D-branes and constant electro-magnetic backgrounds

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Abstract: We probe the effective worldvolume theory of a set of coinciding D-branes by switching on constant electro-magnetic fields on them. The comparison of the mass spectrum predicted by string theory with the mass spectrum obtained from the effective action provides insights in the structure of the effective theory.

1 Introduction

The discovery of D-branes implies a novel way of looking at gauge theories. Indeed, the worldvolume degrees of freedom of a Dp-brane are described by a p + 1-dimensional field theory containing a U(1) gauge field and 9 − p scalar fields¹. The former describes an open string longitudinal to the brane while the latter describe the transversal fluctuations of the Dp-brane. For slowly varying fields, the effective action is known to all orders in α′: it is the ten-dimensional Born-Infeld action, dimensionally reduced to p + 1 dimensions [1], [2].

The situation becomes more interesting when several, say n, Dp-branes are present. The mass of a string stretching between two branes is proportional to the shortest distance between these two branes. Ignoring the transversal coordinates, we have, as long as the branes are well separated, n massless vector fields forming a (U(1))ⁿ gauge multiplet. However, once the branes coincide, n(n − 1) additional massless vector fields appear which correspond to oriented open strings connecting different branes. This enhances the gauge symmetry from (U(1))ⁿ to U(n) [3]. One expects the effective action to be some non-abelian generalization of the Born-Infeld action. However, as the notion of an

¹Throughout the paper, we will ignore the fermionic degrees of freedom as they neither add to nor change our conclusions.
acceleration term is ambiguous in a non-abelian theory,

\[ D_i D_j F_{kl} = \frac{1}{2} \{ D_i, D_j \} F_{kl} - \frac{i}{2} [ F_{ij}, F_{kl} ], \quad (1) \]

the concept of a slowly varying field is ambiguous too. Nonetheless, we do have some information about the \( U(n) \) non-abelian Born-Infeld action (NBI). Since it arises from the calculation of gluon scattering amplitudes in string theory, only one overall trace of the \( U(n) \) matrices should be taken. When switching off the off-diagonal modes of the gauge fields, it should reduce to the sum of \( n \) copies of the abelian Born-Infeld action. Finally, the NBI was explicitly calculated through order \( F^4 \) \cite{4, 5}. Based on this and assuming that all terms proportional to anti-symmetrized products of fieldstrengths should be viewed as acceleration terms which in the limit of slowly varying fields are ignored, a proposal was formulated for the NBI \cite{6}. The action assumes a form similar to the abelian case but, upon expanding it in powers of the fieldstrength, one first symmetrizes over all fieldstrengths and subsequently one performs the group theoretical trace. Alternative possibilities are discussed in \cite{7}.

In the present paper, we review and extend some of the results obtained in \cite{8, 9}. In those papers, the mass spectrum in the presence of constant magnetic background fields was calculated from the effective action and compared to predictions from string theory. As will be demonstrated later on, this shows that the symmetrized trace proposal is flawed from order \( F^6 \) on.

A direct calculation of the effective action at higher order in \( \alpha' \) would involve the analysis of at least a six-gluon scattering amplitude or the calculation of a five-loop \( \beta \)-function. As this does not seem feasible, different approaches are called for. One possibility, which we will review further in this paper, uses the mass spectrum as a guideline \cite{10}. Another possibility uses \( \kappa \)-symmetry to fix the orderings in the effective action \cite{11, 12}.

When finishing this paper, a preprint \cite{13} appeared, where various aspects, mostly complementary to our work, of strings in constant electro-magnetic are studied.

## 2 Electro-magnetic backgrounds and Lorentz transformations

Throughout the remainder of this paper, we mainly focus on D1- and D2-branes. General results can be found in \cite{9}. In addition, we do not consider the transversal coordinates as they provide no additional information.

