Potential Models for Radiative Rare $B$ Decays

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We compute the branching ratios for the radiative rare decays of $B$ into $K$-Meson states and compare them to the experimentally determined branching ratio for inclusive decay $b \to s \gamma$ using nonrelativistic quark model, and form factor definitions consistent with HQET covariant trace formalism. Such calculations necessarily involve a potential model. In order to test the sensitivity of calculations to potential models we have used three different potentials, namely linear potential, screening confining potential and heavy quark potential as it stands in QCD. We find the branching ratios relative to the inclusive $b \to s \gamma$ decay to be $(16.07 \pm 5.2)\%$ for $B \to K^*(892)\gamma$ and $(7.25 \pm 3.2)\%$ for $B \to K_S^0(1430)\gamma$ for linear potential. In the case of the screening confining potential these values are $(19.75 \pm 5.3)\%$ and $(4.74 \pm 1.2)\%$ while those for the heavy quark potential are $(11.18 \pm 4.6)\%$ and $(5.09 \pm 2.7)\%$ respectively. All these values are consistent with the corresponding present CLEO experimental values: $(16.25 \pm 1.21)\%$ and $(5.93 \pm 0.46)\%$.

PACS number(s): 13.40.Hq, 12.37.Pn, 12.39.Hg

1. INTRODUCTION

The flavour changing weak decays of mesons have always been a rich source of information about basic interactions in particle physics. In particular, radiative $B$ decays $B \to K^{(*)}\gamma$ ($K^{**} \sim K^{*}(892)$, $K^{*}(1430)$ etc.) received intensive theoretical studies. The presence of heavy $b$ quark permits the use of Heavy Quark Effective Theory (HQET) in evaluating the relevant hadronic matrix elements where the relevance of the use of a potential model comes in. One purpose of our paper is to test the sensitivity of the branching ratios for $B \to K^{(*)}\gamma$ decays relative to the inclusive $b \to s \gamma$ decay rate to potential models. Among the radiative processes $B \to X_s \gamma$, $B \to K^*(892)\gamma$ and $B \to K_S^0(1430)\gamma$ exclusive branching ratios have been measured experimentally [1]:

$$B(B \to K^*(892)\gamma) = (4.55 \pm 0.34) \times 10^{-5}$$ (1)

$$B(B \to K_S^0(1430)\gamma) = (1.66 \pm 0.13) \times 10^{-5}$$ (2)

and so has been the inclusive rate [2]

$$B(B \to X_s \gamma) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-5}$$ (3)

Several methods have been employed to predict $B \to K^{(*)}\gamma$ decay rate: HQET [3,4], QCD sum rules [5]-[10], quark models [11]-[21], bound state resonances [22] and Lattice QCD [23]-[26]. In this paper we follow the approach of [3,4] in which both $b$ and $s$ quarks are considered heavy. Although the $s$-quark is not particularly heavy and very substantial corrections to the Isgur-Wise functions are to be anticipated, particularly at the symmetry point, yet it has been found that the use of heavy quark symmetry for $s$-quark has not been too bad [4], particularly in connection with prediction of decay rates for $D \to K^{(*)}\ell\nu, D \to K\ell\nu$ from $B \to D^{(*)}\ell\nu$ [3]. It is, therefore, not an unreasonable hope that the static limit may provide results of a comparable accuracy also for the radiative rare $B$-decays. In fact the agreement obtained for $B \to K^{(*)}\gamma$ seems to support this hope. In any case the rates of $K^{**}$ states relative to $B \to K^{*}\gamma$, should not too much depend on $\frac{1}{m_s}$ corrections. In the heavy quark limit, the long distance effects are

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II. THEORETICAL FRAMEWORK

For $B \rightarrow K^{*} \gamma$ decays the effective hamiltonian is well known [31] [33]- [34]:

$$H_{eff} = -\frac{4G_{F}}{\sqrt{2}} V_{tb} V_{ts}^{*} C_{7}(m_{b}) O_{7}(m_{b})$$

where

$$O_{7} = \frac{e}{32\pi^{2}} F_{\mu\nu} [m_{b} \bar{s} \sigma^{\mu\nu}(1 + \gamma_{5}) b + m_{s} \bar{s} \sigma^{\mu\nu}(1 - \gamma_{5}) b]$$

