Dirac Decomposition of Wheeler-DeWitt Equation on the Bianchi Class A Models

Hidetomo Yamazaki *

Department of Physics, Kyoto Sangyo University, Kyoto 603-8555, Japan

Abstract

The Wheeler-DeWitt equation for the Bianchi Class A cosmological models is expressed generally in terms of the second-order differential equation like the Klein-Gordon equation. To obtain the positive-definite probability density, a new method extending the Dirac-Square-Root formalism, which factorizes the Wheeler-DeWitt equation into the first-order differential equation using the Pauli matrices, is investigated. The solutions to the Dirac type equation thus obtained are expressed in terms of two-component spinor form. The probability density defined by the solution is positive-definite and there is a conserved current. The newly found spin-like degree of freedom causes the universe to go through an early quantum stage of evolution with agitated anisotropy-oscillation like Zitterbewegung.

1 Introduction

Arnowitt-Deser-Misner [1](ADM) reformulated the general relativity using the canonical formalism, which makes a 3+1 split of the space-time metric, and introduced some constraints through the variational principle. According to the ADM approach, the Einstein equation is derived as the equation of dynamical evolution. The canonical formalism by the ADM approach is a system of dynamics including two constraints: one is the momentum constraint and the other is the Hamiltonian constraint. The latter governs the time evolution of the system. According to Dirac [2] the procedure of canonical quantization replaces the Hamiltonian constraint with the supplementary condition on the wave function which represents the quantum state of space-time. The Hamiltonian constraint thus becomes the Wheeler-DeWitt equation, which is fundamental to the quantum gravity and determines the quantum state of space-time. The Wheeler-DeWitt equations for specific cosmological models like the Friedmann-Robertson-Walker model [3] or the Bianchi models [4, 5, 6] have been investigated.

The Wheeler-DeWitt equation for the Bianchi models is in general reduced to the second-order differential equation if the operator-ordering is neglected. The resultant Wheeler-DeWitt equation is similar to the Klein-Gordon type, which has a problem that the probability density is not positive-definite. The Wheeler-DeWitt equation has also often this kind of problem that the probability density becomes negative. The negative values of probability density have been shown by Furusawa [7] through numerical calculation, who treated the Wheeler-DeWitt equation for the quantum mixmaster model using wave-packet solution.

*E-mail address: yamazaki@cc.kyoto-su.ac.jp
To avoid the problem, the approach called Dirac-Square-Root formalism has been studied [5, 6, 8, 9]. Since the Hamiltonian for the Bianchi type models is expressed in terms of the quadratic, the first-order differential equation can be derived by applying a procedure similar to the Dirac method. Mallett [8] applied it to the Friedmann model coupled to the charged scalar field. Kim and Oh [9] mainly discussed the Bianchi type-IX model. In their papers, the first-order equation included an unknown function dependent on the time-parameter is postulated and the wave function is required to be a solution of two-component column vector. Iterating the first-order equation thus postulated, the differential equation with respect to the unknown function is yielded so as to be consistent with the Wheeler-DeWitt equation for each component. The differential equation with respect to the time parameter is thus obtained. Indeed, the differential equation is reduced to Riccati or Bernoulli equation in case of the Friedmann model with charged scalar field or the Bianchi type-IX model, respectively. However, one sees that it is difficult to obtain the exact solutions in both cases.

The purpose of this paper is to propose a new method, which factorizes the Wheeler-DeWitt equation into the first-order differential equation, using the Pauli matrices. The resultant differential equation thus obtained is similar to the Dirac equation. As a necessary condition for the Dirac factorization, it is postulated that the resultant Hamiltonian is "self-adjoint". We apply this method to the the Wheeler-DeWitt equation of the Bianchi Class A vacuum models except for type-IX model. The solution of the Dirac type equation is written by two-component column vector. Using the Dirac type equation and the complex conjugate, we can define the probability density. Then, we get the equation of continuity and the conserved current. Here, we restrict ourselves to determine the form of Hamiltonian so as to make the probability density positive-definite. It is to be noted that our formalism cannot be applied to the Bianchi type-IX model, because the structure of anisotropic potential for the Bianchi type-IX model is different from that of the other Bianchi Class A models.

The organization of this paper is as follows. In §2, we review the ADM canonical formulation for the vacuum Bianchi Class A models and introduce the Wheeler-DeWitt equation. In §3, we propose the new method in which the Wheeler-DeWitt equation is factorized into the first-order differential equation in terms of the Pauli matrices and apply the method to the Bianchi type-I, II, VI_{0}, VII_{0} and VIII models. For each of them, the Dirac type equation is derived and solved except for the Bianchi type-VIII model. Then, we assume a trial function which is expressed in terms of two-component spinor. We show that the probability density becomes positive-definite and there is a conserved current. As the time parameter approaches the limit of the singularity of the universe, the Wheeler-DeWitt equation for the Bianchi type-II, VI_{0}, VII_{0} and VIII models are reduced to that of the Bianchi type-I model, so that the Dirac type equations must possess the solution for the Bianchi type-I model. In §4, we discuss the behavior for the Dirac type equation near the singularity in detail. It is shown that from the Heisenberg equation the quantum behavior of the anisotropic parameters $\beta_{k}$ make the quivering motion and the Hamiltonian system has a 'spin-like' degree of freedom. The 'spin-like' degree of freedom causes the universe to go necessarily through an early quantum stage of evolution with agitated anisotropy-oscillation. In §5, summary and discussion are made on the results of our formalism and comparisons are made with the Dirac-Square-Root formalism. Henceforward, we take that $i$, $j$ and $k$ runs from 1 to 3 and the unit $c = \hbar = 16\pi G = 1$. 


