Relational time in generally covariant quantum systems: four models

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(August 27th 2000)

We analyze the relational quantum evolution of generally covariant systems in terms of Rovelli’s evolving constants of motion and the generalized Heisenberg picture. In order to have a well defined evolution, and a consistent quantum theory, evolving constants must be self-adjoint operators. We show that this condition imposes strong restrictions to the choices of the clock variables. We analyze four cases. The first one is non-relativistic quantum mechanics in parametrized form, we show that, for the free particle case, the standard choice of time is the only one leading to self-adjoint evolving constants. Secondly, we study the relativistic case. We show that the resulting quantum theory is the free particle representation of the Klein Gordon equation in which the position is a perfectly well defined quantum observable. The admissible choices of clock variables are the ones leading to space-like simultaneity surfaces. In order to mimic the structure of General Relativity we study the SL(2R) model with two Hamiltonian constraints. The evolving constants depend in this case on three independent variables. We show that it is possible to find clock variables and inner products leading to a consistent quantum theory. Finally, we discuss the quantization of a constrained model having a compact constraint surface. All the models considered may be consistently quantized, although some of them do not admit any time choice such that the equal time surfaces are transversal to the orbits.

I. INTRODUCTION

An old problem in physics is how to eliminate from our physical theories any reference to "absolute elements". Leibnitz already used relational arguments, against Newton’s concept of absolute space and time: "...it is not possible for two things to differ from one another in respect of place and time alone, but it is always necessary that there shall be some other internal difference... in general, place, position and quantity are mere relations...". Absolute elements are elements of the theory whose interpretation requires the existence of things outside the dynamical system described by the theory. The absolute acceleration in Newtonian mechanics and the absolute interval between two events in special relativity are absolute elements of these theories. General relativity allows us to get rid of these elements and proposes in its place a purely relational description of space-time.

Quantum mechanics still requires some absolute elements in order to explain how the quantum world gives rise to events in the (classical) measurement devices. However, it is relational in the sense that quantum objects does not carry pre-established answers for possible measurements, and the events are produced during the measurements and depend upon the relation between the quantum system and the experimental setup.

Time is also an absolute element in standard quantum mechanics. Depending on the representation, states or operators are parametrized by time. The choice of the time variable is independent from the dynamics of the system, and given by the value of some external device used as a clock. Furthermore, there is no time operator in quantum mechanics because time has nothing to do with the expectation value of any observable of the quantum system. However, in the absence of a fundamental time variable as is the case of totally covariant systems, like general relativity, the most natural description of evolution is relational.

Generally covariant systems are a particular kind of constrained systems in which the motion is just the unfolding of a gauge transformation. In other words, these theories do not have a genuine Hamiltonian for describing the evolution of the system. Dynamics is given by the gauge transformations generated by the first class constraints of the theory. In generally covariant theories time plays the role of any other degree of freedom and is not explicitly included within the formalism. The canonical variables follow orbits in phase space that may be described in terms of arbitrary parametrizations. The arbitrary parameter does not possess any physical significance, and therefore the formalism is invariant under reparametrizations.

Here, we shall analyze the quantum evolution of generally covariant systems in terms of Rovelli’s [1] evolving constants of motion and the generalized Heisenberg picture. This procedure allows us to describe the evolution without any gauge fixing. It is given in terms of correlations between the original dynamical variables. Time is identified with some internal clock variable $T(\tau)$, and what is actually measured is not the value of a physical variable $Q$ for certain value of the parameter $\tau$ but the value $Q(T)$ taken by the physical variable when the clock variable takes the value $T$. 

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Since the publication of Rovelli’s seminal papers, evolving constants have received much attention [2–6]. However, several important problems concerning the description of the evolution in terms of evolving constants are still open. In the first place, contrary to what is usually claimed, we shall see that the choice of the clock variable is highly restricted. The fundamental limitation for having a good clock variable arises from the nontrivial conditions required in order to transform the evolving constants into good, possibly unbound, self-adjoint operators of the quantum theory. These conditions turn out to be closely related with the transversality of the equal-time surfaces to the orbits of the system. Secondly, this approach was initially developed for the study of constrained systems with one dimensional orbits. Its extension [6] to the general case of a system with N degrees of freedom and n constraints requires further study. In fact, the n independent variables required to parametrize the evolution of the system cannot be identified as clock variables, and further conditions on the choice of the clock variables appear. In principle, one can choose more than one internal degree of freedom as a clock variable. However, we have explored this possibility in the SL(2,R) case, and we have concluded that, one needs to use a single clock variable, in order to have a consistent description of the evolution. But, we cannot rule out the possibility of having several clock variables in more general systems. Finally, there are systems that are not Hamiltonian, insofar as they do not admit a decomposition of the constraint surface Σ as a direct product Σ = Σ × R where R is the real line. That is the case of systems with a compact constraint surface. As it is well known, these systems are far from being well understood. We will show that it is possible to find time variables leading to consistent quantizations.

This paper is organized as follows. In Section II, we recall what we mean by a generally covariant system, we describe two quantization procedures for this kind of systems and we discuss the role of the evolving constants in the description of the time evolution. In section III, we analyze the simplest model of a quantum generally covariant system, the parametrized non-relativistic particle. We shall analyze quantization in terms of different choices of the clock variable, and show that there is only one choice that leads to self-adjoint evolving constants, the standard choice of time in non-relativistic quantum mechanics. In section IV, we consider the case of the relativistic particle. Quantization in terms of evolving constants leads, for the standard choice of time, to the free particle representation of the Klein Gordon equation. In this representation the position is an unbound self-adjoint operator and therefore a perfectly well defined quantum observable. We also analyze the set of admissible choices of clock variables. In section V, we study the SL(2R) model with two Hamiltonian constraints. This is a model with three-dimensional orbits. The evolving constants approach is extended to this case and some clock variables identified. In section VI we discuss the case of systems without any natural time structure associated with compact constraint surfaces, and show that these quantum systems admit a consistent notion of evolution. Section VII contains the conclusion and some relevant considerations.

II. GENERALLY COVARIANT SYSTEMS

We shall here consider finite dimensional first class totally constrained systems. Let Σ be the phase space of a generally covariant system with coordinates \( \{x^\mu, p_\mu\} \). Its action \( S \) is given by

\[
S = \int d\tau [p_\mu \dot{x}^\mu - N^\alpha C_\alpha]
\]

(1)

where \( \dot{x} \) represents the derivative with respect to \( \tau \), \( \mu = 1 \ldots N \), \( N^\alpha \) are Lagrange multipliers, and \( C_\alpha, \alpha = 1 \ldots n \), are the constraints. Varying \( S \) with respect to \( N^\alpha \) one gets the constraint equations

\[
C_\alpha = 0.
\]

(2)

We assume that they are first class constraints. That is, their Poisson bracket satisfies

\[
\{C_\alpha, C_\beta\} = f_{\alpha\beta\gamma} C_\gamma
\]

(3)

where \( f_{\alpha\beta\gamma}(x, p) \) are the structure functions. In most of the models we shall consider structure constants.

A basic notion of our description will be that of observable. An observable is a function, in general complex, in phase space such that its Poisson Bracket with the constraints vanishes. The observables are the gauge invariant quantities of the system. In a generally covariant system, the observables are constants along the orbits. That is the reason why, in this case, they are also called perennials. The perennials form a Poisson algebra with involution. The evolving constants are a particular kind of observables [1] defined as follows. Let us assume for simplicity that the constrained system is defined by a single constraint \( C \) on a phase space \( x^\mu, p_\mu, \mu = 1 \ldots N \) and let us take one of the variables, say \( x^1 \) as a clock variable. Then we want to introduce observable quantities that represent the evolution of \( x^1 \ i = 2 \ldots N \) and \( p_\mu \) in terms of \( x^1 \). To do that, we consider a one-parameter family of observables. Each observable
of the family represents the value of \( x^i \) at a different value, say \( T \), of the clock variable \( x^1 \). Let us call these observables \( X^i(T) \); they satisfy

\[
\{ X^i(T, x^μ, p_μ), C(x^μ, p_μ) \} = 0 \\
X^i(T, x^1 = T, x^2, x^3 \ldots p_N) = x^i.
\] (4) (5)

The first equation imposes that \( X^i \) is constant along the orbits, the second determines the value taken by the observable along the orbit.

For instance, take the presymplectic system associated with the non-relativistic free particle. The phase space is four dimensional with coordinates \((x^0, x^i, p_0, p_i)\), and the physical trajectories are contained in the constraint surface \( C = p_0 + p^2/2m \). The evolving constant that describes the evolution of \( x^1 \) in terms of \( x^0 \) is given by

\[
X^1 = x^1 - \frac{p}{m}(x^0 - T)
\] (6)

It is immediate to check that equations (4) and (5) are satisfied. Notice that the evolving constant may be interpreted as the expression of a dynamical variable in terms of the initial conditions, which are perennials, and a value of the clock variable. When the clock variable evolves, it reproduces the classical evolution of the dynamical variables.

As we shall see, generally covariant systems with more than one constraint require more than one independent variable to coordinate the orbits. A subset of these variables may be taken as clock variables while the others may be interpreted as spatial coordinates.

In what follows we shall assume that we have identified a complete set of perennials such that any classical perennial is a function of the elements of this set. Let us consider the Poisson algebra with involution \( A \) generated by this set. Evolving constants are generically functions of the complete set of perennials, but they do not necessarily belong to their algebra. The starting point of our description of the time evolution of a generally covariant system will be the extended Poisson algebra with involution \( B^{(\star)}_{\text{phys}} \) that contains \( A \) and a suitable set of evolving constants. Within this approach, the resulting evolution will depend on this choice of evolving constants.