We take two D2-branes wrapped once around a torus with cycles of length \( L_1 \) and \( L_2 \) and switch on magnetic fields \( \mathcal{F}_{12} = b^{(0)} \sigma_0 + b^{(3)} \sigma_3 \), where \( \sigma_a, a \in \{1, 2, 3\} \) are the Pauli matrices and \( \sigma_0 \) is the \( 2 \times 2 \) unit matrix. Flux quantization implies

\[ b^{(0)} \pm b^{(3)} = \frac{2\pi}{L_1 L_2} m_\pm, \quad m_\pm \in \mathbb{Z}. \quad (2) \]

Considering the Wess-Zumino term, which describes the coupling to the type IIA RR background fields, one can show that this is equivalent to the statement that the branes have \( m_+ \) and \( m_- \) D0-branes dissolved in them. We proceed by choosing a gauge\(^2\) such that \( A_1 = 0 \) and \( A_2 = \mathcal{F}_{12} x^1 \). After T-dualizing in the 2 direction, we end up with two

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2We are not very careful about the boundary conditions for the potentials on the torus. A detailed account of this can be found in e.g. \cite{14} or \cite{9}.
tilted D1-branes [15]. The new transversal coordinate is given by [16],

\[ X^2 = 2\pi\alpha'F_{12}x^1 = \left( \begin{array}{cc} m_+\hat{L}_2 x^1_{L_1} & 0 \\ 0 & m_-\hat{L}_2 x^1_{L_1} \end{array} \right), \]

where \( \hat{L}_2 = 4\pi^2\alpha'/L_2 \) is the length of the dual cycle. Eq. (3) clearly shows that the two D1-branes are wrapped once around cycle 1 and \( m_+ \) and \( m_- \) times resp. around cycle 2. The branes are rotated in the 12 plane over angles \( \arctan(2\pi\alpha'(b^{(0)} \pm b^{(3)})) \). The angle \( \phi \) between the two branes is given by

\[
\begin{align*}
\phi &= \arctan(2\pi\alpha'(b^{(0)} + b^{(3)})) - \arctan(2\pi\alpha'(b^{(0)} - b^{(3)})) \\
&= \arctan \frac{4\pi\alpha'b^{(3)}}{1 + (2\pi\alpha')^2((b^{(0)})^2 - (b^{(3)})^2)}. 
\end{align*}
\]

(4)

We now turn to electric backgrounds. Consider a D1-brane wrapped around a circle in the 1 direction and turn on a constant electric field along the brane of the form \( F_{01} = e^{(0)}\sigma_0 + e^{(3)}\sigma_3 \). In the gauge where \( A_0 = 0 \) and \( A_1 = F_{01}x^0 \), we end up, after T-dualizing in the 1 direction, with two D0-branes boosted in the 1 direction. Their speeds \( v_{\pm} \) and rapidities \( \alpha_{\pm} \) are given by \( v_{\pm} = 2\pi\alpha'(e^{(0)} \pm e^{(3)}) \) and \( \alpha_{\pm} = \arctanh(2\pi\alpha'(e^{(0)} \pm e^{(3)})) \) respectively. The fluxes are quantized, \( \cosh\alpha_{\pm} = m_{\pm} \in \mathbb{Z} \), which gives the branes momentum \( 2\pi m_{\pm}/\hat{L}_1 \) in the 1 direction. The relative velocity of the two branes is given by \( \tanh(\alpha_+ - \alpha_-) = (v_+ - v_-)/(1 - v_+v_-) \). Before T-duality, one can view an electric field with flux \( m \) on a D1-brane as a bound state of a D1-brane with \( m \) fundamental strings [3].

Finally, concerning the question of stability, as long as the electro-magnetic field has no component in the \( \sigma_3 \) direction, the resulting configuration is BPS with 16 supercharges preserved. Once a component in the \( \sigma_3 \) direction is turned on, one finds that for certain magnetic configurations, BPS states arise which preserve 2, 4, 6, or 8 supercharges. In the electric case, this never happens [15].

3 The spectrum from Yang-Mills theory

We consider a \( U(2) \) Yang-Mills theory in the presence of a constant background \( F_{\mu\nu} = F_{\mu\nu}^{(0)}\sigma_0 + F_{\mu\nu}^{(3)}\sigma_3 \) with potential \( A_\mu \). After separating the potential in the sum of the background and the fluctuation, \( A_\mu = A_\mu + \delta A_\mu \), one obtains for the part of the Lagrangian quadratic in the fluctuations,

\[
\mathcal{L} = -\frac{1}{2} \left\{ \sum_{i=1}^{2} \partial_\mu \delta A^{(i)} \partial^\mu \delta A^{(i)\nu} + 2D_\mu^+ \delta A^+_\mu D^-\nu \delta A^{-\nu} - 8iF_{\mu\nu}^{(3)}\delta A^{+\mu}\delta A^{-\nu} \right\},
\]

(5)

where we wrote,

\[
\delta A_\mu = \begin{pmatrix} \delta A^{(1)}_\mu & \delta A^{(1)}_\mu \\ \delta A^{(2)}_\mu & \delta A^{(2)}_\mu \end{pmatrix},
\]

and \( D_\mu^\pm = \partial_\mu \mp 2i\varphi_\mu^{(3)} \). We work in a background covariant Lorentz gauge.