2 Canonical Quantization formulation for the Bianchi Class A models

The Einstein-Hilbert action is given by [10]

\[ I = \int \sqrt{g}^{(4)} R \, d^4x, \]  

(1)

where \(^{(4)} g\) and \(^{(4)} R\) are the determinant of the space-time metric and the curvature, respectively; the indices (4) implies 4-geometry quantity. The metric for the Bianchi models is generally given by

\[ ds^2 = -N^2 dt^2 + e^{2\alpha} e^{2\beta_{ij}} \chi^i \chi^j, \]  

(2)

where \(N\), \(\alpha\) and \(\beta_{ij}\) are functions of \(t\) only: \(N\) is the lapse function; \(e^{2\alpha}\) plays the role of a scale factor, so that the universe has the initial singularity as \(\alpha \to -\infty\); \(\beta_{ij}\) are an isotropic parameters of the universe, which gives a matrix form of traceless and \(\chi^i\) are 1-forms. According to Misner [4], anisotropic parameters \(\beta_{ij}\) are expressed in terms of two parameters \(\beta_+\) and \(\beta_-\)

\[ \beta_{ij} = \text{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_, -2\beta_+). \]  

(3)

The 1-forms obey

\[ d\chi^i = \frac{1}{2} C^i_{jk} \chi^j \wedge \chi^k, \]  

(4)

where \(C^i_{jk}\) are the structure constants.

The Bianchi models are characterized by the structure constants. The Bianchi models are classified into two groups from the structure constants, so-called Class A and B [11]. It should be noted that, as was first indicated by Hawking [12], it cannot be reduced to the correct field equations by the ADM canonical formulation in the Bianchi Class B models except for the Bianchi type-V model [13, 14]. The reason is that the spatial divergence term does not vanish. Hence, we must discuss only the Bianchi Class A models because we use the canonical formulation. The characteristic of the Bianchi Class A models is that the structure constants obey the condition

\[ C^i_{ji} = 0. \]  

(5)

We must therefore select the 1-forms so as to satisfy Eq. (5).

After the procedure for the ADM decomposition, we use the coordinates \(\alpha, \beta_+, \beta_-\) and the corresponding canonical momenta as \(p_\alpha, p_+, p_-\). From the variation of \(N\), one obtains the Hamiltonian constraint. The 3-geometry curvature \(^{(3)} R\) is obtained from the structure constants. According to Misner [4], the relation of the curvature \(^{(3)} R\) to the anisotropic potential \(V(\beta_+, \beta_-)\) is introduced by

\[ ^{(3)} R = \frac{3}{2} e^{2\alpha} (1 - V(\beta_+, \beta_-)). \]  

(6)

As a result, the Hamiltonian constraint becomes

\[ H = -p_\alpha^2 + p_+^2 + p_-^2 + e^{4\alpha} (V - 1). \]  

(7)

On the canonical quantization procedure, we treat the operators \(p_\alpha, p_+, p_-\) in the Schrödinger representation:

\[ \hat{p}_\alpha = -i \frac{\partial}{\partial \alpha}, \quad \hat{p}_+ = -i \frac{\partial}{\partial \beta_+}, \quad \hat{p}_- = -i \frac{\partial}{\partial \beta_-}. \]  

(8)
Because the Hamiltonian constraint is the first-class constraint, the quantized Hamiltonian constraint must be satisfied by the supplementary condition

$$\hat{H}\Psi = 0,$$

(9)

$$\hat{H} \equiv \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta^2} - \frac{\partial^2}{\partial \beta_+^2} + e^{4\alpha}(V - 1),$$

(10)

where $\Psi = \Psi(\alpha, \beta_+, \beta_-)$ is a state vector of the system, called by the wave function of the universe. This is the Wheeler-DeWitt equation for the Bianchi models.

### 3 Dirac factorization for the Wheeler-DeWitt equation

According to the Furusawa’s discussion [7], the probability density of Klein-Gordon type

$$\rho(\alpha, \beta_+, \beta_-) = \frac{i}{2} (\Psi^* \partial_\alpha \Psi - \Psi \partial_\alpha \Psi^*)$$

becomes negative for some values $\alpha$. The negative values of the probability density are inconsistent with the physical theories and it is the failure of the structure in the Wheeler-DeWitt equation. To get rid of the problem, we adopt the similar procedure that Dirac factorized the Klein-Gordon equation in order to obtain the positive-definite probability density. That is, we try to factorize the Wheeler-DeWitt equation into the first-order differential equation. We denote the first-order equation as Dirac type equation. As a necessary condition of Dirac factorization, it is postulated that the resultant Hamiltonian is “self-adjoint”, because one tries to think various ways of the factorization into the first-order equation and there is not any rules for the factorization. We suppose that the Wheeler-DeWitt equation for the Bianchi models is replaced by the equation of Dirac type

$$\hat{H}\Psi \equiv \left(\hat{p}_\alpha + \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- + \sigma_3 e^{2\alpha} v\right) \Psi = 0,$$