There are several alternative procedures of quantization of totally constrained systems. The resulting quantum theory may differ depending on the procedure. These methods are not always equivalent due to factor-ordering problems or nontrivial global issues. Here, we are going to consider, two types of quantization procedures. The first type of procedure consists in quantizing first the kinematical variables and after that imposing the constraints.

Algebraic Quantization [7,8] and Refined Algebraic Quantization [9,10] are procedures of this type. In the second type of procedure, the reduced phase space quantization, one is only concerned with the observable quantities. Thus one starts from the classical Poisson algebra with involution \( B^{(\star)}_{\text{phys}} \), and look for a quantum representation of the corresponding operators belonging to \( B^{(\star)}_{\text{phys}} \) on a Hilbert space \( \mathcal{H}^\dagger_{\text{phys}} \) such that their reality properties be preserved by the quantization. Depending on the model, one method could be more convenient than the other.

As it is technically more involved, we will start by making a brief introduction to the refined algebraic quantization procedure. We refer the interested reader to the original references [9,10]. One starts by introducing operators \( \hat{x} \) and \( \hat{p} \) on an auxiliary Hilbert space \( \mathcal{H}_{\text{aux}} = L^2(\mathbb{R}^N) \). Let \( \hat{x}^μ \) be multiplicative and \( \hat{p}_μ \) act by \((-i \times)\) differentiation, so that all the operators are self-adjoint on \( \mathcal{H}_{\text{aux}} \). One represents the constraints \( C_α \) as self-adjoint operators \( \hat{C}_α \) on \( \mathcal{H}_{\text{aux}} \). Now, consider the quantum version \( B^{(\star)}_{\text{phys}} \) of the classical algebra \( B^{(\star)}_{\text{phys}} \), whose elements are operators (perennials) on \( \mathcal{H}_{\text{aux}} \) which commute with the constraints \( \hat{C}_α \). They should be linear, but not necessarily bounded operators. From a physical point of view, the identification of a suitable family of perennials is a key step. In particular, as it was already mentioned, it involves the choice of a set of evolving constants in order to describe the evolution of the system.

The solutions of the constraints define a linear space \( L \). As generically the constraints will not be bounded, it will be impossible to identify \( L \) with \( \mathcal{H}_{\text{aux}} \). For example, if the spectrum of any given constraint is purely continuous, then it has no normalizable eigenvectors on \( \mathcal{H}_{\text{aux}} \) and therefore the elements of \( L \) will not be contained in \( \mathcal{H}_{\text{aux}} \). In fact, the elements of \( L \) will be distributions belonging to the dual of some dense subspace \( S_L \) of \( \mathcal{H}_{\text{aux}} \). One imposes the following conditions on \( S_L \) [10]: i) \( S_L \) is contained in the domain of each constraint \( \hat{C}_α \). ii) \( S_L \) is left invariant under the action of each constraint. These two conditions ensure that the solutions of the constraints belong to the dual of \( S_L \). Furthermore one requires that the elements \( \hat{A} \) of \( B^{(\star)}_{\text{phys}} \) be such that: iii) the domain of \( \hat{A} \) and \( \hat{A}^\dagger \) contains \( S_L \), and iv) \( \hat{A} \) and \( \hat{A}^\dagger \) leave \( S_L \) invariant. These last two conditions imply that it is possible to induce an inner product in the physical space such that the reality properties of the physical operators of \( B^{(\star)}_{\text{phys}} \) are ensured.

The physical Hilbert space is introduced as follows: One defines an inner product in some subspace \( M \) of \( L \) in such a way that its completion is the physical Hilbert space \( \mathcal{H}_{\text{phys}} \). It is always possible under the aforementioned hypothesis to induce an inner product in \( M \) such that the operators of \( B^{(\star)}_{\text{phys}} \) corresponding to real classical perennials are self-adjoint. The explicit construction of the inner product is given in Refs [9,10].
As we shall see in what follows, evolving constants are a powerful tool to study the evolution of generally covariant systems. In particular, they allow us to compare the evolution resulting from the use of different clock variables simply by including in $\mathcal{B}(\ast)_{\text{phys}}$ the evolving constants corresponding to the different choices.

The main assumption is that the operators belonging to $\mathcal{B}(\ast)_{\text{phys}}$ corresponding to real quantities are self-adjoint. Any choice of clock variable leading to self-adjoint evolving constants is valid. We shall show that, contrary to what is usually claimed, this condition imposes strong restrictions to the choices of clock variables. In fact, although general procedures for systematically ordering observables exist [11,12] and they ensure that the resulting operators are symmetric, in the case of unbound operators this is not enough to ensure that they are also self-adjoint [13]. The choice of the clock variables not only fixes the rate of the running clocks, but also determines the set of simultaneous events. An "equal-time" surface $\Sigma$ is called transversal to the orbits, if each orbit intersects $\Sigma$ in one and only one point and the tangent vectors to the orbit are independent from the tangent vectors to $\Sigma$. We shall see that transversality is not required to have self-adjoint evolving constants, and a weaker condition is sufficient to ensure a consistent description of the quantum evolution.

Notice that the quantum description resulting from this procedure corresponds to a Heisenberg representation, i.e., the observables evolve in time. However, one does not assume the existence of a Hamiltonian such that for any perennial $Q(T)$ the following equation holds

$$ih\frac{dQ(T)}{dT} = [Q(T), H]$$

(7)

The resulting quantum description in terms of evolving constants is relational in the sense that it assigns probabilities to events produced in measurement devices when certain dynamical variables taken as clocks take a given value. Depending on the choice of clock variable and on the physical quantity under consideration (represented by an evolving constant) we will have a different experimental setup and a different procedure for the synchronization of the clocks. One can describe the same experimental setup in different states of motion, by changing the parameterization of the evolving constants. It is even possible, by changing the clock variable, to introduce two different orders for the same set of events. If both orders lead to well defined quantum theories, their predictions should be consistent.

### III. PARAMETRIZED NON-RELATIVISTIC QUANTUM MECHANICS

Parametrized non-relativistic quantum mechanics admits a natural time structure. As it was already shown by Rovelli [14], the standard choice of the clock variable $T = x^0$, leads to predictions that agree with usual non-relativistic quantum mechanics. However, the method of evolving constants admits, in principle, a much wider variety of choices for the clock variable. In principle, these choices are only restricted by the requirement that the corresponding evolving constants are well defined self-adjoint operators on some Hilbert space allowing to preserve the probabilistic structure of the resulting quantum theory. In Rovelli’s words, the idea is that "the time axiom can be dropped without compromising the other axioms of the probabilistic interpretation of the theory." Here we intend to analyze whether other choices of the clock variable meet these requirements. In what follows we will show that there is strong evidence suggesting that the only allowed time choice of the form $T(x)$ is the standard one.

Let us consider the elementary case of a non-relativistic free particle in one dimension. Let $\Gamma$ be the phase space of the system with coordinates $(x^0, x)$ and momenta $(p_0, p)$. Its action is given by

$$S = \int d\tau [p_0 \dot{x}^0 + p \dot{x} - N(p_0 + \frac{x^2}{2m})]$$

(8)

that leads to the Hamiltonian constraint

$$\mathcal{C} \equiv p_0 + \frac{x^2}{2m} = 0$$

(9)

which defines the constraint surface $\Gamma'$. A complete set of classical perennials is given by $p$ and $x - x^0 p/m$. Now we choose the auxiliary Hilbert space to be $\mathcal{H}_{\text{aux}} = L^2(\mathbb{R}^2, dp_0 dp)$ and represent $x^\mu$ and $p_\mu$ by the usual expressions of the momentum representation, so that the corresponding operators are self-adjoint. Let us now choose clock variables

\[\text{this problem was also analyzed by Hartle [4] who also concluded based on physical arguments that the allowed time functions should be restricted for the predictions to coincide with those of the usual quantum theory.}\]
and the corresponding evolving constants that will be used to describe the evolution, and include them among the elements of $\mathcal{B}^{(*)}_{\text{phys}}$.

Our first choice will be $T = x^0$, then $\mathcal{B}^{(*)}_{\text{phys}}$ will include the perennials $\{ \hat{q} := x - x^0 p / m, \hat{p}, \hat{X}(T) := \hat{q} + \hat{p} T / m \}$ The spectrum of $\hat{X}(T)$ will characterize the possible outcomes of a measurement of $\hat{X}$ when $x^0$ takes the value $T$. Our physical states will be associated with generalized eigenstates of $\hat{C}$ with vanishing eigenvalue. These states may be considered as distributions acting on a dense subspace $S_L \subset \mathcal{H}_{\text{aux}}$ In this case, the natural choice for $S_L$ is the space of smooth functions with compact support. Notice that, for this choice of the clock variable, $S_L$ satisfies the requirements established in the previous section. The physical states are

$$\psi(p, p_0) = \delta(p_0 + \frac{p^2}{2m}) f(p) \quad (10)$$

The perennials $\hat{p}, \hat{q}, \hat{X}(T)$ are self-adjoint on $\mathcal{H}_{\text{aux}}$ and satisfy the following generalized eigenvalue equations:

$$\hat{p} \psi_{p_1}(p, p_0) = p \psi_{p_1}(p, p_0) = p_1 \psi_{p_1}(p, p_0) \quad (11)$$
$$\hat{q} \psi_{q_1}(p, p_0) = (i \frac{d}{dp} - ip/m \frac{\partial}{\partial p}) \psi_{q_1}(p, p_0) = q_1 \psi_{q_1}(p, p_0) \quad (12)$$
$$\hat{X}(T) \psi_{x_1, T}(p, p_0) = [i \frac{d}{dp} - ip/m \frac{\partial}{\partial p} + T p / m] \psi_{x_1, T}(p, p_0) = x_1 \psi_{x_1, T}(p, p_0), \quad (13)$$

