line (and a fortiori in the wings). Eqs. 46, 50, 51 and the relations between the Einstein coefficients, give $\chi_0 = 2.310^{-10} N_1$, $N_1$ can be written:

$$\frac{N_1}{N(\text{Fe}^{+13})} = \frac{N(\text{Fe}^{+13})}{N(\text{Fe})} \frac{N(\text{Fe})}{N_H} \frac{N_H}{N_e} N_e,$$

where $N_H$ is the number of hydrogen nuclei per unit volume. The iron abundance, $\frac{N(\text{Fe})}{N_H}$, is equal to 4 $10^{-5}$, and the ratio $\frac{N_H}{N_e}$ is equal to 0.83 in the corona (owing to electrons from helium and heavy elements). Let us make the conservative assumption that iron is entirely in the form of Fe$^{+13}$, and that $N_1$ is much larger than the populations of all the other levels, i.e. $N_1 = N(\text{Fe}^{+13})$. Both assumptions are reasonable in the region emitting the FeXIV line, actually. One finds then $\chi_0 \sim 10^{-23} N_e$, or $\tau_0 \sim 10^{-23} N_e$, where $N_e$ is the number of electrons on the line of sight per unit surface, in the emission region. Assuming that the whole corona is emitting the line
(which is clearly an overestimation) one gets $N_e = \langle N_e \rangle > 2\sqrt{H R_{\odot}}$, where $\langle N_e \rangle$ is the density averaged on the scale height, $N_e \sim 5 \times 10^8$ cm$^{-3}$, and finally one gets for $H \sim 5 \times 10^9$ cm: $\tau_0 \leq 10^{-4} \ll 1$.

At this stage it is also possible to show that the diffusion coefficient neglected in all this chapter is indeed negligible compared to the absorption coefficient in the line. Since diffusion is due to Thomson scattering, the diffusion coefficient is equal to $\sigma_T N_e = 0.66 \times 10^{-24} N_e$, and we immediately see that it is smaller than $\chi_0$. However, it is not possible to neglect the emissivity due to diffusion in the continuum, as it is the only emission process for the continuum underlying and surrounding the line. According to Eq. 22 for an optically thin purely diffusing medium,

$$I_{\nu} = \frac{E_\nu}{4\pi} \frac{N_2}{A_2} = \frac{E_\nu}{4\pi} \frac{N_2}{A_2}$$

is indeed the intensity of the visible continuum observed at the basis of the corona. And according to Eq. 23, the intensity in the line should simply be added to the continuum intensity, since the absorption coefficient dominates on the diffusion one. So finally we see that the computation of the line intensity can be completely decoupled from the diffusion process.

The intensity in the line is therefore equal to $\int \eta_\nu(z)dz$ (as the emissivity is strongly dependent on the position, one cannot use the homogeneous approximation), It is proportional to the emissivity at the line center and to the emissivity integrated on the line profile (cf. Eqs. 41 and 45), which writes, using Eq. 43:

$$\eta_{\text{line}} = \frac{E_\nu}{4\pi} \frac{N_2}{A_2} = \frac{E_\nu}{4\pi} \frac{N_2}{A_2} + B_1 J_\nu$$

Eq. 86 gives thus, in the two extreme cases of $N_e$ much larger and much smaller than the critical density of $10^9$ cm$^{-3}$:

1. $N_e > 10^9$ cm$^{-3}$: $\eta_\nu \sim \frac{E_\nu}{4\pi} C_{12} \frac{N_e}{N_0} [N(\text{Fe}^{+13})/N(\text{Fe})]$.
2. $N_e < 10^9$ cm$^{-3}$: $\eta_\nu \sim \frac{E_\nu}{4\pi} B_1 J_\nu N_e [N(\text{Fe}^{+13})/N(\text{Fe})]$.

The first case corresponds to dense coronal loops, and the second one is close to the quiet corona.

Let us assume that the density is larger than the critical one (actually it is often the case for other forbidden coronal lines). The intensity writes thus:

$$I_{\text{line}} \sim 4 \times 10^{-5} \frac{E_\nu}{4\pi} \int N_e^2 C_{12}(T_e) \frac{N(\text{Fe}^{+13})}{N(\text{Fe})} dz.$$
The intensity, and consequently the flux measured at the Earth, is therefore a function of the density and of the temperature in the region where iron is in the form of Fe$^{+13}$. But $N(\text{Fe}^{+13})/N(\text{Fe})$ depends so strongly on the temperature, that one considers it equal to unity in a small range of temperature, and zero outside. Moreover $C_{12}(T_e)$ is not strongly dependent on $T_e$. One can thus approximate the intensity by

$$I_{\text{line}} \sim 4 \times 10^{-5} \frac{E_{12}}{4\pi} C_{12} \int E(T_e) dT_e$$

where $\int E(T_e) dT_e = \int N_e^2 d\ell$ is called the emission measure.

Therefore, measuring other visible forbidden coronal lines corresponding to different ions should allow to map the emission measure as a function of the temperature. More sophisticated methods involving inversion procedures can also be used to avoid the rough approximations made above.

We have focussed here on a forbidden line, but it is clear that the same kind of computation can be performed for other types of lines, such as resonance UV or X-ray lines. The equations giving the population of the upper level of the transition would not necessarily have been as simple as in the particular case considered here, and consequently the dependence of the emissivity on the temperature and on the density would have been more complicated, but the whole procedure would have been similar. Then, by comparing forbidden lines with resonance lines of the same ion in the extreme UV and in the X-ray, or more generally pairs of lines of the same ion, one can get powerful diagnostics of the temperature and/or of the density in the region where this ion is abundant. Besides, the EUV and X-ray lines are observed also on the disk, where the forbidden visible lines cannot be observed. For instance Si\textsc{ix} and Si\textsc{x} lines which are density sensitive have been observed with the coronal diagnostic spectrometer (CDS) on board SOHO and have allowed to obtain the density and the temperature in active regions on the limb and on the disk as a function of the radius and the position angle. One can guess that many other important results will be obtained from the data gathered in this mission.

Finally, one should not forget that the simple study of the line profiles using lines of different elements allows to determine both the kinetic temperature and the turbulent velocity, as shown also in this chapter.

Acknowledgements

I wish to thank Petr Heinzel for kindly reading the manuscript, which benefitted from the improvements he suggested.
References