(11)

where $v \equiv \sqrt{V - 1}$ and the Pauli matrices $\sigma_i$ are expressed as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We here rewrite the Dirac type equation (11) in the Schrödinger form

$$i \frac{\partial}{\partial \alpha} \Psi = \left[\sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- + \sigma_3 e^{2\alpha} v\right] \Psi.$$

(12)

The self-adjointness of the Hamiltonian is easily proved:

$$\hat{H}^\dagger = \hat{p}_\alpha + \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- + \sigma_3 e^{2\alpha} v = \hat{H},$$

(13)

where the momentum operators are self-adjoint and commute with the Pauli matrices.

With the help of relations (8), the Dirac type equation (11) is rewritten as

$$-i \frac{\partial \Psi}{\partial \alpha} - i\sigma_1 \frac{\partial \Psi}{\partial \beta_+} - i\sigma_2 \frac{\partial \Psi}{\partial \beta_-} + \sigma_3 e^{2\alpha} v \Psi = 0.$$

(14)

Taking the complex conjugate and multiplying $\Psi^\dagger$ to Eq. (14) from the left-hand side, we have

$$i \Psi^\dagger \frac{\partial \Psi}{\partial \alpha} + i \Psi^\dagger \sigma_1 \frac{\partial \Psi}{\partial \beta_+} + i \Psi^\dagger \sigma_2 \frac{\partial \Psi}{\partial \beta_-} + \Psi^\dagger \sigma_3 e^{2\alpha} v \Psi = 0.$$

(15)
Subtracting Eq. (15) from Eq. (14) multiplied $\Psi^\dagger$ from the left-hand side, we obtain
\[
\frac{\partial}{\partial \alpha} \Psi^\dagger \Psi + \frac{\partial}{\partial \beta_1} (\Psi^\dagger \sigma_1 \Psi) + \frac{\partial}{\partial \beta_-} (\Psi^\dagger \sigma_2 \Psi) = 0.
\] (16)

Setting the probability density $\rho$ and the current vectors $j_+$ and $j_-$ as
\[
\rho = \Psi^\dagger \Psi, \quad j_+ = \Psi^\dagger \sigma_1 \Psi, \quad j_- = \Psi^\dagger \sigma_2 \Psi,
\] (17)
we get the equation of continuity
\[
\frac{\partial \rho}{\partial \alpha} + \frac{\partial j_+}{\partial \beta_1} + \frac{\partial j_-}{\partial \beta_-} = 0.
\] (18)

Then, one sees that the probability density is positive-definite. We apply our formalism to solve the Bianchi type-I, II, VI$_0$, VII$_0$ and VIII models below.

### 3.1 The Bianchi type-I model

The metric of the Bianchi type-I model is given in Eq. (2), in which the 1-forms are given by [15]
\[
\chi^1 = dx^1, \quad \chi^2 = dx^2, \quad \chi^3 = dx^3.
\] (19)

The structure constants in the model obviously supply
\[
C^i_{jk} = 0.
\] (20)

It leads to the zero curvature of 3-dimensional hypersurfaces $^{(3)} R_I = 0$, which implies the spatial flat structure. From this, the anisotropic potential becomes $V_I = 1$. Then, the Wheeler-DeWitt equation is reduced to
\[
(-\hat{p}_\alpha^2 + \hat{p}_+^2 + \hat{p}_-^2) \Psi = 0.
\] (21)

We postulate here that the Dirac type equation for the Bianchi type-I model holds:
\[
\mathcal{H}_I \Psi_I(\alpha, \beta_+, \beta_-)(\hat{p}_\alpha + \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- \Psi = 0.
\] (22)

The solutions to this equation have been already found [5, 6]. They are plane wave solutions given by two-component spinor form
\[
\Psi_I(\alpha, \beta_+, \beta_-) = e^{i(k_+ \beta_+ + k_- \beta_- - \omega \alpha)} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
\] (23)

where $\omega$ and $k_\pm$ are arbitrary real constants. Substituting the solution (23) into the Dirac type equation (22), we get the components
\[
\psi_1 = \frac{k_+ - ik_-}{\omega} \psi_2 \quad \text{and} \quad \psi_2 = \frac{k_+ + ik_-}{\omega} \psi_1.
\] (24)

It implies obviously $|\psi_1|^2 = |\psi_2|^2$ and the relation among $\omega$ and $k_\pm$ is
\[
\omega = \pm \sqrt{k_+^2 + k_-^2}.
\] (25)

Thus, the parameter $\omega$ corresponding to the frequency of the plane wave depends on the parameters $k_\pm$ corresponding to the wave number, which vary continuously.
The probability density is
\[ \rho_I(\alpha, \beta_+, \beta_-) = \Psi_I^\dagger \Psi_I = |\psi_1|^2 + |\psi_2|^2 = 2|\psi_1|^2, \] (26)
which is positive-definite. The current vectors \( j_{\pm} \) are respectively
\[ j_+ = \psi^*_2 \psi_1 + \psi^*_1 \psi_2 \] (27)
and
\[ j_- = i\psi^*_2 \psi_1 - i\psi^*_1 \psi_2, \] (28)
which take some constants. Therefore, the equation of continuity (18) holds and there is a conserved current. From the parameters (3) and the result of Eq. (27) and (28), we interpret that the evolution of the quantized universe is that whether one of the three spatial axes is expanding and two axes are contracting or one axis is contracting and two axes are expanding.