Matrix elements of bilinear currents of two heavy quarks ($J(q) = \bar{Q} TQ$) are most conveniently evaluated within the framework of covariant trace formulasm. Denoting $\omega = \nu', \nu$, where $\nu$ and $\nu'$ are the four velocities of the two mesons, we have

$$\langle \Psi'(\nu')| J(q)| \Psi(\nu) \rangle = Tr[\tilde{M}' (\nu') \Gamma M(\nu)] M(\omega)$$

where $M'$ and $M$ denote matrices describing states $\Psi'(\nu')$ and $\Psi(\nu)$, $\tilde{M} = \gamma^{0} M'^{\dagger} \gamma^{0}$, and $M(\omega)$ represents the LDF. $M$, $M'$ and $M(\omega)$ can be found in [4,30] and using Eq.(5) we can write:

$$\langle K^{*} \gamma | O_{7}(m_{b}) | B \rangle = \frac{e}{16\pi^{2}} \eta_{\nu} q_{\nu} Tr[\tilde{M}' (\nu') \Omega^{\mu\nu} M(\nu)] M(\omega)$$

where the factor $q_{\nu} = m_{B} v_{\nu} - m_{K} \cdot v_{\nu}'$ comes from the derivative in the field strength $F_{\mu\nu}$ of Eq.(4), $\eta_{\nu}$ is the photon polarization vector and

$$\Omega^{\mu\nu} = m_{B} \sigma^{\mu\nu}(1 + \gamma_{5}) + m_{K} \cdot \sigma^{\mu\nu}(1 - \gamma_{5})$$

Using the mass shell condition of the photon ($q^{2} = 0$) and polarization sums for spin-1 and spin-2 particles, the decay rates are calculated in [4,31]:

$$\Gamma(B \rightarrow K^{*}(892)\gamma) = \Omega |\xi_{C}(\omega)|^{2} \frac{1}{y} [(1 - y)^{3}(1 + y)^{5}(1 + y^{2})]$$

$$\Gamma(B \rightarrow K_{1}(1270)\gamma) = \Omega |\xi_{E}(\omega)|^{2} \frac{1}{y} [(1 - y)^{5}(1 + y)^{3}(1 + y^{2})]$$

$$\Gamma(B \rightarrow K_{1}(1400)\gamma) = \Omega |\xi_{F}(\omega)|^{2} \frac{1}{24y^{2}} [(1 - y)^{5}(1 + y)^{7}(1 + y^{2})]$$

$$\Gamma(B \rightarrow K_{2}^{*}(1430)\gamma) = \Omega |\xi_{F}(\omega)|^{2} \frac{1}{8y^{3}} [(1 - y)^{5}(1 + y)^{7}(1 + y^{2})]$$
\[ \Gamma(B \rightarrow K^\ast(1680)\gamma) = \Omega |\xi_G(\omega)|^2 \frac{1}{24y} (1 - y)^7(1 + y)^3(1 + y^2) \] (12)

\[ \Gamma(B \rightarrow K_2(1580)\gamma) = \Omega |\xi_G(\omega)|^2 \frac{1}{8y^3} (1 - y)^7(1 + y)^5(1 + y^2) \] (13)

\[ \Gamma(B \rightarrow K^\ast(1410)\gamma) = \Omega |\xi_G(\omega)|^2 \frac{1}{y} (1 - y)^3(1 + y)^5(1 + y^2) \] (14)

\[ \Gamma(B \rightarrow K_1(1650)\gamma) = \Omega |\xi_G(\omega)|^2 \frac{1}{y} (1 - y)^5(1 + y)^3(1 + y^2) \] (15)

where

\[ y = \frac{m_{K^\ast}}{m_B} \] (16)

\[ \Omega = \frac{\alpha}{128\pi^4} G_F^2 m_B^5 |V_{tb}|^2 |V_{ts}|^2 |C_7(m_B)|^2 \] (17)

and the argument of the Isgur-Wise (IW) function is fixed by the mass shell condition of the photon \( (q^2 = 0) \),

\[ \omega = \frac{1 + y^2}{2y} \] (18)