Focussing on the magnetic case, \( F_{12} \neq 0 \), we immediately read off the spectrum for the diagonal fluctuations,

\[
M^2 = \left( \frac{2\pi m_1}{L_1} \right)^2 + \left( \frac{2\pi m_2}{L_2} \right)^2, \quad m_1, m_2 \in \mathbb{Z}.
\]

(7)
The spectrum of the off-diagonal fluctuations is most easily obtained by passing to complex coordinates, \( z = (x^1 + ix^2)/\sqrt{2} \). The equations of motion of \( \delta A^- \) become

\[
\begin{align*}
\left( \Box + 2D^-_z D^z_- - (2 + 4)F^{(3)}_{12}\right) \delta A^-_z &= 0, \\
\left( \Box + 2D^-_\bar{z} D^{\bar{z}}_- - (2 - 4)F^{(3)}_{12}\right) \delta A^-_{\bar{z}} &= 0,
\end{align*}
\]

where we used that \([D^-_z, D^-_{\bar{z}}] = -2F^{(3)}_{12}\) and \( \Box \) is the d’Alambertian for the non-compact directions. As shown in [17], a complete set of eigenfunctions for the compact part of this operator in a gauge where \( \partial_z A_z = 0 \), is of the form

\[
|m> = (D^-)^m|0>, \quad m \in \mathbb{N},
\]

where

\[
|0> = e^{-2izA^{(3)}_{\infty}} \zeta(z),
\]

with \( \zeta(z) \) an anti-holomorphic function which satisfies appropriate boundary conditions on the torus. It is expressed in terms of \( \theta \)-functions [17]. From the fact that \( D^-|0> = 0 \), one obtains the spectrum for the off-diagonal modes:

\[
M^2 = 2(2m + 1 \pm 2)F^{(3)}_{12}, \quad m \in \mathbb{N}.
\]

For an electric background field \( F_{01} \), the situation is radically different. Instead of using complex coordinates, one uses light-cone coordinates \( x^\pm = (x^1 \pm x^0)/\sqrt{2} \). The equations of motion for \( \delta A^\pm \) are now

\[
\left( \vec{\nabla} \cdot \vec{\nabla} + 2D^- D^+_\mp + 2i(1 \mp 2)F^{(3)}_{01}\right) \delta A^\pm = 0,
\]

with \( \vec{\nabla} \) the gradient in the transversal direction. In order to diagonalize it, we expand the fluctuations in [18],

\[
|m, y^\perp> = (D^-)^m e^{-2ix^+ A^{(3)}_{\infty}(x^-)} \zeta(x^-, y^\perp), \quad m \in \mathbb{N},
\]

where \( y^\perp \) denote the transversal coordinates. This gives,

\[
\left( \vec{\nabla} \cdot \vec{\nabla} + 2i(2m + 1 \mp 2)F^{(3)}_{01}\right) |m, y^\perp> = 0.
\]

The imaginary part reflects the inherent instability of this system which manifests itself by Schwinger’s pair production in an electric field [19], [13], [20].

### 4 The spectrum from string theory

The calculation of the mass spectrum for strings beginning and ending on a tilted brane is straightforward. Assume that the brane is rotated over an angle \( \gamma \) into the 12 plane. The length of the D1-brane is \( \tilde{L}_1 = L_1/\cos \gamma \) and the length of a string attached to the brane and wrapped once around the torus is \( \tilde{L}_2 = \cos \gamma \hat{L}_2 \). With this we get the mass formula,

\[
M^2 = \left( \frac{2\pi m_1}{\tilde{L}_1} \right)^2 + \left( \frac{m_2 \tilde{L}_2}{2\pi \alpha'} \right)^2 = \frac{1}{1 + \tan^2 \gamma} \left\{ \left( \frac{2\pi m_1}{L_1} \right)^2 + \left( \frac{2\pi m_2}{L_2} \right)^2 \right\}, \quad m_1, m_2 \in \mathbb{Z}.
\]
Using the results of previous section, we get $\tan \gamma = 2\pi \alpha' (b^{(0)} \pm b^{(3)})$. In order to calculate the mass spectrum for strings stretching between the two D1-branes at an angle $\phi$, we take one of them along the 1 axis and the other one rotated over an angle $\phi$ in the 12 plane. In this way, the boundary conditions become
\[
\partial_\sigma X^1|_{\sigma=0} = \partial_\tau X^2|_{\tau=0} = 0, \quad (\partial_\sigma X^1 + \tan \phi \partial_\tau X^2)|_{\sigma=\pi} = (\partial_\tau X^2 - \tan \phi \partial_\sigma X^1)|_{\tau=\pi} = 0.
\] (16)