It should be noted that the quantity
\[ \int \rho_I(\alpha, \beta_+, \beta_-) d\beta_+ d\beta_- = 2|\psi_1|^2 \int d\beta_+ d\beta_- \] (29)
is not convergent and the wave function (23) is not the square integrable function. Therefore, it is impossible to normalize the wave function (23) without an appropriate boundary condition and we do not discuss the normalization constant in this paper. Here, the quantity (29) does not directly determine the probability, so that the probability must be regarded as a quantity proportional to this quantity (29). Moreover, since \( \partial \rho / \partial \alpha = 0 \), the probability density is independent of the time-parameter and quantity (29) is conserved.

As mentioned above, it should be noted here that since the Wheeler-DeWitt equation and the Dirac type equation for the Bianchi type-II, VI\textsubscript{0}, VII\textsubscript{0} and VIII models are reduced to that of the Bianchi type-I model at the singularity (\( \alpha \rightarrow -\infty \)), all models have the same property that the behavior near the singularity becomes that of the plane wave solution. We shall discuss the quantized Hamiltonian system of the Bianchi type-I model in \( \S \)4.

3.2 The Bianchi type-II model
The metric of the Bianchi type-II model is expressed as Eq. (2) and the 1-forms are given by [15]
\[ \chi^1 = dx^1 - x^3 dx^2, \quad \chi^2 = dx^2, \quad \chi^3 = dx^3. \] (30)
Substituting the 1-forms (30) into Eq. (4), we get the structure constants
\[ C_{23}^1 = -C_{32}^1 = 1. \] (31)
Other components are zero. The curvature is
\[ {^{(3)}}R_{II} = -\frac{1}{2}e^{2\alpha+4\beta_++4\sqrt{3}\beta_-}, \] (32)
and the anisotropic potential \( V_{II}(\beta_+, \beta_-) \) is obtained from the relation (6) as
\[ V_{II} - 1 = \frac{1}{3}e^{4\beta_++4\sqrt{3}\beta_-}. \] (33)
Then, the Wheeler-DeWitt equation is expressed as
\[ \left( -\dot{p}_\alpha^2 + \dot{p}_\beta^2 + \dot{p}_\gamma^2 + \frac{1}{3}e^{4\alpha+4\beta_++4\sqrt{3}\beta_-} \right) \Psi = 0. \] (34)
We postulate here that the factorized Hamiltonian is expressed as

$$H_{II} = \hat{p}_\alpha + \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- + \frac{\sigma_3}{\sqrt{3}} e^{2\alpha + 2\beta_+ + 2\sqrt{3} \beta_-}$$

and that the Dirac type equation holds:

$$H_{II} \Psi_{II} = 0. \quad (36)$$

With the help of the relation (8), it becomes

$$i \frac{\partial \Psi_{II}}{\partial \alpha} = \begin{pmatrix} \frac{1}{\sqrt{3}} e^{2\alpha + 2\beta_+ + 2\sqrt{3} \beta_-} & -i \frac{\partial}{\partial \beta_+} - \frac{\partial}{\partial \beta_-} \\ -i \frac{\partial}{\partial \beta_+} + \frac{\partial}{\partial \beta_-} & -\frac{1}{\sqrt{3}} e^{2\alpha + 2\beta_+ + 2\sqrt{3} \beta_-} \end{pmatrix} \Psi_{II}. \quad (37)$$

Now we look for a solution to this equation. We take a trial function which gives the two-component spinor form and notices the form of the Dirac type equation as

$$\Psi_{II} = e^{x p \left[ -\frac{1}{6} e^{2\alpha + 2\beta_+ + 2\sqrt{3} \beta_-} \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \right]},$$

where the spinor components $\psi_1$ and $\psi_2$ are arbitrary constants. Substituting the trail function (38) into Eq. (37), we get the relation of the spinor components

$$\psi_2 = -\psi_1. \quad (39)$$

Here we use the condition that the Dirac type equation (36) for the Bianchi type-II model is reduced to that for the Bianchi type-I model as the limit of the singularity ($\alpha \to -\infty$). The trail function (38) would include such an asymptotic behavior. The trail function involving the plane wave part is expressed as

$$\Psi_{II} = e^{i(k_+ \beta_+ + k_- \beta_- - \omega \alpha)} \exp \left[ -\frac{1}{6} e^{2\alpha + 2\beta_+ + 2\sqrt{3} \beta_-} \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \right]. \quad (40)$$

Although $\omega$, $k_\pm$ are arbitrary real constants, the coefficients $\omega$ and $k_\pm$ are not free at all. Substituting the solution (40) into Eq. (37), we have

$$\begin{pmatrix} -\omega \\ k_+ + ik_- \\ -\omega \end{pmatrix} \begin{pmatrix} \psi_1 \\ -\psi_1 \end{pmatrix} = 0. \quad (41)$$