**III. POTENTIAL MODELS FOR THE ISGUR-WISE FUNCTIONS**

Following [31] for the evaluation of IW form factors needed for the decay rates we assume that we can describe heavy-light mesons using some non-relativistic potential models; the rest frame LDF wave function can then be written as

\[ \phi_{jL}(x) = \sum R_{\alpha L}(r) Y_{LmL}(\Omega) \chi_{m_s} \langle L, m_s | jL, 1/2 \rangle \] (19)

where \( \chi_s \) represent the rest frame spinors normalized to one, \( \chi_s^\dagger \chi_s = \delta_{m_s, m_s} \) and \( \alpha \) represents all other quantum numbers. In [31] following expressions for the form factors are obtained:

\[ \xi_C(\omega) = \frac{2}{\omega + 1} (j_0(ar))_{00}, \quad 0^- \rightarrow (0^-, 1^-) \] (20)

\[ \xi_F(\omega) = \frac{2}{\sqrt{\omega^2 - 1}} (j_1(ar))_{10}, \quad 0^- \rightarrow (0^+, 1^+) \] (21)

\[ \xi_F(\omega) = \frac{2}{\sqrt{\omega^2 - 1}} (j_1(ar))_{10}, \quad 0^- \rightarrow (1^-, 2^+) \] (22)

\[ \xi_C(\omega) = \frac{2\sqrt{3}}{\omega^2 - 1} (j_2(ar))_{20}, \quad 0^- \rightarrow (1^+, 2^+) \] (23)

where denoting the energy of LDF as \( E_q \):

\[ a = (E_q + E_q) \sqrt{\frac{\omega - 1}{\omega + 1}} \] (24)
\begin{equation}
\langle F(ar)\rangle_{L' L}^{\alpha' \alpha} = \int r^2 dr R_{\alpha' L'}^* (r) R_{\alpha L} (r) F(ar)
\end{equation}

To find the form factors we use the method of [31] i.e. solve the Schrödinger Equation numerically. We use three different potentials

Linear potential:
\begin{equation}
V = -\frac{4\alpha_s}{3r} + br + c,
\end{equation}

Screening confining potential [19]:
\begin{equation}
V = \left(-\frac{4\alpha_s}{3r} + br\right) \frac{1 - e^{-\mu r}}{\mu r},
\end{equation}

and Heavy quark potential [32]:
\begin{equation}
V = br - \frac{8C_F}{r} u(r)
\end{equation}

Following [31], we fix \( b = 0.18 \text{ GeV}^2 \) and vary \( \alpha_s \) and \( c \) for a given value of \( m_{u,d} \) (in the range 0.30 – 0.35 GeV) and \( m_s \) (in the range 0.5 – 0.6 GeV), until a good description of the spin averaged spectra of \( K \)-meson states is obtained. Following this procedure, our \( \alpha_s \) ranges from 0.37 to 0.48, while \( c \) takes values from \(-0.83 \text{ GeV} \) to \(-0.90 \text{ GeV} \). These parameters are in good agreement with the original ISGW values [36] (\( \alpha_s = 0.5 \) and \( c = -0.84 \text{ GeV} \) for \( m_{u,d} = 0.33 \text{ GeV} \) and \( m_s = 0.55 \text{ GeV} \)). For the screening confining potential: \( \sigma = 0.18 \pm 0.02 \text{ GeV}^2 \) and \( \mu^{-1} = 0.8 \pm 0.2 \text{ fm} \), while for the heavy quark potential [32] \( C_F = \frac{N^2 - 1}{2N_c} \) and \( [a(q^2)] \) is defined in [32]

\begin{equation}
u(r) = \int_0^\infty dq \frac{\partial}{q} \left( a(q^2) - \frac{k}{q^2} \right) \sin(qr)
\end{equation}

which is calculated numerically at \( r \geq 0.01 \text{ fm} \) and represented in the MATHEMATICA file in the format of notebook at the site http://www.ihep.su/~kiselev/Potential.nb. The short distance behaviour of the potential is purely perturbative, so that at \( r \leq 0.01 \text{ fm} \) we can put

\begin{equation}
V(r) = -C_F \frac{\alpha_v (1/r^2)}{r}
\end{equation}

where the value of the running coupling constant \( \alpha_v (1/r^2) \) at \( r_s = 0.01 \text{ fm} \) is \( \alpha_v (1/r_s^2) = 0.22213 \).