The solutions belonging to the kernel of the constraint are

$$\psi_{p_1}(p, p_0) = \delta(p_0 + \frac{p^2}{2m}) \delta(p - p_1) \quad (14)$$
$$\psi_{q_1}(p, p_0) = \delta(p_0 + \frac{p^2}{2m}) \exp ipq_1 \quad (15)$$

and

$$\psi_{x_1, T}(p, p_0) = \delta(p_0 + \frac{p^2}{2m}) \exp i(px_1 - \frac{p^2}{2m} T). \quad (16)$$

It is now possible to introduce [9] an inner product in the physical space of solutions of the constraint. One considers solutions of the form $\psi(p, p_0) = \delta(p_0 + p^2/2m) f(p, p_0)$, with $f(p, p_0) \in S_L$. Then, the inner product is given by:

$$\langle \psi_1 | \psi_2 \rangle_{\text{phys}} = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp_0 \delta(p_0 + \frac{p^2}{2m}) f_1^*(p, p_0) f_2(p, p_0) = \int_{-\infty}^{\infty} dp \tilde{f}_1^*(p) \tilde{f}_2(p) \quad (17)$$

where $\tilde{f}(p) = f(p, -p^2/2m)$. Then, by considering the completion of $S_L$ one gets a Hilbert space $\mathcal{H}_{\text{phys}}$ with the standard inner product in the momentum representation. All the elements of $\mathcal{B}^{(*)}_{\text{phys}}$ corresponding to real quantities will be trivially defined self-adjoint operators in this space. They will be given by $\hat{\psi} = i \partial \psi / \partial p, \hat{\psi} = p \hat{\psi}$, and $X(T) = \hat{q} + p/m T$. Notice that in this simple case, if we had followed the second procedure and taken as starting points the algebra of observables we would have recovered in a simpler way the same quantum operators and the inner product (17).

Summarizing, the choice of clock variable $T = x^0$ leads to the standard form of the quantum free particle in the Heisenberg representation. In particular one may recover from Eqs(16) and (17) the standard transition amplitude

$$\langle \psi_{x, T} | \psi_{x', T'} \rangle = \left( \frac{2\pi i (T - T')}{m} \right)^{-1/2} \exp \frac{im(x - x')^2}{2(T - T')} \quad (18)$$

Now, we would like to analyze other choices of the clock variable for this simple model and determine whether they lead to self-adjoint evolving constants and admit a probabilistic interpretation. Let us first consider the choice of the clock variable $x = T$. The corresponding evolving constant is $X^0(T) = \{ -m / p \} (q - T)$. It obviously commutes with the constraint $\hat{C}$ and $X^0(x = T) = x^0$. The inclusion of $X^0(T)$ in $\mathcal{B}^{(*)}_{\text{phys}}$ will immediately lead to difficulties. The space $S_L$ of smooth functions of constant support does not satisfy the conditions iii) and iv) of the previous section. Due to the division by $p$, $S_L$ is not contained in the domain of $X^0(T)$. Only the wave functions with zero amplitude for $p = 0$ might belong to $\mathcal{H}_{\text{aux}}$. The resulting operator is not self-adjoint and does not admit self-adjoint extensions. Let us show that explicitly by following the reduced phase space quantization procedure. The standard inner product

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This choice has been extensively analyzed in Ref. [15]
in momentum representation ensures that the complete set of perennials \( \hat{q} \), and \( \hat{p} \) is self-adjoint. Let us consider the symmetric form of the evolving constant \( \hat{X}^0(T) \) in the momentum representation given by:

\[
\hat{X}^0(T)\psi(p) = \left[ -\frac{1}{2} \left( \frac{m}{p} \frac{i\partial}{\partial p} + \frac{i\partial m}{\partial p} \frac{p}{m} \right) + \frac{mT}{p} \right] \psi(p)
\]  

(19)

In order to find out whether this operator is self-adjoint or at least has self-adjoint extensions, we are going to use the following simple and powerful method \([16,17]\): Check the dimensionality of the two subspaces: \( K_- = \ker(\hat{X}^0(T) - i) \) and \( K_+ = \ker(\hat{X}^0(T) + i) \). If they do not have the same dimensionality, the operator is not self-adjoint and has no self-adjoint extensions.

It is immediate to see that

\[
dim \ker(\hat{X}^0(T) + i) = \dim \{ c \sqrt{\frac{p}{2\pi}} \exp(-ipT - \frac{p^2}{2m}) \} = 1,
\]  

(20)

because the elements of this kernel belong to \( L^2[-\infty, \infty, dp] \), while

\[
dim \ker(\hat{X}^0(T) - i) = \dim \{ c \sqrt{\frac{p}{2\pi}} \exp(-ipT + \frac{p^2}{2m}) \} = 0.
\]  

(21)

Thus, the evolving constant \( X^0 \) is not self-adjoint, and it does not correspond to any observable quantity of the quantum theory. In physical terms it is not possible to assign probability amplitudes to the generalized eigenvalues of the operator. The physical origin of this problem is the following: The equal "time" surfaces \( x = T \) are not transversal to the orbits and the measurement devices lying at different values of \( x^0 \) are causally connected. In particular, a particle may be found, at equal \( T \), at two different values of \( x^0 \) as it can be seen by computing the inner product of two eigenfunctions corresponding to different values of \( x^0 \). In fact, the eigenfunctions of \( X^0(T) \) with eigenvalue \( x^0 \) are given by:

\[
\psi_{T,x^0}(p) = \sqrt{\frac{p}{2\pi}} \exp(-ipT + \frac{p^2}{2m} x^0)
\]  

(22)

Therefore the functions corresponding to different eigenvalues are not orthogonal. In fact

\[
< \psi_{T,x^0} | \psi_{T,x^0'} >= \frac{1}{2\pi} \int_{-\infty}^{\infty} dp |p| \exp \frac{ip^2}{2m} (x^0' - x^0)
\]  

(23)

and introducing the variable \( p_0 = \frac{p}{2m} \) one gets

\[
< \psi_{T,x^0} | \psi_{T,x^0'} >= \frac{m}{2\pi} \int_0^{\infty} dp_0 \exp ip_0 (x^0' - x^0) = \frac{m}{2} [\delta(x^0' - x^0) - 2\pi i \mathcal{P} \frac{1}{x^0' - x^0}],
\]  

(24)

where \( \mathcal{P} \) is the principal part. One may wonder whether requiring the evolving constant to be self-adjoint is too stringent when we are considering operators associated to causally connected measurements. After all, as we are taking \( x = T \) the states are prepared by measurements performed for all the possible values of \( x^0 \) at a given \( x \), and the standard free particle propagator (18) leads to nonvanishing transition amplitudes between points with coordinates \( x, x^0 \) and \( x, x^0' \). However, we don’t see any consistent procedure for the assignment of probabilities when the operators are not self-adjoint. In fact, the natural way for the assignment of probabilities to the measurement of a quantum observable, is via the quadratic form \( < \phi | A | \phi > \). This expression represents the mean value of the Heisenberg observable only if \( A \) is a self-adjoint operator with a spectral decomposition for any value of the time variable \( T \). If that is not the case, one looses the natural interpretation for the proper values as a possible outcome of the measurement. Notice, furthermore, that if one simply attempts to identify the inner product (24) with the free particle kernel (18) evaluated for \( x = x' = T \), one notices that the inner product does not reproduce the standard transition amplitude.

Other choices of the clock variable leading to equal time surfaces which are not transversal to the orbits also lead to the same kind of difficulties. For instance the choice \( T = x^0 - ax \), where \( a \) is an arbitrary constant, corresponds to the evolving constant

\[
X(T) = \frac{q + \sqrt{mT}}{1 - \frac{a}{m}}
\]  

(25)
and leads to a non self-adjoint operator. Again the smooth functions of \( p \) with compact support lay outside the domain of \( \hat{X} \) and the kernels of \( \hat{X} \pm i \) have different dimension. Notice that this choice of clock variable do not differ from the standard choice in the rate of the running clocks, but define different equal time surfaces. In fact this choice corresponds to an array of measurement devices at rest, where the clocks have been synchronized on the surface \( x^0 = ax \), and therefore ”simultaneous” measurements of the position are causally connected by orbits with speed \( 1/a \).

One could have considered an array of measurement devices in uniform motion. In order to do that, one may introduce the Galileo transformation \( x' = x + v_0 t \) and \( x^0 = x^0' \). Then if one defines an evolving constant \( \hat{X}' \) associated with \( x' \) and choose the clock variable \( x^0 = T \), one gets a perfectly well behaved self-adjoint operator and a consistent quantum mechanical description. It is also possible to extend this analysis to the case of non-relativistic particles under the action of simple conservative forces. One is led to the same conclusion for all the cases considered: The requirements for the existence of a consistent quantum mechanical description of the non relativistic particle are fulfilled if, and only if, the choice of the clock variable leads to equal time surfaces which are transversal to the orbits. As we shall see in the following sections, one should not hasten to conclude, from these simple models with a natural time structure, that in the case of generally constrained systems without a natural time structure of transversal surfaces it will be impossible to have a consistent quantum mechanical description of the evolution.