Therefore, the coefficients must satisfy

$$k_+ = -\omega, \quad k_- = 0. \quad (42)$$

Consequently, the solution is finally expressed in terms of the product of the amplitude and oscillation parts restricted by Eq. (42):

$$\Psi_{II} = e^{-i\omega (\beta_+ + \alpha)} \exp \left[ -\frac{1}{6} e^{2\alpha + 2\beta_+ + 2\sqrt{3} \beta_-} \left( \begin{array}{c} \psi_1 \\ -\psi_1 \end{array} \right) \right]. \quad (43)$$

The probability density is expressed, by using the wave function (43), as

$$\rho_{II}(\alpha, \beta_+, \beta_-) = \Psi_{II}^* \Psi_{II} = 2|\psi_1|^2 \exp \left[ -\frac{1}{3} e^{2\alpha + 2\beta_+ + 2\sqrt{3} \beta_-} \right]. \quad (44)$$

7
This is always positive-definite. One sees that this probability density goes to zero as \( \beta_+ + \sqrt{3} \beta_- \to \infty \), while it approaches a certain constant as \( \beta_+ + \sqrt{3} \beta_- \to -\infty \). The current vectors \( j_\pm \) given by Eq. (17) are

\[
j_+ = -2|\psi_1|^2 \exp \left[ -\frac{1}{3} e^{2\alpha + 2\beta_+ + 2\sqrt{3} \beta_-} \right]
\]

and

\[
j_- = 0.
\]

Hence, the equation of continuity (18) holds and there is the conserved current. Moreover, it is shown that the wave part propagates to the direction of \( \beta_+ \) axis only and it agrees with the result of Eq. (42).

### 3.3 The Bianchi type-VI\(_0\) model

The metric of the Bianchi type-VI\(_0\) model is expressed as Eq. (2) and the 1-forms are given by [15]

\[
\chi^1 = \cosh x^3 dx^1 - \sinh x^3 dx^2, \quad \chi^2 = -\sinh x^3 dx^1 + \cosh x^3 dx^2, \quad \chi^3 = dx^3.
\]

The structure constants are

\[
C^1_{23} = -C^1_{32} = 1, \quad C^2_{31} = -C^2_{13} = -1
\]

and the rest are zero. The anisotropic potential becomes

\[
V_{VI_0} - 1 = \frac{4}{3} e^{4\beta_+} \left( \cosh(4\sqrt{3} \beta_-) + 1 \right).
\]

Then, the Wheeler-DeWitt equation for the Bianchi type-VI\(_0\) model is expressed as

\[
\left[ -\hat{p}_\alpha^2 + \hat{p}_+^2 + \hat{p}_-^2 + \frac{4}{3} e^{4\beta_+} \left( \cosh(4\sqrt{3} \beta_-) + 1 \right) \right] \Psi = 0.
\]

We postulate that the Dirac type equation for the Bianchi type-VI\(_0\) model is expressed as

\[
\mathcal{H}_{VI_0} \Psi_{VI_0} \equiv \left[ \hat{p}_\alpha + \sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_- + \frac{2}{\sqrt{3}} \sigma_3 e^{2\alpha + 2\beta_+} \left( e^{2\sqrt{3} \beta_-} + e^{-2\sqrt{3} \beta_-} \right) \right] \Psi_{VI_0} = 0.
\]

As in the case of the Bianchi type-II model, we assume a trial function of the form

\[
\Psi_{VI_0} = \exp \left[ -\frac{1}{3} e^{2\alpha + 2\beta_+} \left( e^{2\sqrt{3} \beta_-} - e^{-2\sqrt{3} \beta_-} \right) \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
\]

where the components \( \psi_1 \) and \( \psi_2 \) are arbitrary constants. Substituting the trial function (52) into the Dirac type equation (51), we get

\[
\psi_2 = -\psi_1.
\]

From the condition that the wave function (52) should be reduced to the plane wave solution at the limit of the singularity, we obtain

\[
k_+ = -\omega, \quad k_- = 0.
\]
The wave function involving the plane wave part is finally expressed as
\[
Ψ_{V\!I\!V\!I_0} = e^{-i(β_+ + α)} \exp \left[ -\frac{1}{3} e^{2α + 2β_+} \left( e^{2\sqrt{3}β_-} - e^{-2\sqrt{3}β_-} \right) \right] \begin{pmatrix} ψ_1 \\ -ψ_1 \end{pmatrix}. \tag{55}
\]

The probability density becomes
\[
ρ_{V\!I\!V\!I_0}(α, β_+, β_-) = Ψ_{V\!I\!V\!I_0}^\dagger Ψ_{V\!I\!V\!I_0} = 2|ψ_1|^2 \exp \left[ -\frac{2}{3} e^{2α + 2β_+} \left( e^{2\sqrt{3}β_-} - e^{-2\sqrt{3}β_-} \right) \right], \tag{56}
\]
which is positive-definite. The current vectors are
\[
j_+ = -2|ψ_1|^2 \exp \left[ -\frac{2}{3} e^{2α + 2β_+} \left( e^{2\sqrt{3}β_-} - e^{-2\sqrt{3}β_-} \right) \right], \quad j_- = 0. \tag{57}
\]

One sees that the equation of continuity holds and there is a conserved current. The plane wave part of the wave function (55) propagates to the direction of $β_+$ axis only. It is to be noted that since the wave function (55) and the probability density (56) are both divergent as $β_- \to -∞$, this solution is unphysical in such a region.