As in [4] the definition of the LDF energy for a \( K^{**} \) meson, proposed to account for the fact that \( s \) mesons are not particularly heavy, is

\begin{equation}
E = \frac{m_{K^{**}} \times m_{u,d}}{m_s + m_{u,d}}
\end{equation}

Another definition which is consistent with heavy quark symmetry is

\begin{equation}
E = m_{K^{**}} - m_s
\end{equation}

These two definitions are not equivalent in the heavy quark limit, so we have done all calculations employing both of these two definitions and at the end we have quoted the broadest possible range of the results obtained. Finally, \( E \) for the \( B \) meson has been taken to be the same as for the \( K^{**} \) meson, consistent with heavy quark symmetry. It turns out that this is actually a very reasonable assumption. To find the size of a meson, which we need for the evaluation of the integral in Eq.(25), we investigate for the asymptotic behavior of Schrödinger Equation for a particular potential model. As an example we display the asymptotic behavior of LDF wave function for linear potential model(for \( \ell = 1 \)) in Fig 1.
In Table 1 we present our results for the ratio \( R = \frac{\Gamma(B \rightarrow K^{*\gamma})}{\Gamma(B \rightarrow X_s\gamma)} \) for various \( K \) meson states; the inclusive branching ratio is usually taken to be QCD improved quark decay rate for \( B \) which can be written as \[29,33\]

\[
\Gamma(B \rightarrow X_s\gamma) = 4 \Omega (1 - \frac{m^2}{m_b^2})^3 (1 + \frac{m^2}{m_b^2})
\]

(33)
giving the prediction for \( BR(b \rightarrow s\gamma) \) to be \((2.8 \pm 0.8) \times 10^{-4}\) \[31\], where the uncertainty is due to the choice of the QCD scale.

We find that the radiative decays of \( B \) into \( K \) meson states saturate 30\% to 50\% the inclusive decay rate. We cannot reach more quantitative conclusions due to errors involved in theoretical estimates, due to which also we cannot at present distinguish between the potential models used. However, even with the present accuracy it should be possible to make such a distinction when data for the other \( K \)-states listed in Table 1 become available.

[17] D.Du and C.Liu, The \( 1/m_s \) Corrections to the Exclusive Rare \( B \) Meson Decays, preprint BIHEP-TH-92-41
[25] K.C.Bowler et al., A lattice calculation of the branching ratio for some of the exclusive modes of \( b \rightarrow s\gamma \), preprint EDINBURGH-94-94-4 (hep-lat/9407013)
[26] D.R.Burford et al., Form-factors for \( B \rightarrow \pi l\nu_l \) and \( B \rightarrow K^{*\gamma} \) decays on the lattice, preprint FERMILAB-PUB-95/023-T (hep-lat/9503002)
<table>
<thead>
<tr>
<th>Meson</th>
<th>$J^P$</th>
<th>Linear Pot.</th>
<th>Scr. Pot.</th>
<th>Heavy Quark</th>
<th>Ref [31]</th>
<th>Experimental value</th>
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<tbody>
<tr>
<td>$K_0^-$</td>
<td>0$^-$</td>
<td>Forbidden</td>
<td>19.75±5.3</td>
<td>11.18±4.6</td>
<td>16.82±6.4</td>
<td>16.25±1.21</td>
</tr>
<tr>
<td>$K_0^+$</td>
<td>1$^+$</td>
<td>16.07±5.2</td>
<td>19.75±5.3</td>
<td>11.18±4.6</td>
<td>16.82±6.4</td>
<td>16.25±1.21</td>
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<tr>
<td>$K_1^+$</td>
<td>0$^+$</td>
<td>Forbidden</td>
<td>4.68±0.8</td>
<td>8.33±2.4</td>
<td>1.63±0.8</td>
<td>4.28±1.6</td>
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<tr>
<td>$K_1^0$</td>
<td>1$^+$</td>
<td>2.43±0.8</td>
<td>3.83±0.8</td>
<td>5.49±0.8</td>
<td>2.07±0.9</td>
<td></td>
</tr>
<tr>
<td>$K_2^+$</td>
<td>2$^+$</td>
<td>7.25±3.2</td>
<td>4.74±1.2</td>
<td>5.09±2.7</td>
<td>6.18±2.9</td>
<td>5.93±0.46</td>
</tr>
<tr>
<td>$K_2^0$</td>
<td>1$^+$</td>
<td>0.5±0.2</td>
<td>3.79±1.1</td>
<td>1.37±0.4</td>
<td>0.54±0.2</td>
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<tr>
<td>$K_3^-$</td>
<td>0$^-$</td>
<td>Forbidden</td>
<td>1.89±0.6</td>
<td>0.74±0.2</td>
<td>1.8±0.4</td>
<td>1.64±0.4</td>
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<td>$K_0^0$</td>
<td>1$^-$</td>
<td>5.5±0.9</td>
<td>6.4±1.2</td>
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<td>$K_1^-$</td>
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<td>1.78±0.7</td>
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FIG. 1. Low r and high r (asymptotic) behavior of LDF wavefunction