IV. PARAMETRIZED RELATIVISTIC QUANTUM MECHANICS

As a second example of systematic application of the evolving constants technique, we are going to study the quantization of the relativistic particle. This system is conceptually more involved than the previous one due to several issues. In the first place, while the non-relativistic particle only admits a natural time structure, the relativistic system should admit a wider class of clock variables associated to space like slicings. Secondly, the phase space of the relativistic particle with quadratic Hamiltonian constraint has two disconnected sectors. When one attempts to quantize the theory including both sectors, for instance using the Klein Gordon procedure, one runs into difficulties. The inner product is not positive definite, the position is not self-adjoint, and when the particle is subject to external forces which are not slowly varying on a few Compton wavelengths, the system does not admit any probabilistic interpretation. We are going to show that the evolving constants technique sheds new light on these phenomena.

Let us briefly consider, in the first place, the simplest case of a relativistic particle with positive energy. The action is given by

\[
S = \int dt[p_0 x^0 + p \dot{x} - N(p_0 + \sqrt{p^2 + m^2})]
\] (26)

We follow steps parallel to the case of the non-relativistic particle. Again the auxiliary Hilbert space will be \( \mathcal{H}_{aux} = L^2(R^2, dp_0 dp) \) A complete set of classical perennials is given by \( p \) and \( q := x - x^0 p/\sqrt{p^2 + m^2} \). In this case, the system admits several consistent choices of clock variables. The first and trivial choice is \( x^0 = T \). We take as an evolving constant the one giving the value of the coordinate \( x \) when the clock takes the value \( T \). It is given by

\[
X(T) = q + pT/\sqrt{p^2 + m^2}
\]

which should be included together with the perennials \( q \) and \( p \) in \( B^{(x)}_{phys} \). These quantities are trivially quantized on \( \mathcal{H}_{aux} \). They satisfy conditions i) to iv) when acting on the space of smooth functions with compact support \( S_L \). It is therefore not surprising that they be self-adjoint operators. For instance, it may be immediately checked that the dimensionality of the subspaces \( \mathcal{K}_+ \) and \( \mathcal{K}_- \) is the same and equal to zero. This is therefore a consistent choice of time, and reproduces the standard results for a relativistic scalar particle with positive energy. One can, for instance, compute the generalized eigenfunctions of \( \hat{X}(T) \), which satisfy orthonormality and closure relations and lead to the standard positive energy propagator of the relativistic particle.

This model admits other consistent choices of clock variable or evolving constants. For instance, from the choice of clock variable \( T = x^0 - ax \), where \( a \) is a real constant, one may define two evolving constants

\[
X(T) = \frac{q + \frac{pt}{\sqrt{p^2 + m^2}}}{1 - \frac{aq}{\sqrt{p^2 + m^2}}}
\] (27)

and

\[
X^0(T) = \frac{-aq + T}{1 + \frac{ap}{\sqrt{p^2 + m^2}}}
\] (28)

The first one takes the value \( x \) for \( T = x^0 + ax \), while the second one takes, for the same value of the clock variable, the value \( x^0 \). It is not difficult to check that at the quantum mechanical level these choices lead to consistent theories
The spectrum of $\hat{\epsilon}$ is given by the eigenvalue equation:

$$\hat{X}(T)|\psi(p)\rangle = \left[\frac{i}{2} \frac{\partial}{\partial p} \left( \frac{1}{\sqrt{p^2 + m^2}} \right) + \frac{i}{2} \frac{1}{\sqrt{p^2 + m^2}} \frac{\partial}{\partial \sqrt{p^2 + m^2}} + \frac{pT}{\sqrt{p^2 + m^2} - pa} \right]|\psi(p)\rangle. \quad (29)$$

Looking at the dimensions of the two subspaces $\mathcal{K}_+$ and $\mathcal{K}_-$ one gets:

$$\dim \ker(\hat{X}(T) + i) = \dim \{ c \sqrt{1 + \frac{pa}{\sqrt{p^2 + m^2}}} \exp(i \sqrt{p^2 + m^2} T + \sqrt{|a| p^2 + m^2 - p}) \} = 0 \quad (30)$$

and

$$\dim \ker(\hat{X}(T) - i) = \dim \{ c \sqrt{1 + \frac{pa}{\sqrt{p^2 + m^2}}} \exp(i \sqrt{p^2 + m^2} T - \sqrt{|a| p^2 + m^2 - p}) \} = 0 \quad (31)$$

for $a \leq 1$, because these elements do not belong to $L^2[-\infty, \infty, dp]$. For $a > 1$ the dimensions are different, while the kernel $\mathcal{K}_+$ has dimension zero, the second kernel $\mathcal{K}_-$ has dimension 1.

The self-adjoint evolving constants $\hat{X}(T)$ and $\hat{X}^0(T)$ are associated with the production of events in the same measurement devices. In both cases, we choose to perform a T simultaneous measurement that means that we are opening the detectors synchronised on the T-surface. However, while in the first case the values of the position of the event are registered, in the second what is registered is the proper time of the corresponding measurement devices.

In both cases, the simultaneity surfaces $x^0 + ax = T_0$ coinide, and for $a \leq 1$ they are transversal to the orbits. Thus, as in the non-relativistic case, in order to have a consistent quantum mechanical description one need to choose clock variables leading to equal time surfaces transversal to the orbits. This seems to be a general feature of the evolving constant technique when applied to systems having a natural Hamiltonian evolution.

When both sectors of the relativistic free particle phase space are included at the same time, the system does not have a natural Hamiltonian evolution. As it is well known, the standard Klein Gordon quantization is far from satisfactory, leading to an inner product which is not positive definite. Generically, it is not consistent with a probabilistic interpretation. Here, we would like to analyze whether the description of the evolution in terms of the evolving constants which in principle does not require any Hamiltonian structure could give a new insight to this old problem.

Let $\Gamma$ be the phase space of the system with coordinates $(x^0, x)$ and momenta $(p_0, p)$. Its action is given by

$$S = \int d\tau [p_0 x^0 + p \dot{x} - N(p_0^2 - p^2 - m^2)] \quad (32)$$

that leads to the Hamiltonian constraint

$$\mathcal{C} := p_0^2 - p^2 - m^2 = 0 \quad (33)$$

which defines the constraint surface $\Gamma'$. A complete set of classical perennials is given by $p, q := x + x^0 p/p_0$ and $\epsilon := p_0/\sqrt{p^2 + m^2} = \text{sgn}(p_0)$. Now we choose the auxiliary Hilbert space to be $\mathcal{H}_{aux} = L^2(\mathbb{R}^2, dp_0 dp)$ and represent $x^\mu$ and $p_\mu$ by the usual expressions of the momentum representation, so that the corresponding operators are self-adjoint.

Let us now choose clock variables and the corresponding evolving constants that will be used to describe the evolution, and include them among the elements of $\mathcal{B}^{(s)}_{phys}$.

We will analyse the usual choice $T = x^0$: then $\mathcal{B}^{(s)}_{phys}$ will include the operators $\{\hat{q}, \hat{p}, \hat{\epsilon}, \text{ and } \hat{X}(T) := \hat{q} + \hat{p} T/p_0\}$

The spectrum of $\hat{X}(T)$ will characterise the possible outcomes of a measurement of $\hat{X}$ when $x^0$ takes the value $T$. It is given by the eigenvalue equation:

$$\hat{X}(T)|\psi_{x_1, T}(p, p_0)\rangle = \left[ i \frac{\partial}{\partial p} + \frac{p}{p_0} \frac{\partial}{\partial p_0} - \frac{i}{2} \frac{p}{p_0^2} + \frac{pT}{p_0} \right]|\psi_{x_1, T}(p, p_0)\rangle = x_1|\psi_{x_1, T}(p, p_0)\rangle \quad (34)$$

Notice that, as $[\hat{\epsilon}, \hat{X}(T)] = [\hat{\epsilon}, \hat{p}] = 0$, one may consider simultaneous eigenfunctions of $\hat{X}$ and $\hat{\epsilon}$. The generalized eigenfunctions belonging to the kernel of the constraint equation are,

$$\psi_{x_1, T, x_1}(p, p_0) = \delta_{x_1, \epsilon} \delta(p_0^2 - p^2 - m^2) \sqrt{|p_0|} \exp(-ipx - ip_0 T) \quad (35)$$
where $\epsilon = p_0/\sqrt{p^2 + m^2}$. As in the non-relativistic case, the inner product in $\mathcal{H}_{aux}$ induces [9] an inner product in the physical state space.

$$<\psi|\psi'>_{phys} = \int dpdp_0\delta(p_0^2 - p^2 - m^2)\psi^*(p, p_0)\psi'(p, p_0)$$ (36)

With this inner product, the operators $\hat{X}(T)$ and $\hat{\epsilon}$ are self-adjoint and the corresponding eigenfunctions orthonormal. Notice, however, that with this inner product the expectation value of the energy $<\psi| - p_0|\psi>$ may take negative values. If one is interested in avoiding states with negative energy, one may introduce a modified inner product given by:

$$<\psi|\psi'>_{phys}^M = -\int dpdp_0\delta(p_0^2 - p^2 - m^2)\text{Sg}(p_0)\psi^*(p, p_0)\psi'(p, p_0)$$ (37)

where $-\text{Sg}(p_0)$ is the sign of the energy. Thanks to the commutation of $\hat{X}(T)$ with $\hat{\epsilon}$, the self-adjointness of the evolving constant $\hat{X}$ is preserved, but the new inner product is not positive definite. In fact the resulting formulation coincides with the free particle representation [18] of the Klein Gordon theory. The quantum version of the evolving constant turns out to be automatically self-adjoint and coincides with the Newton-Wigner [19] position operator.

From the relational point of view adopted in this paper, each system is treated as a closed system. It that sense, the relativistic particle is considered as a cosmological system \(^3\), and makes no reference to an order in time, there is no before or after, only events labelled by different values of the parameter. Within this context, the first inner product given by (36) is more natural. It leads to a consistent quantization with a positive definite inner product and self adjoint observable quantities, treats on an equal footing the particles with positive energy, travelling forwards in time, and the particles with negative energy, travelling backwards.