3.4 The Bianchi type-VII₀ model

The metric of the Bianchi type-VII₀ model is expressed as Eq. (2) and the 1-forms are given by [15]
\[
χ^1 = \cos x^1 dx^2 + \sin x^1 dx^3, \quad \chi^2 = -\sin x^1 dx^2 + \cos x^1 dx^3, \quad χ^3 = dx^1. \tag{58}
\]

The structure constants are
\[
C_{23}^1 = -C_{32}^1 = -1, \quad C_{31}^2 = -C_{13}^2 = -1 \tag{59}
\]
and the rest are zero. The anisotropic potential is
\[
V_{V\!I\!V\!I_0} - 1 = \frac{4}{3} e^{4β_+} \left( \cosh(4\sqrt{3}β_-) - 1 \right). \tag{60}
\]

Then, the Wheeler-DeWitt equation for the Bianchi type-VII₀ model is expressed as
\[
\left[ \tilde{p}_α^2 + \tilde{ρ}_+^2 + \tilde{ρ}_-^2 + \frac{4}{3} e^{4β_+} \left( \cosh(4\sqrt{3}β_-) - 1 \right) \right] Ψ = 0. \tag{61}
\]

Here, we postulate that the Dirac type equation with a factorized Hamiltonian is expressed as
\[
\mathcal{H}_{V\!I\!V\!I_0}Ψ_{V\!I\!V\!I_0} = -\tilde{p}_α + σ_1 \tilde{ρ}_+ + σ_2 \tilde{ρ}_- + \frac{2}{\sqrt{3}} σ_3 e^{2α + 2β_+} \left( e^{2\sqrt{3}β_-} - e^{-2\sqrt{3}β_-} \right) Ψ_{V\!I\!V\!I_0} = 0. \tag{62}
\]

As in the case of previous models, we can find out the wave function
\[
Ψ_{V\!I\!V\!I_0} = e^{-iω(β_+ + α)} \exp \left[ -\frac{1}{3} e^{2α + 2β_+} \left( e^{2\sqrt{3}β_-} - e^{-2\sqrt{3}β_-} \right) \right] \begin{pmatrix} ψ_1 \\ -ψ_1 \end{pmatrix}. \tag{63}
\]

The probability density becomes
\[
ρ_{V\!I\!V\!I_0}(α, β_+, β_-) = Ψ_{V\!I\!V\!I_0}^\dagger Ψ_{V\!I\!V\!I_0} = 2|ψ_1|^2 \exp \left[ -\frac{2}{3} e^{2α + 2β_+} \left( e^{2\sqrt{3}β_-} - e^{-2\sqrt{3}β_-} \right) \right], \tag{64}
\]
which is positive-definite. The current vectors have
\[ j_+ = 2|\psi|^2 \exp \left[ -\frac{2}{3} e^{2\alpha + 2\beta} \left( e^{2\sqrt{3}\beta} + e^{-2\sqrt{3}\beta} \right) \right], \quad j_- = 0. \] (65)

One sees that the equation of continuity holds and there is a conserved current. The plane wave part of the wave function (63) propagates to the direction of \( \beta_+ \) axis only. At the limit of \( \beta_+ \to -\infty \), the value of the probability density approaches a certain constant, while converges to zero as \( \beta_+ \to \infty \) or \( \beta_- \to \pm \infty \).

### 3.5 The Bianchi type-VIII model

The metric of the Bianchi type-VIII model is expressed as Eq. (2) and the 1-forms are given by [15]
\[
\begin{align*}
\chi_1 &= \cosh x^2 \cos x^3 dx^1 - \sin x^3 dx^2, \\
\chi_2 &= \cosh x^2 \sin x^3 dx^1 + \cos x^3 dx^2, \\
\chi_3 &= \sinh x^2 dx^1 + dx^3.
\end{align*}
\] (66)

The structure constants are
\[
C_{23}^1 = -C_{32}^1 = -1, \quad C_{31}^2 = -C_{13}^2 = -1, \quad C_{12}^3 = -C_{21}^3 = 1 \quad (67)
\]
and the rest are zero. The anisotropic potential becomes
\[
V_{\text{VIII}} - 1 = \frac{1}{3} \left( e^{-8\beta} - 2e^{2\beta} + 4e^{-2\beta} \cosh(2\sqrt{3}\beta) + 2e^{4\beta} \cosh(4\sqrt{3}\beta) \right). \quad (68)
\]

However, since the anisotropic potential (68) is a very complicated expression, we cannot present the Dirac type equation of the form as Eq. (11).