Solving the equations of motion with these boundary conditions give $Z(\sigma, \tau) = f(\sigma^+) + \bar{f}(\sigma^-)$, with $Z = (X^1 + iX^2)/\sqrt{2}$, $\sigma^\pm = (\tau \pm \sigma)$ and
\[
f(\sigma^+) = i\sqrt{\alpha'/2} \sum_{m \in \mathbb{Z}} \frac{a^{+}_{m-\beta}}{m-\beta} e^{-i(m-\beta)\sigma^+}
\]
\[
\bar{f}(\sigma^-) = i\sqrt{\alpha'/2} \sum_{m \in \mathbb{Z}} \frac{a^{-}_{m+\beta}}{m+\beta} e^{-i(m+\beta)\sigma^-},
\] (17)

where $\beta \equiv \phi/\pi$. Upon quantizing this theory, we get low-lying states in the NS sector of the form
\[
|m >_{\pm} = (a^{+}_{-\beta})^m \psi_{-\frac{1}{2}+\beta} |0 >_{NS}, \quad m \in \mathbb{N},
\] (18)

with mass [15], [9],
\[
M^2 = (2\pi \alpha')^{-1}(2m + 1 \pm 2)\phi.
\] (19)

Note that the modes $\psi$ arise from the fermions in the NS sector. The angle $\phi$ is given in eq. (4) in terms of the magnetic fields. The appearance of a tachyon is a consequence of the fact that this configuration breaks all supersymmetry.

T-dualizing back along the 2 direction is equivalent to interchanging $\partial_\tau X^2 \leftrightarrow \partial_\sigma X^2$. The boundary conditions in eq. (16) turn into those for a charged open string with no magnetic field at the $\sigma = 0$ side and a magnetic field $F_{12} = (2\pi \alpha')^{-1}\tan \phi$ at the $\sigma = \pi$ side.

When calculating the spectrum for two D0-branes with relative rapidity $\alpha$, one solves the equations of motion combined with the boundary conditions
\[
\partial_\sigma X^0|_{\sigma=0} = \partial_\tau X^1|_{\tau=0} = 0, \quad (\partial_\sigma X^0 + \tanh \alpha \partial_\tau X^1)|_{\sigma=\pi} = (\partial_\tau X^1 + \tanh \alpha \partial_\sigma X^0)|_{\tau=\pi} = 0.
\] (20)

Once this is done, one passes to the light-cone gauge and quantizes the system. In the case of D-branes with a relative velocity, the light-cone gauge becomes quite subtle. We refer to [18] for more details.

5 Towards the effective action

Comparing the spectrum obtained from an ordinary $U(2)$ Yang-Mills theory, as in section 3, to that predicted by string theory, as in the previous section, one obtains exact agreement only for small values of the background fields. In order to reproduce the string
predictions correctly for arbitrary values of the backgrounds, one should use the Born-Infeld action instead of the Yang-Mills action. It is not hard to see that the diagonal fluctuations only probe the abelian part of the Born-Infeld action,

\[ \mathcal{L} = -\sqrt{\det(\delta_{\mu}^{\nu} + 2\pi\alpha' F_{\mu}^{\nu})}. \]  

(21)

For a D2-brane with a constant magnetic background field \( F_{12} \), the part quadratic in the fluctuations is given by

\[ \mathcal{L} = \frac{1}{2}(\det G)^{-1} \left( G^{ij} \delta \tilde{F}_{0i} \delta \tilde{F}_{0j} - \frac{1}{2} G^{ij} G^{kl} \delta \tilde{F}_{ik} \delta \tilde{F}_{jl} \right). \]  

(22)

where \( \tilde{F} \equiv 2\pi\alpha' F \) and \( G^{ij} = (1 + \tilde{F}^2)^{-1} \delta^{ij} \). It is clear that this does reproduce the correct spectrum.