If one wants to include interactions one needs to start from the evolving constants of the system including the interaction. It is well known that the relativistic systems present a highly anomalous behaviour, the so-called Klein paradox, when the system is coupled with an external field with abrupt variations at distances of the order of the Compton wavelength. Let us conclude this section with a brief discussion of the evolving constant formalism applied to a particle interacting with an step-function potential. The Hamiltonian constraint is given by

$$C = (p_0 - \phi(x))^2 - p^2 - m^2$$ (38)

with $\phi(x) = -eU(x - a)$, $U$ is the step function and we will take $c > m$. In this case one can see that the evolving constant $\hat{X}(T)$ does not commute with the sign of the energy characterized by $\hat{\epsilon}$. Thus, even though $\hat{X}$ is self-adjoint with the symmetric inner product that is the extension of (36) to the case of the relativistic particle in an external field, it will be impossible to define a self-adjoint operator with the modified inner product (37) that corresponds, in the free particle case, to the Klein Gordon representation. Thus, from the viewpoint of the evolving constants formalism, the Klein paradox can be associated to the impossibility of defining a self-adjoint operator for the position when step potentials are present. To conclude the analysis of the relativistic particle, the evolving constant procedure allows treating both sectors of the classical phase space at the same time. From a purely relational and therefore cosmological point of view, the theory with the first inner product should be preferred because it admits a probabilistic interpretation, even when the system interacts with external fields. This theory gives a symmetric treatment to states with positive and negative "energy" and should be reinterpreted as the cosmological description of a "Universe" classically described by the relativistic particle. Kuchar [21] have shown that the relativistic particle moving in a Riemannian spacetime presents a multiple choice problem: There are different admissible time functions leading to incompatible quantum mechanical schemes. We are now studying these systems making use of the point of view advocated in this paper.

**V. THE SL(2,R) MODEL WITH TWO HAMILTONIANS**

Up to now, we have analyzed the relational formalism in systems with a single Hamiltonian constraint and one-dimensional orbits. The SL(2,R) model [5] is a constrained system with two Hamiltonian constraints that have non-vanishing Poisson bracket with each other. The model has three constraints and their algebra mimics the structure

\(^3\)A similar cosmological interpretation of the relativistic particle was proposed by Marolf [20]
of the constraint algebra of general relativity. Although a complete set of evolving constants associated with the
dynamical variables of the systems has been identified [6], it has not been possible, up to now, to identify clock
variables that allow to promote this quantities to well defined self-adjoint operators. We shall see that algebraic
quantization and reduced phase space quantization lead, in this case, to different quantum theories. The second
procedure will allow us to introduce a consistent time structure and to promote the evolving constants to self-adjoint
operators.

A. Classical dynamics.

Let us start with the classical action of the model.

$$S = \int d\tau [p.\dot{u} + \pi.\dot{v} - NH_1 - MH_2 - \lambda D]$$

(39)

Where

$$H_1 = \frac{1}{2}(p^2 - v^2), \quad H_2 = \frac{1}{2}(\pi^2 - u^2), \quad D = u.p - v.\pi;$$

(40)

the dynamical variables $u = (u^1, u^2)$ and $v = (v^1, v^2)$ are two-dimensional real vectors; $N$, $M$ and $\lambda$
are Lagrange multipliers, and $u^2 = u.u = (u^1)^2 + (u^2)^2$. The momenta conjugate to $u$ and $v$ are $p$ and $\pi$ and the
dynamics of the system is completely given by the constraints $H_1$, $H_2$ and $D$. This is a totally constrained system with
three first-class constraints and a single degree of freedom. The Poisson algebra of the constraints is

$$\{H_1, H_2\} = D \quad \{H_1, D\} = -2H_1 \quad \{H_2, D\} = 2H_2$$

(41)

which is the algebra $sl(2, R)$. We now consider the observables. The six functions [5]

$$O_{12} := u^1 p^2 - u^2 p^1, \quad O_{23} := u^2 v^1 - p^2 \pi^1,$$

$$O_{13} := u^1 v^1 - p^1 \pi^1, \quad O_{24} := u^2 v^2 - p^2 \pi^2,$$

$$O_{14} := u^1 v^2 - p^1 \pi^2, \quad O_{34} := v^2 \pi^1 - v^1 \pi^2,$$

(42)

commute with the constraints and form a closed Poisson bracket algebra $o(2, 2)$. This algebra is isomorphic to the
Lie algebra $sl(2, R) \times sl(2, R)$. The combinations

$$\tau^\eta_0 := \frac{1}{2}(O_{12} - \eta O_{34}),$$

$$\tau^\eta_1 := \frac{1}{2}(O_{13} - \eta O_{24}),$$

$$\tau^\eta_2 := \frac{1}{2}(O_{23} - \eta O_{14}).$$

(43)

with $\eta \in \{1, -1\}$, are the elements of the basis adapted to the algebra $sl(2, R) \times sl(2, R)$ with Poisson brackets

$$\{\tau^\eta_1, \tau^{\eta'}_2\} = -\delta^{\eta, \eta'} \tau^\eta_0,$$

$$\{\tau^{\eta}_2, \tau^{\eta'}_0\} = \delta^{\eta, \eta'} \tau^\eta_1,$$

$$\{\tau^{\eta}_0, \tau^{\eta'}_1\} = \delta^{\eta, \eta'} \tau^{\eta'}_2.$$

(44)

Since the physical phase space is two-dimensional and the system has one degree of freedom, there are at most two
independent continuous observables. In fact, one can show that [5] the reduced physical phase space has the topology
of four cones connected at their vertices and all the observables evaluated at regular points of each of the cones can
be parametrized [22] by

$$\tau^\eta_0 = \frac{1}{2} \epsilon_1 (1 + \eta \epsilon_2)r,$$

$$\tau^\eta_1 = \frac{1}{2} (1 + \eta \epsilon_2) r \cos \phi,$$

$$\tau^\eta_2 = -\frac{1}{2} \epsilon_1 (1 + \eta \epsilon_2) r \sin \phi.$$

(45)
where \( r \) is a positive real parameter, \( \phi \) is an angle, and \( \epsilon_1, \epsilon_2 = \pm 1 \) are two discrete quantities. The Poisson brackets between \( r \) and \( \phi \) in the reduced phase space reads
\[
\{r, \phi\} = 1
\]  
(46)
while \( \epsilon_1 \) and \( \epsilon_2 \) commute with everything. The functions on the phase space \( \tau^{ij} \) satisfy for each \( \eta \) the identity
\[
-(\tau^{00})^2 + (\tau^{11})^2 + (\tau^{22})^2 = H_1 H_2 + \frac{1}{4} D^2
\]  
(47)
which vanishes on the physical phase space. The evolving constants may be easily determined from the following identity among the six observables \( O_{ij} \) and the Lagrangian variables \( \vec{a}, \vec{v} \)
\[
u^a(\tau)\nu^b(\tau)\epsilon_{abc}(\nu^c(\tau)\nu^d(\tau) - p^c(\tau)p^d(\tau)) = O_{12} O_{34}
\]  
(48)
where \( \epsilon_{ab} \) are the Levi Civita symbols. By noticing that each component of \( u^c v^d - p^c p^d \) corresponds to one observable, and making use of parameterisation (45) this relation takes the form
\[
(\epsilon_2 u^1(\tau)v^1(\tau) - u^2(\tau)v^2(\tau))\cos \phi - (u^1(\tau)v^2(\tau) + \epsilon_2 u^2(\tau)v^1(\tau))\epsilon_1 \sin \phi = r
\]  
(49)
The Lagrangian variables obey this relation at any time. From here one can define an evolving constant \( U^1 \) that takes the value \( u^1 \) when \( u^2 \) and \( v^2 \) have assigned values: \( u^2 = x, v^1 = y, v^2 = z \). It is given by
\[
U^1 = \frac{x(z \cos \phi + \epsilon_2 \epsilon_1 y \sin \phi) + \epsilon_2 y \cos \phi - \epsilon_1 x \sin \phi}{\epsilon_2 y \cos \phi - \epsilon_1 x \sin \phi}
\]  
(50)
By considering the gauge orbits of Eq(49) generated by the constraints \( H_1, H_2 \) and \( D \) one can obtain [6] the remaining evolving constants.
\[
P_1 = \epsilon_2 y \sin \phi + \epsilon_1 z \cos \phi
\]  
(51)
\[
P_2 = -z \sin \phi + \epsilon_1 \epsilon_2 y \cos \phi
\]  
(52)
\[
\Pi_1 = \frac{xy + \epsilon_1 \epsilon_2 z r \sin \phi}{\epsilon_1 \epsilon_2 y \cos \phi - z \sin \phi}
\]  
(53)
\[
\Pi_2 = \frac{xz + r \cos \phi}{\epsilon_1 \epsilon_2 y \cos \phi - z \sin \phi}
\]  
(54)
Thus, the relational description of this system involves three independent variables \( x, y, z \). Notice that not all the classical evolving constants are independent. In fact, given \( U^1 \) and its conjugate momentum \( P_1 \), the other quantities are determined up to a sign factor by the constraints. In what follows, we shall discuss if, at least for some choice of clock variables, the evolving constants may be promoted to self-adjoint operators on some Hilbert space leading to a consistent description of the quantum evolution.

**B. Algebraic quantization.**

The SL(2,R) model has been previously quantized by following the algebraic quantization approach. However, it has not been possible up to now to give a satisfactory description of the quantum evolution. We shall briefly describe this procedure following closely the analysis of [5,22] and we shall explore whether the evolving constants may be included in the algebra of observables \( B^{(\ast)}_{\text{phys}} \).