### 4 The behavior near the singularity

It is interesting for us to discuss the behavior of the wave function near the singularity. Misner [4] discussed the Bianchi type-IX (Mixmaster) model which is represented by the classical dynamical system where an imaginary particle moves in the anisotropic potential of the approximated regular triangle and regarded the dynamical system as the billiard in the \( \beta_+ - \beta_- \) plane. In this way Misner’s picture is very helpful to discuss the universe through the anisotropic parameters.

The vacuum Bianchi type-I universe is so-called Kasner Universe. The evolution of the model in classical region presents an anisotropically expanding universe in which two axises are expanding and the other is contracting. While, in quantum region the quadratic Hamiltonian is derived by the ADM approach and the evolution of the quantized universe is presented by the Wheeler-DeWitt equation like Klein-Gordon type. There is the problem that the probability density can become negative by the Klein-Gordon approach.

Let us consider to change the point of view to our formalism. Near the singularity \( (\alpha \to -\infty) \), the Dirac type equations for the Bianchi Class A models are reduced to that of Bianchi type-I
\[
i \frac{\partial \Psi}{\partial \alpha} = (\sigma_1 \hat{p}_+ + \sigma_2 \hat{p}_-) \Psi \equiv \hat{h}' \Psi. \quad (69)
\]

This dynamical system describes a massless imaginary particle moving in zero potential. The equations of motion in the Heisenberg representation for \( \hat{\beta}_+ \) and \( \hat{\beta}_- \) are
\[
\frac{\partial \hat{\beta}_+}{\partial \alpha} = i \left[ \hat{h}', \hat{\beta}_+ \right] = i \sigma_1 \left[ \hat{p}_+, \hat{\beta}_+ \right] = \sigma_1 \quad (70)
\]
and

\[ \frac{\partial \hat{\beta}_-}{\partial \alpha} = i \left[ \hat{h}', \hat{\beta}_- \right] = i \sigma_2 \left[ \hat{p}_-^\prime, \hat{\beta}_- \right] = \sigma_2, \tag{71} \]

where the commutative relation \( [\hat{h}_a, \hat{p}_a] = i \delta_{ab} \) \( (a, b = +, -) \) is used. It is similar to the Zitterbewegung [16] (a rapidly oscillating motion) of electron, so that the imaginary particle moves a trembling motions near the singularity of the universe. This behavior in quantum mechanics with factorized Hamiltonian is very different from that for the Wheeler-DeWitt equation. We emphasize here that this kind of behavior cannot be obtained from the Wheeler-DeWitt equation.

Here, we set an orbital angular momentum \( \hat{m}' \) as

\[ \hat{m}' = \hat{\beta}_+ \hat{p}_- - \hat{\beta}_- \hat{p}_+. \tag{72} \]

The Heisenberg equation of motion for the orbital angular momentum is

\[ \frac{d\hat{m}'}{d\alpha} = i \left[ \hat{h}', \hat{m}' \right] = \sigma_1 \hat{p}_- - \sigma_2 \hat{p}_+. \tag{73} \]

We have, furthermore,

\[ \frac{d\sigma_3}{d\alpha} = i \left[ \hat{h}', \sigma_3 \right] = 2 \sigma_2 \hat{p}_+ - 2 \sigma_1 \hat{p}_-. \tag{74} \]

The quantity \( \hat{m}' + \frac{1}{2} \sigma_3 \) is a constant of motion. This result one can interpret by saying a massless imaginary particle has a ‘spin-like’ angular momentum \( \frac{1}{2} \sigma_3 \). The Dirac type equation (69) is a similar type to the Weyl equation, which has two-dimension of the spatial part here. We know that the Weyl equation describes the behavior of the massless neutrino with a half spin. On the analogy of this characteristic with the Weyl equation, the massless imaginary particle would have a half spin. Accordingly, we think that the Dirac type equation for the Bianchi type-I model describes the universe as a massless imaginary particle with ‘spin-like’ degree of freedom. Thus, the ‘spin-like’ degrees of freedom cause the universe to go through an early quantum stage of evolution with agitated anisotropy-oscillation. However, the physical interpretation of the ‘spin-like’ degree of freedom is not comprehensible yet. In conclusion, we obtain that the Dirac type equation for the Bianchi type-I model shows the interesting results on quantum behavior, unlike Misner’s negative conclusion that the status of the wave function remains classical near the initial singularity [4].