In [9], the spectrum of the off-diagonal fluctuations was calculated using the NBI action based on the symmetrized trace proposal. The result does not agree with the string calculation. This clearly showed that from order \( F^6 \) on, the symmetrized trace proposal for the NBI, is flawed.

Following an initial exploration of possible modifications at order \( F^6 \), [21], a systematic investigation of the NBI through order \( F^6 \), was performed in [10]. The mass spectrum is used as a guideline. We assume that the following properties hold for the \( U(2) \) NBI:

- Only the field strength should appear, not its derivatives.
- The part quadratic in the fluctuations should reduce to eq. (5) after performing a suitable coordinate transformation.

The first ansatz is a translation of our definition of slowly varying field configurations [10]. The second requirement is based on the observation that the spectrum is a rescaled Yang-Mills spectrum, compare eqs. (11,19). In addition, because we use constant magnetic fields which are block diagonal in the Lorentz indices, only terms in the effective action containing even numbers of Lorentz traced field strengths contribute to our calculation [10]. Under these assumptions the \( F^2 \) and \( F^4 \) terms are fully determined. For the \( F^6 \) terms in the NBI, we find three different types:

- type 1: \( tr(F_{\mu_1\mu_2} F_{\mu_3\mu_4} F_{\mu_5\mu_6} F_{\mu_7\mu_8}) \): 14 inequivalent orderings,
- type 2: \( tr(F_{\mu_1\mu_2} F_{\mu_3\mu_4} F_{\mu_5\mu_6} F_{\mu_7\mu_8}) \): 9 inequivalent orderings,
- type 3: \( tr(F_{\mu_1\mu_3} F_{\mu_2\mu_4} F_{\mu_5\mu_6} F_{\mu_7\mu_8} F_{\mu_9\mu_10}) \): 5 inequivalent orderings.

Calculating the contribution of these terms to the mass spectrum under the assumptions outlined above and comparing it to the string calculation fixes 21 of the 28 parameters. Making the additional assumption that for D5-branes with static self-dual field strength configurations the whole NBI collapses to the Yang-Mills action, [22], fixes an additional 2 parameters.

It is not so surprising that the mass spectrum does not fix all parameters at order \( F^6 \). At this order one has for the first time terms of the form \( tr([F,F][F,F][F,F]) \), which for backgrounds living in the torus of \( U(2) \) do not contribute to the spectrum.
6 Conclusions

The precise structure of the NBI remains an enigma. A direct calculation through gluon scattering amplitudes or $\beta$-functions looks very hard. As we saw, just by using the mass spectrum, the $F^2$ and $F^4$ are easily and fully calculable. To a large extent, the $F^6$ term is determined as well.

A different approach uses invariance under $\kappa$-symmetry as a way to determine the orderings. It remains to be seen how much information this method will extract. However, it is encouraging that the $\kappa$-invariant Lagrangian deviates at low order from the symmetric trace proposal. We refer to elsewhere in this volume for more details [12].

Perhaps one of the most interesting avenues is the study of non-abelian BPS states. These configurations should be solutions of the equations of motion. As explained in [9], [12], some of these BPS configurations relate different orders of $F$ in the equations of motion. In such a way, one might hope for recursion relations to emerge.

In fact, a combination of both previous points might be considered. Indeed, it would be interesting to investigate which non-linear extensions of the Yang-Mills action are consistent with supersymmetry. In 9+1 dimensions, the supersymmetry algebra should be very restrictive, and clues to the appropriate modifications of the susy-variations can be obtained from the demand that the known BPS conditions, which translate non-trivially in terms of background field strengths, solve the condition for a trivial variation of the gluino.

Finally, a completely different approach would be to use some of the ideas in [23]. Consider the abelian case in the presence of a background. Instead of performing the calculations in this paper, one could just push the background field in the open string metric and a non-commutativity parameter. When calculating the mass spectrum in the limit of slowly varying fields, the non-commutative Born-Infeld reduces to a $U(1)$ Yang-Mills theory with the open string metric instead of the flat one. Calculating the spectrum with this modified metric indeed reproduces the spectrum as predicted by string theory. As shown in [23], one can reconstruct from this, order by order, the Born-Infeld action in the commutative limit. The generalization of these observations to the non-abelian case is quite subtle and presently under study [24].

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