One works in a "coordinate representation" with wavefunctions \( \psi(\vec{u}, \vec{v}) \), and considers the \( \ast \)-algebra of physical observables \( B^{(\ast)}_{\text{phys}} \) generated by the real operators \( \hat{\tau}^\eta_i \). If one defines \( \hat{\tau}^\eta_{\pm} := \hat{\tau}^\eta_0 \pm i \hat{\tau}^\eta_1 \) they satisfy the following commutation relations
\[
[\hat{\tau}^\eta_0, \hat{\tau}^\eta_{\pm}] = \pm \delta^{\eta, \eta'} \hat{\tau}^\eta_{\pm},
\]  
(55)
\[
[\hat{\tau}^\eta_0, \hat{\tau}^\eta_{\pm}] = -2 \delta^{\eta, \eta'} \hat{\tau}^\eta_0
\]  
(56)
One can solve the quantum constraints by separation of variables [5]. For this purpose, it is convenient to introduce polar coordinates defined by \( u^1 + i u^2 = u e^{i \alpha}, \quad v^1 + v^2 = v e^{i \beta} \). One finds the set of smooth solutions given by:
\[ \psi_{m,\epsilon} := e^{im(\alpha + i\beta)}J_m(uv) \]

where \( m \) is an integer, \( \epsilon \in \{-1,1\} \), and \( J_m \) is the Bessel function of the first kind. Notice that the functions \( \psi_{m,\epsilon} \) are linearly independent with the exception that \( \psi_{0,+} = \psi_{0,-} \). One can represent the action of the generators of the \( \ast \)-algebra on the linear space spanned by \( \psi_{m,\epsilon} \) by

\[
\begin{align*}
\hat{\tau}_0^n \psi_{m,\epsilon} &= \delta^{n,\epsilon} m \psi_{m,\epsilon} \\
\hat{\tau}_\pm^n \psi_{m,\epsilon} &= \delta^{n,\epsilon} m \psi_{m \pm 1,\epsilon}
\end{align*}
\]

The \( \ast \)-relations induce the adjoint operations

\[
\begin{align*}
(\hat{\tau}_0^n)^\dagger &= \hat{\tau}_{-n}^0 \\
(\hat{\tau}_\pm^n)^\dagger &= \hat{\tau}_{\mp n}^n
\end{align*}
\]

Now one can define an inner product among the elements of this linear space. The eigenfunctions \( \psi_{m,\epsilon} \) of the self-adjoint operator \( \hat{\tau}_0^n \) should be orthogonal, and taking into account equations (58) and (59) one gets a recurrent relation between \( (\psi_m, \psi_m) \) and \( (\psi_{m\pm 1}, \psi_{m\pm 1}) \) that allows one to determine the inner product up to a positive constant \( a \)

\[ (\psi_m, \psi_{m'}) = a|m|\delta_{m,m'}. \]

Notice that the states \( \psi_{0,\epsilon} \) have vanishing norm. As this state is annihilated by every operator in \( B_{\text{phys}}^{(s)} \), it can be excluded from the physical space. One can thus define four linear spaces \( V_{\epsilon_1,\epsilon_2} := \text{span}\{ \psi_{m,\epsilon_2} : \epsilon_1 m > 0 \} \). Each of them carries an irreducible representation of \( B_{\text{phys}}^{(s)} \). The completion of these spaces yields the four Hilbert spaces \( \mathcal{H}_{\epsilon_1,\epsilon_2} \). Finally, notice that the Casimir operators \( (\hat{\tau}_0^n)^2 = -(\hat{\tau}_0^n)^2 + (\hat{\tau}_1^n)^2 + (\hat{\tau}_2^n)^2 \) vanish on each Hilbert space \( \mathcal{H}_{\epsilon_1,\epsilon_2} \) and therefore, the quantum theory preserves the classical identity (47).

Now, if one attempts to analyse the problem of the relational description of the evolution in the context of the algebraic quantization on \( \mathcal{H}_{\epsilon_1,\epsilon_2} \) one finds obstacles from the outset. For that analysis, we would like to include among the elements of the algebra of perennials at least some of the evolving constants. One should include in \( B_{\text{phys}}^{(s)} \) the independent evolving constants \( U^1 \) and \( P_1 \). Recall that at the classical level they allow us to determine the other evolving constants on the constraint surface. One can easily check that \( P_1 \) and \( U^1 \) are not well defined self-adjoint operators on \( \mathcal{H}_{\epsilon_1,\epsilon_2} \). In fact, now the troublesome vectors \( \psi_{0,\epsilon} \) of vanishing norm are not annihilated by the new elements of the algebra and therefore they cannot be dropped at the outset. Furthermore, if one attempts to restrict the action of these operators to one of these sectors, the inverse of \( P_2 \) appearing in the definition of \( U^1 \) is not defined on any sector of \( \mathcal{H}_{\epsilon_1,\epsilon_2} \). Therefore, the quantum theory resulting from the algebraic quantization does not seem to admit a natural time structure induced by the evolving constant formalism.

C. Reduced phase space quantization and relational evolution of the SL(2,R) model.

We shall see in this subsection that contrary to what we have observed previously, it is possible to find in this case a consistent description of the time evolution. It will be convenient to begin by introducing a reparameterisation of the reduced phase space of the model. We start from Eq.(45) and define \( J = \epsilon_1 \tau \) and \( \varphi = -\epsilon_1 \phi + \frac{11 + \epsilon_2}{4} \pi \). This parameterisation allows us to rewrite the generators of the \( sl(2,R) \) algebra as follows:

\[
\begin{align*}
\tau_0^n &= \frac{1}{2}(1 + \eta \epsilon_2)J, \\
\tau_1^n &= -\frac{1}{2}(1 + \eta \epsilon_2)J \cos \varphi, \\
\tau_2^n &= -\frac{1}{2}(1 + \eta \epsilon_2)J \sin \varphi.
\end{align*}
\]

The Poisson brackets between \( J \) and \( \varphi \) in the reduced phase space read

\[ \{ \varphi, J \} = 1 \]

This new parametrization allows us to completely absorb the \( \epsilon_1 \) factors and therefore, for each value of \( \epsilon_2 \), describes simultaneously two opposite cones and the common tip of the reduced phase space [5]. One can easily check that
the complete set of evolving constants (50) and (54) may be written without any reference to \( \epsilon_1 \). Thus, within this parameterisation, the natural subsets of the reduced phase space are not smooth manifolds. This will have important consequences at the quantum level. We are interested in finding a quantum representation, including in the fundamental *-algebra of observables—besides the generators of the \( \text{sl}(2, \mathbb{R}) \) algebra—the evolving constants \( U_1(x, y, z) \) and \( P_1(y, z) \), at least for some choice of clock variable. When written in terms of the new parameters, the evolving constants take the form:

\[
U^1 = \frac{x(z \cos \varphi - \epsilon_2 y \sin \varphi) - J}{\epsilon_2 y \cos \varphi + z \sin \varphi}
\]

\[
P_1 = \epsilon_2 y \sin \varphi - z \cos \varphi
\]

These quantities take a more compact form by considering the following change of the independent coordinates \( y = \epsilon_2 \rho \cos \varphi_0 \) and \( z = -\rho \sin \varphi_0 \)

\[
P_1 = \rho \sin(\varphi + \varphi_0)
\]

\[
U^1 = \frac{-x \rho \sin(\varphi + \varphi_0) + J}{\rho \cos(\varphi + \varphi_0)}
\]

Now we are ready to quantize this system. We consider a linear space of functions \( \psi(\varphi) \). We then promote the Poisson bracket relation (62) to a commutator by taking \( \hat{\varphi} \) as a multiplicative operator and \( \hat{J} = -i \hbar \) and define the quantum generators of the \( \text{sl}(2, \mathbb{R}) \) algebra by making use of the parametrization (45) as:

\[
\hat{\tau}_0^n = \frac{1}{2}(1 + \eta \epsilon_2) \hat{J}
\]

\[
\hat{\tau}_\pm^n = -\frac{1}{4}(1 + \eta \epsilon_2)[\hat{J} e^{\pm i \varphi} + e^{\pm i \varphi} \hat{J}]
\]

One can immediately check that they satisfy the commutation relations (55) and (56) and therefore the \( \text{sl}(2, \mathbb{R}) \) algebra. Since we are working directly on the reduced phase space of perennials we may wonder if the quantum theory preserves the classical identity (47). To answer this question we compute the Casimir operators. One gets

\[
(\tau^n)^2 = - (\hat{\tau}^n_0)^2 + \frac{1}{2} (\hat{\tau}^n_+ \hat{\tau}^n_- + \hat{\tau}^n_- \hat{\tau}^n_+) = \frac{1}{8}(1 + \eta \epsilon_2) \hbar^2,
\]

Thus, we see that the quantization of the reduced phase space, given by the singular manifold composed by two opposite cones with a common tip, introduces a quantum anomaly and the Casimir invariant only vanishes in the limit \( \hbar \to 0 \).

As we have already mentioned, we are interested in including among the algebra of observables the evolving constants \( \hat{P}_1 \) and \( \hat{U}^1 \) on the linear space of functions of \( \varphi \). The first operator has a trivial, multiplicative action on this space, while the natural candidate for the second is given by the symmetric combination:

\[
\hat{U}^1 \psi(\varphi) = -\left\{ \frac{1}{2} \hat{J} \frac{1}{\rho \cos(\varphi + \varphi_0)} + \frac{1}{\rho \cos(\varphi + \varphi_0)} \hat{J} \right\} + x \tan(\varphi + \varphi_0) \}
\]

In order to introduce a time structure in the system, one needs to analyze several questions. Firstly, one needs to find an inner product such that the *-algebra be preserved. In particular, one needs to ensure that the real evolving constants turn out to be, at the quantum level, self-adjoint operators on the corresponding Hilbert space. Secondly, one needs to analyze the issue of the clock variables in the case of multidimensional orbits. Is it possible to consider all the independent coordinates \( x, \rho, \varphi \) as time variables, or do some of them play a different role? If there is a time structure, is that structure again related with the existence of surfaces transversal to the orbits or may this condition be somewhat relaxed in this case?