5 Conclusions and Discussion

We have investigated the first-order equation using the Pauli matrices by factorizing the Wheeler-DeWitt equation for the vacuum Bianchi Class A models except for the Bianchi type-IX. We have derived the solution expressed in terms of two-component spinor on the Bianchi type-I, II, VI\textsubscript{0} and VII\textsubscript{0} models. It has been shown that the probability density becomes positive-definite and the equation of continuity holds with a conserved current. However, we could not find the solution to the Bianchi type-VIII model because the form of anisotropic potential is very complicated expression. It is to be noted that we could not normalize the wave function and the probability density for each model. In the Bianchi type-II model, it is able to factorize the quadratic Hamiltonian of the Wheeler-DeWitt equation into self-adjoint Hamiltonians other than the Hamiltonian (35). However, we cannot extract physically interesting solutions to the Dirac type equations by such Hamiltonians. The similar results are obtained for the Bianchi type-VI\textsubscript{0} and type-VII\textsubscript{0} models.
The Dirac type equations for the Bianchi type-II, VI$_0$ and VII$_0$ model are reduced to that for the Bianchi type-I model near the singularity ($\alpha \to -\infty$). The quantized universe near the singularity is expressed by the plane wave solution with two-component spinor and the anisotropic parameters $\hat{\beta}_+$ and $\hat{\beta}_-$ make the motion of Zitterbewegung. The Zitterbewegung can be related to the 'spin-like' degree of freedom and thus the 'spin-like' degree of freedom causes the universe to go necessarily through an early quantum stage of evolution with agitated anisotropy-oscillation. The 'spin-like' degree of freedom which does not appear when we deal with the Wheeler-DeWitt equation for the Bianchi models may play an important role when we discuss the anisotropy of the universe in the initial stage. In such a sense, the Zitterbewegung is a significant subject how the big-bang or the inflation is related. To solve the problem of the initial singularity we must find out the physical interpretation of the 'spin-like' degree of freedom newly found by the Dirac factorization.

As mentioned in §1, our method cannot be applied to the Bianchi type-IX model. Because, the anisotropic potential of the Bianchi type-IX $V_{IX}$ is expressed in terms of $V_{IX} = \frac{1}{3} e^{-8\beta_+} - \frac{4}{3} e^{-2\beta_+} \cosh 2\sqrt{3}\beta_- + \frac{2}{3} e^{4\beta_+} (\cosh 4\sqrt{3}\beta_- - 1) + 1 > 0$. (75)

The quantity $v^2 = V_{IX} - 1$ takes negative value as well as positive value here. It means that the quantity $v = \sqrt{V_{IX} - 1}$ which appears in the factorized theory turns out to be imaginary as well as real. Therefore, we cannot factorize the quadratic Hamiltonian into a fixed form. Hence, we must not use the Pauli matrices but the Dirac matrices. To remedy this defect, we notice the fact that the anisotropic potential $V_{IX}$ itself is non-negative, and carry out the Dirac procedure of factorization with anticommuting $\gamma^\mu$ $(\mu = 0, 1, 2, 3)$ and $\gamma_5$ matrices. One possibility is to make the factorization in the form

$$\left(\gamma^0 p_\alpha + \gamma^1 p_+ + \gamma^2 p_- + \gamma^3 v' e^{2\alpha} + \gamma_5 e^{2\alpha}\right) \Psi = 0,$$

(76)

where we set $v' = \sqrt{V_{IX}}$. However, as we have seen, because the anisotropic potential $V_{IX}$ is a very complicated expression, we could not have found out the solution to the Dirac type equation (76) with the factorized Hamiltonian.

We compare our formalism with the Dirac-Square-Root (DSR) formalism [8, 9]. On the DSR, it is postulated that the factorized Dirac type equation for the Bianchi type-I model is expressed as

$$\left(\frac{\partial}{\partial \alpha} - i\sigma_1 \frac{\partial}{\partial \beta_+} - i\sigma_2 \frac{\partial}{\partial \beta_-}\right) \Psi(\alpha, \beta_+, \beta_-) = 0.$$ (77)

The Dirac type equation (77) then implies that the Wheeler-DeWitt equation for the Bianchi type-I model yields

$$(-\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta_+^2} + \frac{\partial^2}{\partial \beta_-^2}) \Psi(\alpha, \beta_+, \beta_-) = 0,$$

(78)

for each component of the wave function. According to Mallett [8], it is postulated that the Dirac type equation is expressed as

$$\left(i\frac{\partial}{\partial \alpha} - i\sigma_1 \frac{\partial}{\partial \beta_+} - i\sigma_2 \frac{\partial}{\partial \beta_-} + iW(\alpha)\right) \Psi(\alpha, \beta_+, \beta_-) = 0,$$

(79)

where $W(\alpha)$ is an unknown function depended on $\alpha$ only. In Kim and Oh [9] paper, it has been applied the first-order equation (79) to the Bianchi type-IX model.

However, it is not valid to apply the DSR formalism to the Wheeler-DeWitt equation including the anisotropic potential term like (75) and to factorize in the form of the Dirac
type equation (79). Because, it is inconsistent with the original Wheeler-DeWitt equation in the point that the differential operators with the anisotropic parameters $\beta_+$ and $\beta_-$ do not act on the unknown function $W(\alpha)$. The incorrect equation yields if one assume the equation (79) when the Wheeler-DeWitt equation has the anisotropic potential term included the anisotropic parameters. While, our formalism is valid to apply to the Bianchi Class A models because it is able to factorize the Wheeler-DeWitt equation included the anisotropic potential term into the Dirac type equation (11). This is the extended DSR formalism.

When we yield the original expression from the Dirac type equation discussed by our formalism and the iterating equation, we also obtain the extra terms which include the time parameter and the anisotropic parameters. Regarding the extra terms as a new constraint to the wave function and solving the constraint, we cannot get solutions which satisfy the Dirac type equation. As a consequence, we only discuss the Dirac type equation without imposing the constraint condition on the wave function.

Acknowledgments

The author is grateful to Profs. Tetsuya Hara and Ikuo S. Sogami for the motivation of this work and helpful discussion.

References