Before being more specific with the inner product, let us give a first glance to the issue of the clock variables. We first notice that, being \( \hat{P}_1 \) a multiplicative operator, its corresponding eigenfunctions will have the form \( \psi_{\varphi_1}(\varphi) = \delta(\varphi - \varphi_1) \), and therefore they do not depend on the independent coordinates that label the evolution along the orbits. In what concerns the evolving constant \( \hat{U}^1 \), its eigenvalue equation is

\[
\hat{U}^1 \psi_{u^1}(\varphi) = u^1 \psi_{u^1}(\varphi)
\]

with solutions
\[ \psi_{u_1}(\varphi) = A \sqrt{|\rho \cos(\varphi + \varphi_0)|} \exp -i [u_1^1 \rho \sin(\varphi + \varphi_0) + x \rho \cos(\varphi + \varphi_0)] \]  

(71)

Since we have not introduced an inner product, we do not know which of these solutions appear in the spectral decomposition of \( \hat{U}^1 \). However, notice that \( T \) seems to play the role of a time variable, in the sense that the transition amplitudes \( < \psi_u | \psi_v > \), computed with any sesquilinear inner product, will be "time" independent. If that is the case, then \( \cos(\varphi + \varphi_0) \) will play the role of the "energy" of the system.

One can easily check that the standard inner product of square integrable functions of \( \varphi \) in the interval \([-\pi, \pi]\) does not lead to a self-adjoint evolving constant \( \hat{U}^1 \). However, if one divides the interval in two, and considers the Hilbert space \( \mathcal{H}_{\varphi_0} \) of periodic functions in the interval \(-\pi/2 \leq \varphi + \varphi_0 < \pi/2\) with inner product

\[ < \psi | \psi' > = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} d\varphi \psi^*(\varphi) \psi'(\varphi) \]  

(72)

which corresponds to the sector of positive "energy", the elements of the quantum algebra \( \mathfrak{B}_{\text{phys}}^{(s)} \) are self-adjoint on \( \mathcal{H}_{\varphi_0} \). In order to prove that the generators of the \( \mathfrak{sl}(2, \mathbb{R}) \) algebra \( \tilde{\tau}^u \) are self-adjoint, it is enough to take into account Eq.(61) and to show that \( \hat{J} \) is self-adjoint. This results from the fact that \( \hat{J} \) admits the following spectral decomposition in \( \mathcal{H}_{\varphi_0} \)

\[ < \varphi | \hat{J} | \varphi' > = \sum_p 2p \pi e^{i2p(\varphi - \varphi')} = -i \frac{\partial}{\partial \varphi} \delta_\pi(\varphi - \varphi') \]  

(73)

where \( \delta_\pi \) is the periodic Dirac delta. In what concerns the evolving constants, \( \hat{P}_1(\rho, \varphi_0) \) is trivially self-adjoint while \( \hat{U}^1(x, \rho, \varphi_0) \) has the following set of orthonormal eigenvectors:

\[ < \varphi | n, x \rho > = \sqrt{\pi/2} \sqrt{|\cos(\varphi + \varphi_0)|} \exp i[n \pi \sin(\varphi + \varphi_0) + x \rho \cos(\varphi + \varphi_0)], \]  

(74)

corresponding to the eigenvalues \( u^1_n = \frac{2n \pi}{\rho} \). The solutions of Eq.(70) for the other values of \( u^1 \) do not belong to the domain of the symmetric operator \( \hat{U}^1 \). Notice that, while the operator \( \hat{U}^1 \) has a continuum spectrum on the kinematical space of functions \( \psi(u, v) \) its counterpart has a discrete spectrum on the reduced physical space. This eigenfunctions define a complete basis, in fact

\[ \delta_\pi(\varphi - \varphi') = \frac{1}{\pi} \sum_n < \varphi | n, x \rho > < n, x \rho | \varphi' >, \]  

(75)

and therefore, \( \hat{U}^1 \) is self-adjoint in \( \mathcal{H}_{\varphi_0} \) with spectral decomposition

\[ < \varphi' | \hat{U}^1 | \varphi > = \sum_n \frac{n \pi^2}{2 \rho} \sqrt{\cos \varphi' \cos \varphi} e^{in \pi (\sin \varphi - \sin \varphi')} \]  

(76)

Thus, the reduced phase space quantization leads to a satisfactory description of the time evolution of the \( \mathfrak{sl}(2, \mathbb{R}) \) model. The variable \( T = x \rho \) plays the role of a clock variable. The inner product between eigenstates of the evolving constant \( \hat{U}^1 \) is time independent, that is \( < n, T | n', T > = < n, T' | n', T' > \) which corresponds to a unitary evolution. Furthermore, it is immediate to check from the definition of the inner product (72), that the transition probabilities \( < n, T | n', T' > \) are independent of \( \varphi_0 \). In other words, all the Hilbert spaces \( \mathcal{H}_{\varphi_0} \) give equivalent descriptions of the evolution. Also notice that the transition amplitudes are independent of \( \rho \), the original dependence on these parameter of the eigenfunctions of \( \hat{U}^1 \) given by Eq.(71) was absorbed in the normalization and the only remnant dependence on this parameter appears in the spectrum of the evolving constant. This ambiguity in the spectrum may be easily understood by noticing that by considering the partial gauge fixing \( v^1 = v^1_0, v^2 = v^2_0 \) the original model may be interpreted as a constrained system with Hamiltonian constraint \( (p_1)^2 + (p_2)^2 = \rho^2 \). The parameter \( \rho \) plays the role of the mass of this constraint system. Different mass values obviously correspond to different representations.

---

4Recall that \( A \) is said symmetric if, its domain \( D_A \) is dense in \( \mathcal{H} \) and for \( f, g \in D_A \), \( (Af, g) = (f, Ag) \).

5Notice that after a partial gauge fixing, the \( \mathfrak{sl}(2\mathbb{R}) \) model takes the form of a Barbour Bertotti [23] model. Our description of the quantum evolution naturally includes these models.
of the \(*\)-algebra of observables. Notice that the equal time surfaces are not transversal to the orbits. By using the classical form of the evolving constants in terms of \(\phi\) and \(J\) one can see that the orbit corresponding to the value of \(\varphi = -\frac{\pi}{2} - \varphi_0\) lays completely on the equal time surface \(T = J\). Thus, it is possible to get a satisfactory description of the "time" evolution of the SL(2R) model in spite of the fact that the equal time surfaces are not globally transversal to the orbits. Notice, however, that although the equal-time surfaces contain causally connected points, for the resulting quantum theory the probability amplitudes \(<\pm \frac{\pi}{2} - \varphi_0|n, T>\) vanish for any value of \(n\). If one excludes these pathological orbits, all the remaining orbits are transversal to the equal time surfaces.

The region of the reduced phase space described by a particular choice of \(\varphi_0\) corresponds to \(J \in [-\infty, \infty]; \varphi + \varphi_0 \in [-\pi/2, \pi/2]\) and covers one half of the two opposite cones, for each value of \(\epsilon_2\). The dynamics of the other half may be easily included. It is enough to consider wave functions with two components \(\psi_{a}(\varphi) a = 1, 2\) with \(\varphi + \varphi_0 \in [-\pi/2, \pi/2]\) and inner product

\[
<\psi|\psi' > = \frac{1}{\pi} \int_{\varphi_0}^{\varphi_0 + \pi} d\varphi \sum_{a} \psi^*_a(\varphi)\psi'_a(\varphi). \tag{77}
\]

The first component corresponds to the sector with positive energy, while the second corresponds to the negative energy sector. The generators of the \(sl(2R)\) algebra of observables now take the form \(\hat{\tau}^0 \otimes I\), \(\hat{\tau}^1 \otimes I\) where the \(\hat{\tau}\)s are given by Eq.(67) and \(I\) is the identity in the 2-dimensional space. Finally the evolving constants are also diagonal with \(P_{22}(\varphi, \rho, \varphi_0) = P_{11}(\varphi + \pi, \rho, \varphi_0)\) and \(U^{112}(J, x, \rho, \varphi_0) = U^{111}(J, \varphi + \pi, x, \rho, \varphi_0)\).

To conclude, we have shown that it is possible to introduce a consistent notion of quantum evolution in a system with more than one Hamiltonian constraint. Dirac’s quantization and reduced phase space quantization lead in this case to inequivalent quantum theories. Only the last procedure allowed us to introduce a time structure. However, one cannot discard the possibility of finding a consistent notion of evolution within the context of the Dirac’s quantization.

VI. A MODEL WITH A COMPACT PRESYMPLECTIC PHASE SPACE

In this section we are going to consider a system with a compact constraint surface in phase space. This system does not contain any \(\Sigma \times R\) structure, and therefore it does not correspond to a Hamiltonian system. It was first analysed by Rovelli [14]. He concluded that although in a certain regime, which stands for certain states and a certain range of measurements, the system behaves as a standard quantum mechanical system, while outside this regime unitarity is lost. The lessons learnt in this paper will allow us to introduce a time structure that will preserve the unitarity of the evolution and the standard probabilistic interpretation.

The model is given by a phase space with canonical coordinates \(q_1, q_2, p_1, p_2\) and canonical commutation relations \(\{q_i, p_j\} = \delta_{ij}\). The constraint is

\[
C = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) - M. \tag{78}
\]

The equations of motion may be easily integrated leading to the parametric equations for the orbits

\[
q_1 = \sqrt{2A}\sin(\tau) \\
q_2 = \sqrt{2M - 2A}\sin(\tau + \phi) \tag{79}
\]

The orbits are labelled by the integration constants

\[
0 \leq A \leq M \quad a \leq \phi \leq 2\pi \tag{80}
\]

For \(\pi \leq \phi \leq 2\pi\) one has the same orbits as for \(0 \leq \phi \leq \pi\), but followed with the opposite orientation. Geometrically, the orbits are ellipses that are inscribed in a rectangle with a diagonal equal to \(M\). The observables \(A\) and \(\phi\) form a complete set and the topology of the reduced phase space has the topology of a sphere. The explicit form of the observables in terms of the original phase space variables is

\[
A = \frac{1}{4}(2M + p_1^2 - p_2^2 + q_1^2 - q_2^2) \\
\tan \phi = \frac{p_1 q_2 - p_2 q_1}{p_1 p_2 + q_1 q_2} \tag{81}
\]

The relational evolution of some of the coordinates in terms of the others may be easily determined from the following identity among the two observables and the Lagrangian variables \(\dot{q}\)
where $\mu = \sqrt{A/M - A}$. If we simply try to describe the evolution of one of the coordinates, say $q_2$ in terms of $q_1$ that will play the role of clock variable, one runs into problems [14]. In fact,

$$q_2(q_1) = \sqrt{M/A - 1}\cos \phi q_1 \pm \sin \phi (2A - q_1^2)^{1/2}$$

and we see that due to the square root, the family of orbits going through a certain equal time surface will depend on $q_1 = T$. As $q_1$ grows, less orbits reach the "equal-time" surface. At the quantum level this implies that, the evolving constant cannot be self-adjoint operators, and the evolution will not be unitary.

We shall see that there are, however, choices of clock variables and evolving constants, leading to self-adjoint observables and unitary evolutions. The main observation is that, as for the model analysed in the previous section, it is possible to define clock variables such that the equal time surfaces, up to isolated pathological orbits, are transversal to the orbits. We shall call this kind of surfaces locally transversal. To do that, we define new configuration variables $r$ and $\alpha$ such that

$$q_1 = r \sin \alpha$$

and

$$q_2 = r \cos \alpha$$

which are related by

$$L_x = \frac{1}{2}(p_1q_2 + q_1p_2),$$

$$L_y = \frac{1}{2}(p_1q_2 - q_1p_2),$$

$$L_z = \frac{1}{2}(p_1^2 - p_2^2 + q_1^2 - q_2^2),$$

which are related by

$$L_x^2 + L_y^2 + L_z^2 = \frac{M^2}{4}$$

In terms of the original variables $A$ and $\phi$, they take the following form

$$L_x = \sqrt{A(M - A)} \cos \phi,$$

$$L_y = \sqrt{A(M - A)} \sin \phi,$$

$$L_z = A - \frac{M}{2}$$

and obey the angular momentum Poisson algebra.

We are interested in the quantization of the *-algebra of observables generated by the observables $\hat{L}$ and the evolving constant $R(T)$. The quantization of the observables $\hat{L}$ is straightforward, using the standard representations of the $su(2)$ algebra. It follows that

$$\hat{L}^2 = \frac{M^2}{4} = j(j + 1)$$

where $j$ is integer or half-integer, and therefore the classical limit is preserved. We shall see in what follows that only the half-integer values of $j$ are admissible. The corresponding Hilbert space will have a standard basis $|m>\rangle$ of eigenvectors of $L_z$, with $-j \leq m \leq j$. In order to have a well defined description of the evolution at the quantum level, we need to promote $R(T)$ to a self-adjoint operator. As we are working on a finite dimensional Hilbert space, it is enough to show that $R(T)$ is a well defined symmetric operator. Thus, we need to show that the pathological orbits $A = 0$ and $\sin \phi = 0$ have vanishing probabilities.

Notice that
\[
\hat{A}|m > (M/2 + m)|m > = (\sqrt{j(j + 1)} + m)|m > \neq 0 \quad (89)
\]

for all \( m \) such that \(-j \leq m \leq j\), and therefore \( A = 0 \) is not an eigenvalue of \( \hat{A} \). Concerning the possible values of \( \sin \phi \), we notice that it takes the following form, in terms the \( \hat{L} \) operators.

\[
\sin \phi = (M^2/4 - \hat{L}_z^2)^{-1/4} \hat{L}_x (M^2/4 - \hat{L}_z^2)^{-1/4} \quad (90)
\]

This is a self-adjoint operator which for half-integer \( j \) has a well defined inverse. In fact, in the half-integer representations \( \hat{L}_x \) does not have vanishing eigenvalues and is invertible. Therefore, zero is not included among the possible eigenvalues of \( \sin \phi \). Thus, this quantum description assigns zero probability to the pathological orbits. From here, it is not difficult to prove that the symmetric form of the evolving constants

\[
\hat{R}(T) = [(\hat{L}_z + M/2) \cos \alpha + \hat{L}_x \sin \alpha]^2 + \hat{L}_y^2 \sin^2 \alpha |^{-1/4} (\hat{L}_z + M)^{1/4} \sqrt{\hat{L}_y^2}
\]

\[
\times (2\hat{L}_z + M)^{1/4} [(\hat{L}_z + M/2) \cos \alpha + \hat{L}_x \sin \alpha]^2 + \hat{L}_y^2 \sin^2 \alpha |^{-1/4} \quad (91)
\]

\[
\hat{R}(T) = [(\hat{L}_z + M/2) \cos \alpha + \hat{L}_x \sin \alpha]^2 + \hat{L}_y^2 \sin^2 \alpha |^{-1/4} (\hat{L}_z + M)^{1/4} \sqrt{\hat{L}_y^2}
\]

\[
\times (2\hat{L}_z + M)^{1/4} [(\hat{L}_z + M/2) \cos \alpha + \hat{L}_x \sin \alpha]^2 + \hat{L}_y^2 \sin^2 \alpha |^{-1/4} \quad (92)
\]

is a positive definite self-adjoint operator. To see this, it is enough to check that the negative powers are well defined because they involve positive definite self-adjoint operators. Thus, if we label the eigenvalues of \( R(T) \) by \( r_n(T) \) and the eigenvectors by \( |n, T > \) they form an orthonormal basis of the \( 2j + 1 \) dimensional Hilbert space, such that \( < n, T | m, T > = \delta_{n,m} \) for any \( T \). The eigenvectors corresponding to different \( T \) are related by the unitary transformation \( U(n, T; n', T') = < n, T | n', T' > \) and one recovers the standard probabilistic interpretation of the quantum theory. The only peculiarity of this model lays in the fact that the eigenvalues of \( R(T) \) are "time dependent". If one uses the unitary transformation to go from the Heisenberg representation to the Schroedinger representation, one gets a time dependent Schroedinger operator.

### VII. CONCLUSIONS

In this paper, we have addressed the issue of the relational description of the evolution of generally covariant systems. We have shown that Rovelli’s evolving constants are useful tools for the analysis of the evolution in terms of relational time variables. Consistency with quantum mechanics and its probabilistic interpretation puts stringent constraints to the admissible choices of clock variables. The description in terms of evolving observables allows us to compare the descriptions arising from different time choices simply by introducing in the fundamental \( \ast \)-algebra the corresponding evolving observables. Generically, each of these choices corresponds to non-commuting observables and different experimental setups. Therefore, the probability amplitudes assigned to the same event by two different choices will be, in general, also different. This was already noticed by Hartle [4], who concluded that the probabilities of a history may differ due to the time choice.

We were able to give consistent quantum mechanical descriptions for the evolution of systems with a compact reduced phase space and for systems with more than one Hamiltonian constraint. Although the orbits of these kind of systems depend on several parameters, it was possible to describe them with a single clock variable. Some of these systems do not admit equal time surfaces globally transversal to the orbits. In fact, we have considered systems such that most of the orbits are transversal, but they have some exceptional isolated orbits laying completely on an equal-time surface. We have shown that such systems may be consistently quantized. The resulting quantum theory assigns a vanishing probability to these pathological orbits.

To what extent can we generalize these conclusions to more realistic relational systems? The models we have considered are certainly very simple but certain tentative conclusions can be drawn. The fundamental limitation to the choices of clock variables seems to be the causal independence of the points laying on an equal time surface. One can imagine two kinds of extensions of this analysis compatible with this restriction. In first place one may have systems in which for any choice of the clock variable \( Q \), the number of orbits reaching the equal time surface \( Q = T \) varies with \( T \). For certain orbits, the question "where is the dynamical variable \( O \) when the clock variable takes the value \( T \)?" has no answer, because \( Q \) never reaches the value \( T \) in that orbit. In other words, in this kind of systems only a "time dependent" portion of the reduced phase space is covered at each time \( T \). At the quantum level, these systems lead naturally to evolving Hilbert spaces and the computation of transition amplitudes between eigenvectors

\[\text{As it was noticed by Rovelli [24], the choice of time variable } T = q_1 \text{ in the model considered in the previous section leads to a system of this type.}\]
of the evolving observables at different values of the time variable is problematic. One can also extend this analysis
to systems that do not admit a global clock variable. In this case, one is able to find equal time surfaces that are
locally transversal to the orbits, only for certain range of the time variable that does not cover the entire evolution of
the system. In this case, one needs more than one clock variable to cover the complete evolution of the system. We
are now studying models having these two kinds of behavior.

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